Enumerative and Algebraic Combinatorics in the 1960's and 1970's

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(version of 17 June 2021)

The period 1960–1979 was an exciting time for enumerative and algebraic combinatorics (EAC). During this period EAC was transformed into an independent subject which is even stronger and more active today. I will not attempt a comprehensive analysis of the development of EAC but rather focus on persons and topics that were relevant to my own career. Thus the discussion will be partly autobiographical.

There were certainly deep and important results in EAC before 1960. Work related to tree enumeration (including the Matrix-Tree theorem), partitions of integers (in particular, the Rogers-Ramanujan identities), the Redfield-Pólya theory of enumeration under group action, and especially the representation theory of the symmetric group, $GL(n, \mathbb{C})$ and some related groups, featuring work by Georg Frobenius (1849–1917), Alfred Young (1873–1940), and Issai Schur (1875–1941), are some highlights. Much of this work was not concerned with combinatorics *per se*; rather, combinatorics was the natural context for its development. For readers interested in the development of EAC, as well as combinatorics in general, prior to 1960, see Biggs [14], Knuth [77, §7.2.1.7], Stein [147], and Wilson and Watkins [153].

Before 1960 there are just a handful of mathematicians who did a substantial amount of enumerative combinatorics. The most important and influential of these is Percy Alexander MacMahon (1854-1929). He was a highly original pioneer, whose work was not properly appreciated during his lifetime except for his contributions to invariant theory and integer partitions. Much of the work in EAC in the 60's and 70's can trace its roots to MacMahon. Some salient examples are his anticipation of the theory of P-partitions, his development of the theory of plane partitions, and his work on permutation enumeration. It is also interesting that he gave the Rogers-Ramanujan identities a combinatorial interpretation.

William Tutte (1917–2002) was primarily a graph theorist, but in the context of graph theory he made many important contributions to EAC. He developed the theory of matroids, begun by Hassler Whitney (1907–1989), into a serious independent subject.[1] He was involved in the famous project of squaring the square (partitioning a square into at least two squares, all of different sizes) and in the enumeration of planar graphs. He also played a significant role as a codebreaker during World War II, and he remained mathematically active almost until his death.

Three further mathematicians who were active in enumerative combinatorics prior to 1960 (and afterwards) are John Riordan (1903–1988), Leonard Carlitz (1907–1999), and Henry Gould (1928–). Although all three were quite prolific (especially Carlitz), it would be fair to say that their work in combinatorics cannot be compared to MacMahon's and Tutte's great originality. Riordan wrote two books and over 70 papers on EAC. Riordan also has the honor of originating the name "Catalan numbers" [146, pp. 186–187]. He is to be admired for achieving so much work in combinatorics, as well as some work in queuing theory, without a Ph.D. degree. I met him once, at Rockefeller University. Carlitz, whom I also met once (at Duke University), was exceptionally prolific, with about 770 research papers and 45 doctoral students in enumerative combinatorics and number theory. His deepest work was in number theory but was not appreciated until many years after publication. David Hayes [66] states that "(t)his unfortunate circumstance is sometimes attributed to the large number of his research papers." Gould has over 150 papers. His most interesting work from a historical viewpoint is his collection [54] of 500 binomial coefficient summation identities. Nowadays almost all of them would be subsumed by a handful of hypergeometric function identities, but Gould's collection can nevertheless be useful for a non-specialist. Gould also published bibliographies [53] of Bell numbers and Catalan numbers. A curious book [109] of Gould originated from more than 2100 handwritten pages of notes (edited by Jocelyn Quaintance) on relating Stirling numbers of the first kind to Stirling numbers of the second kind via Bernoulli numbers.

I will not attempt a further discussion of the many mathematicians prior

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[1]One further early contributor to matroid theory is Richard Rado (1906–1989). For the early history of matroid theory, see Cunningham [34].

to 1960 whose work impinged on EAC. The "modern" era of EAC began with the rediscovery of MacMahon, analogous to (though of course at a much smaller scale) the rediscovery of ancient Greek mathematics during the Renaissance. Basil Gordon inaugurated (ignoring some minor activity in the 1930's) a revival of the theory of plane partitions. His first paper [27] in this area (with M. S. Cheema) was published in 1964, followed by a series of eight papers, some in collaboration with his student Lorne Houten, during the period 1968–1983. Gordon's contributions were quite substantial, e.g., the generating function for symmetric plane partitions, but he did miss what would now be considered the proper framework for dealing with plane partitions, namely, P-partitions, the RSK algorithm, and representation theory. These will be discussed later in this paper. Gordon also began the "modern era" of the combinatorics of integer partitions with his 1961 combinatorial generalization of the Rogers-Ramanujan identities [52]. Following soon in his footsteps was George Andrews, whose 1964 Ph.D. thesis [3] marked the beginning of a long and influential career devoted primarily to integer partitions, including the definitive text [4].

The 1960's and 1970's (and even earlier) also saw some miscellaneous gems that applied linear algebra to combinatorics, e.g., [12][19][56][57][91], paving the way to later more systematic and sophisticated developments, and the development of spectral graph theory, e.g., Hoffman and Singleton [70] (to pick just one random highlight). For further applications of linear algebra to combinatorics, see Matoušek [100]. For further early work on spectral graph theory, see for instance the bibliographies of Brouwer and Haemers [25] and Cvetković, Rowlinson, and Simić [35].

Another watershed moment in the modern history of EAC is the amazing 1961 paper [125] of Craige Schensted (1927–2021). Schensted was a mathematical physicist with this sole paper on EAC. He was motivated by a preprint of a paper on sorting theory (published as [5]) by Robert Baer and Paul Brock. Schensted defines the now-famous bijection between permutations w in the symmetric group $\mathfrak{S}_n$ and pairs (P, Q) of standard Young tableaux of the same shape $\lambda \vdash n$. (The notation $\lambda \vdash n$ means that λ is a partition of the nonnegative integer n.) This bijection was extended to multiset permutations by Knuth (discussed below) is now most commonly called the RSK algorithm (or just RSK). The letter R in RSK refers to Gilbert de Beauregard Robinson, who (with help from D. E. Littlewood) had previously

given a rather vague description of RSK. For further details on the history of RSK see [142, pp. 399–400]. For $w \in \mathfrak{S}_n$ we denote this bijection by $w \overset{\text{RSK}}{\longrightarrow} (P, Q)$. Schensted proves two fundamental properties of RSK: (a) the length of the longest increasing subsequence of w is equal to the length of the first row of P or Q, and (b) if $w = a_1 a_2 \cdots a_n \overset{\text{RSK}}{\longrightarrow} (P, Q)$, then the "reverse" permutation $a_n \cdots a_2 a_1$ is sent to $(P^t, (Q^*)^t)$, where ^{t} denotes transpose and Q^* is discussed below. It is immediate from (a) and (b) that the length of the longest decreasing subsequence of w is equal to the length of the first column of P.[2]

It was Marcel-Paul Schützenberger (1920–1996) who first realized that RSK was a remarkable algorithm that deserved further study. Beginning in 1963 [127] he developed many properties of RSK and its applications to the representation theory of the symmetric group, including the fundamental symmetry $w \overset{\text{RSK}}{\longrightarrow} (P, Q) \Rightarrow w^{-1} \overset{\text{RSK}}{\longrightarrow} (Q, P)$, the combinatorial description of the tableau Q^* defined above, and the theory of *jeu de taquin*, a kind of two-dimensional refinement of RSK. He realized that the formula $Q^{**} = Q$ (obvious from the definition) can be extended to linear extensions of any finite poset, leading to his theory of promotion and evacuation [128]. I can remember being both mystified and enthralled when I heard him lecture on this topic at M.I.T. on April 21, 1971. Eventually I became familiar enough with the subject to write a survey [143]. The writing style of Schützenberger (and his later collaborator Alain Lascoux) could be very opaque. Schützenberger once said (perhaps tongue in cheek) that he deliberately wrote this way because mathematics should be learned through struggle; it shouldn't be handed to someone on a silver platter.[3] Beginning in 1978 [84] Schützenberger began a long and fruitful collaboration with Alain Lascoux (1944–2013). Highlights of this work include the plactic monoid and the theory of Schubert polynomials.

Dominique Foata (1934–) was perhaps the first modern researcher to look

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[2]Joel Spencer once informed me that he rediscovered the RSK algorithm and the results of Schensted in order to solve a problem [116] in the *American Mathematical Monthly* proposed by Stanley Rabinowitz. Since Spencer's solution was not published he received no recognition, other than having his name listed as a solver, for this admirable accomplishment.

[3]Recall Nietzsche's famous quote, "Was ist Glück?—Das Gefühl davon, daß die Macht wächst, daß ein Widerstand überwunden wird." (Happiness is the feeling that power increases—that resistance is being overcome.)

seriously at the work of MacMahon on permutation enumeration. Foata's first paper [43] in this area appeared in 1963, followed in 1968 by his famous bijective proof [44] of the equidistribution of the number of inversions and the major index of a permutation in $\mathfrak{S}_n$. Foata and various collaborators did much further work to advance enumerative combinatorics, involving such areas as multiset permutations, tree enumeration, Eulerian polynomials, rook theory, etc. Other researchers in the flourishing French school of EAC before 1980 include Dominique Dumont, Jean Françon, Germain Kreweras, Yves Poupard, Schützenberger, and Xavier Gérard Viennot. In particular, Kreweras (1918–1998) [79] and Poupard [113] launched the far-reaching subject of noncrossing partitions.

The theory of parking functions is another thriving area of present-day EAC. The first paper on parking functions *per se* was published in 1966 by Alan Gustave Konheim and Benjamin Weiss [78], though the basic result that there are $(n+1)^{n-1}$ parking functions of length n is equivalent to a special case of a result of Ronald Pyke [115, Lemma 1] in 1959. Some further work was done in the 1970's by Foata, Françon, Riordan and others, but the subject did not really take off until the 1990's, when connections were found with diagonal harmonics, symmetric functions, Lagrange inversion, hyperplane arrangements, polytopes, Tutte polynomials, noncrossing partitions, etc. See Yan [155] for a survey of much of this more recent work.

A major impetus to the development of EAC was Gian-Carlo Rota (1932–1999). He began his career in analysis but after a while was seduced by the siren call of combinatorics. His first paper [47] in EAC (with Roberto Frucht) appeared in 1963 and was devoted to the computation of the Möbius function of the lattice of partitions of a set. Rota was very ambitious and always wanted to see the "big picture." He realized that the Möbius function of a (locally finite) partially ordered set, first appearing in the 1930's in the work of Philip Hall and Louis Weisner, had tremendous potential for unifying much of enumerative combinatorics and connecting it with other areas of mathematics. This vision led to his seminal paper [122] on Möbius functions, with the rather audacious title "On the foundations of combinatorial theory. I. Theory of Möbius functions." This paper had a tremendous influence on the development of EAC and placing posets in a central role. For some further information on Rota's influence on EAC, see the introductory essays

by Bogart, Chen, Goldman, and Crapo in [81][4].

Finite posets, despite their simple definition, have a remarkably rich theory. In addition to the vast number of interesting examples and special classes of posets, there are a surprising number of deep results and questions that are applicable to *all* finite posets. These include incidence algebras and Möbius functions, Möbius algebras, the connection with finite distributive lattices [144, §3.4], *P*-partitions [144, §3.15], poset polytopes [140][144, Exer. 4.58], the order complex and order homology [16, §3], Greene's theory [58] of chains and antichains, evacuation and promotion [143], correlation inequalities (in particular, the XYZ conjecture [130]), the $\frac{1}{3}$-$\frac{2}{3}$ conjecture (surveyed by Brightwell [24]), the Chung-Fishburn-Graham conjecture on heights of elements in linear extensions [138, §3], dimension theory [148], etc. I should also mention the text [15] on lattice theory by Garrett Birkhoff (1911–1996)[5], first published in 1940 with three editions altogether. Most of the book deals with infinite lattices and was not combinatorial, but the early chapters have some interesting combinatorial material on finite lattices and posets. (The term "poset" is due to Birkhoff.)

The 1960's saw the development of an EAC infrastructure, in particular, conferences, journals, textbooks, and prizes. The first conference on matroid theory took place in 1964 (see [34]). On January 1, 1967, the Faculty of Mathematics was founded at the University of Waterloo (in Waterloo, Ontario, Canada), which included a Department of Combinatorics and Optimization. To this day it remains the only academic department in the world devoted to combinatorics (though the Center for Combinatorics at Nankai University functions somewhat similarly). The University of Waterloo became a center for enumerative combinatorics with the arrival of David Jackson in 1972, followed by Ian Goulden, who received his Ph.D. from Jackson in 1979 and has a long and fruitful collaboration with him. Much of their early work appears in their book [55], which is jam-packed with engaging results.

The Department of Combinatorics and Optimization at the University of Waterloo sponsored three early conferences in combinatorics. The Third

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[4]For my own account on how I was influenced by Foundations I and decided to work with Rota, see [145]

[5]Three persons significant for combinatorics died in 1996: Schützenberger, Birkhoff, and Pál (Paul) Erdős.

Waterloo Conference on Combinatorics, held in 1968, was the only one to have a Proceedings [149] and was the first combinatorics conference that I attended. At that time I was a graduate student at Harvard University. Most of the talks were on graph theory and design theory. Jay Goldman and Rota discussed the number of subspaces of a vector space over $\mathbb{F}_q$ [51]. For me it was a fantastic opportunity to meet such legendary (to me, at any rate) mathematicians as Elwyn Berlekamp (later to become my official host when I was a Miller Fellow at U.C. Berkeley, 1971–73), Branko Grünbaum, William Tutte, Alan Hoffman, Crispin Nash-Williams, Johan Seidel, Nicolaas de Bruijn, Richard Rado, Nathan Mendelsohn, Ralph Stanton, Richard Guy, Frank Harary, Roberto Frucht, Claude Berge, Horst Sachs, Denis Higgs, *et al.* One incident sticks in my mind that illustrates the nature of algebraic combinatorics at that time. During an unsolved problem session, an expert on graph automorphisms presented the conjecture that a vertex-transitive graph Γ with a prime number p of vertices has an automorphism which cyclically permutes all the vertices. I instantaneously saw (though I did not mention it until after the problem session) that since a transitive permutation group acting on an n-element S set has a subgroup of index n (the subgroup fixing some element of S), the order of $\mathrm{Aut}(\Gamma)$ is divisible by p. Hence by elementary group theory (Cauchy's theorem), $\mathrm{Aut}(G)$ contains an element of order p, which can only be a p-cycle. Rota later told me that the word had spread about my proof and that many participants were impressed by it.

Another early conference (which I did not attend) that involved some EAC was the Symposium in Pure Mathematics of the American Mathematical Society, held in 1968 at UCLA.[6] Of the 24 papers appearing in the conference proceedings [102], six or so can be said to concern EAC. A further early conference was the two-week June, 1969, meeting in Calgary entitled Combinatorial Structures and Their Applications [62], organized by Richard Guy and Eric Milner. Participants related to EAC included David Barnette, Henry Crapo, Jack Edmonds (who introduced the concept of a polymatroid at the meeting), Curtis Greene, Branko Grünbaum, Peter McMullen, John Moon, Leo Moser, Tutte, and Dominic Welsh.

The University of North Carolina had some strength in combinatorics

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[6]Since the late 1940's UCLA has had a lot of combinatorial activity, as summarized by Bruce Rothschild [123].

headed by the statistician Raj Chandra Bose (1901–1987). Thomas Dowling was a student of Bose who received his Ph.D. in 1967 and stayed at UNC for several years afterwards. Bose and Dowling were the organizers of a 1967 combinatorics conference at UNC. Of the 33 papers in the proceedings [20] of the 1967 conference, perhaps three or four of them could be said to belong to EAC (by Henry Mann, Riordan, Lentin-Schützenberger, and Hoffman). There were also some papers on coding theory which had a strong EAC flavor, though nowadays coding theory is rather tangential to mainstream EAC.

Bose and his collaborator K. R. Nair [22] founded the theory of *association schemes* in 1939, though the term "association scheme" is due to Bose and T. Shimamoto [23] in 1952. They arose in the applications of design theory to statistics. They became of algebraic interest (primarily linear algebra) with a 1959 paper by Bose and Dale Mesner [21] and are a kind of combinatorial analogue of the character theory of finite groups. A major contribution was made in the 1973 doctoral thesis by Philippe Delsarte at the Université Catholique de Louvain, reprinted as a Philips Research Report [39]. Today association schemes remain an active subject, but there is little interaction between its practitioners and "mainstream" EAC.

A second, more elaborate combinatorics conference was held at UNC in 1970 (which I had the pleasure of attending). Its participants included Martin Aigner, George Andrews, Edward Bender, Raj Chandra Bose, Vašek Chvátal, Henry Crapo, Philippe Delsarte, Jack Edmonds, Paul Erdős, Ray Fulkerson, Jean-Marie Goethals, Jay Goldman, Solomon Golomb, Ralph Gomory, Ron Graham, Branko Grünbaum, Richard Guy, Larry Harper, Daniel Kleitman, Jesse MacWilliams, John Moon, Ronald Mullin, Crispin St. John Alvah Nash-Williams, Albert Nijenhuis, George Pólya, Richard Rado, Herbert Ryser, Seymour Sherman, Neil Sloane, Gustave Solomon, Joel Spencer, Alan Tucker, Neil White, and Richard Wilson. Of the 53 papers in the Proceedings [114], around 18 were in EAC and another four on the boundary. (Of course there is some subjectivity in evaluating which papers belong to EAC.) One can see quite a significant increase in EAC activity from 1967 to 1970. In particular, the 1970 proceedings had a number of papers on matroid theory, a rapidly growing area within EAC. Rota was paying some visits to UNC at this time and managed to make many combinatorial converts, including Robert Davis, Ladnor Geissinger, William Graves, and Douglas Kelly. Rota's

student Tom Brylawski also arrived at UNC in 1970.

We can mention two more meetings of note, both occurring in 1971. The University of Waterloo had a Conference on Möbius Algebras. The *Möbius algebra* of a finite lattice (or more generally, finite meet-semilattice) over a field K is the semigroup algebra over K of L with respect to the operation $\wedge$ (meet). It was first defined by Louis Solomon, who developed its basic properties and extended the definition to arbitrary finite posets. Davis and Geissinger made further contributions, and Curtis Greene [59] gave an elegant presentation of the theory with additional development, including a more natural and transparent treatment of the extension to finite posets. An exposition (for meet-semilattices) is given in [144, §3.9]. Möbius algebras allow a unified algebraic treatment of the Möbius function of a finite lattice. One can see how much EAC progressed since Rota's *Foundations I* paper [122] in 1964 by the existence of a conference on the rather specialized topic of Möbius algebras (though the actual meeting was not really this specialized). For the Proceedings, see [33]. One amusing aspect of this conference is that, due to a tight budget, the participants from the USA shared a single hotel suite. The suite had a large living room and two adjoining bedrooms. Those unfortunate enough not to get a bedroom slept in sleeping bags in the living room. Naturally one of the bedrooms went to Rota. The second person (nameless here, but not myself) received this honor because he snored loudly.

The other 1971 meeting was the National Science Foundation sponsored Advanced Seminar in Combinatorial Theory, held for eight weeks during the summer at Bowdoin College in Brunswick, Maine. The existence of this conference further exemplifies the tremendous progress made by EAC since the early 1960's. Figure 1 shows an announcement of the meeting. The main speaker was Rota, who gave a course on combinatorial theory and its applications, assisted by Greene. There were also nine featured speakers, one per week except two the last week.

Rota and Greene soon decided that it would be better to alter their lecture arrangement and for Greene to give instead a parallel set of lectures on combinatorial geometries, with lecture notes written up by Dan Kennedy. "Combinatorial geometry" was a term introduced by Rota to replace "ma-

troid."[7]. Rota thought that the term "matroid" was cacaphonous and had a frivolous connotation. However, Rota's terminology never caught on, so eventually he accepted the term "matroid." But he was right on the button about the importance of matroids in EAC and in mathematics in general. His seminal book [32] with Henry Crapo (Rota's first Ph.D. student in combinatorics) established the foundations of the subject for EAC. Sadly only a preliminary edition was published. One of the topics of this book is the *critical problem*, which Rota hoped would be a fruitful new approach toward the coloring of graphs. This hope, however, has not been realized.

We have mentioned Rota's great foresight in assessing the role of posets and matroids within EAC. Two other subjects for which Rota had high hopes are finite operator calculus (which began with a joint paper [103] with Ronald Mullin) and classical invariant theory. Finite operator calculus can be appreciated by considering the two operators D and Δ on the space of all polynomials in one variable x, say over the reals. They are defined by

$$
\begin{aligned}
Df(x) &= f'(x) \quad \text{(differentiation)} \\
\Delta f(x) &= f(x+1) - f(x) \quad \text{(difference)}.
\end{aligned}
$$

They have many analogous properties; here are a sample. We use the falling factorial notation $(y)_m = y(y-1)\cdots(y-m+1)$.

D	Δ
$De^x = e^x$	$\Delta 2^x = 2^x$
$Dx^n = nx^{n-1}$	$\Delta(x)_n = n(x)_{n-1}$
$f(x+t) = \sum_{n\geq 0} D^n f(t)\frac{x^n}{n!}$	$f(x+t) = \sum_{n\geq 0} \Delta^n f(t)\frac{(x)_n}{n!}$.

Moreover, from the Taylor series expansion

$$
f(x+1) = \sum_{n\geq 0} \frac{D^n f(x)}{n!}
$$

we get the formal identity $\Delta = e^D - 1$. Finite operator calculus gives an elegant explanation for and vast generalization of these results. It also encompasses *umbral calculus*, a seemingly magical technique made rigorous by

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[7]More accurately, "(combinatorial) pregeometry" was the new term for matroid. A (combinatorial) geometry is a loopless matroid for which all two-element subsets are independent. Such matroids are called "simple matroids."

Combinatorial Theory Advanced Science Seminar

for Graduate and Postgraduate Students of Mathematics

Financed by the National Science Foundation

AT

Bowdoin College

Brunswick, Maine

JUNE 22 TO AUGUST 12, 1971

OBJECTIVES

The aim of this Seminar is to enrich mathematics education and stimulate research by providing opportunities for

(1) **Graduate students** to acquire, in a research environment, an understanding of combinatorial theory, and its applications, and to discover new topics for research as well as new tools for research in progress;

(2) **Postdoctoral students** to assist and direct graduate students in studies and investigations, while pursuing their own work amidst mathematicians of similar interest;

(3) **Experienced research mathematicians** to exchange ideas and to work in an environment pleasant for their families and lively for themselves.

PROGRAM

A Course entitled **Combinatorial Theory and Applications** given by Gian-Carlo Rota, assisted by Curtis Greene. A tentative course outline follows.

The theme of the lectures will be the study of some of the classical combinatorial problems by conceptual methods drawn from all branches of mathematics, especially by those developed in combinatorial theory itself. The solution of problems, however amusing or important, by specific, ad hoc devices will be systematically discouraged, in favor of the perception of analogies between problems which on the surface appear totally different; it will be shown that the discovery of such analogies often leads to understanding of the problem.

The main topics to be treated will be

(1) Enumeration theory, its connection with group theory; theorem of Polya and its follow-ups; incidence algebras; Möbius inversion; combinatorial interpretation of special functions.

(2) Combinatorial Geometry, its connections with classical invariant theory, with the coloring problem and coding problems; the critical problem, matching theory. The Tutte representation theory. The Tutte-Grothendieck group.

Among shorter topics to be treated will be: statistical mechanics for the mathematician, the combinatorics of optimization, topics in convexity, and the combinatorics of the symmetric group.

Seminars at various levels on appropriate topics.

A Research Colloquium in Combinatorial Mathematics. An approximate tentative schedule follows. Several of the visiting lecturers will be in Brunswick for an extended period.

Week of June 22	F. Harary (Michigan)	"All About Trees"
Week of June 28	D. R. Fulkerson (RAND)	"Blocking and Anti-Blocking Pairs of Polyhedra"
Week of July 5	B. Grünbaum (Washington)	"Combinatorial Geometry of the Projective Plane"
Week of July 12	H. H. Crapo (Waterloo)	"Constructions in Combinatorial Geometry"
Week of July 19	H. S. Wilf (Penn.)	"The Permanent Function in Combinatorial Analysis"
Week of July 26	H. J. Ryser (Caltech)	"Intersection Properties of Finite Sets"
Week of Aug. 2	W. T. Tutte (Waterloo)	"The Enumerative Theory of Chromatic Polynomials"
Week of Aug. 9	M. Hall, Jr. (Caltech)	"Construction of Combinatorial Designs"
	A. W. Tucker (Princeton)	"Pivotal Algebra"

SELECTION OF PARTICIPANTS

Graduate students will be selected primarily from the nominees of departments at Ph.D. granting institutions. Nominations should be on the basis of proven ability and the appropriateness of this program for the student. An interested graduate student should ask his chairman or probable dissertation advisor to write a letter of endorsement. When a department makes more than one nomination, the candidates should be ranked. The applicant should provide a list of mathematics courses taken to date, with special detail about his background in relevant topics, and a statement of reasons for entering this particular program. In general we expect to select graduate students who need instruction and experience in combinatorial theory at this particular time, and who will have an opportunity in their own graduate schools to put the benefits of the program to effective use.

Postdoctoral students will assist in the program for graduate students and engage in research. Some will be nominees of the senior staff. Each candidate should submit, in addition to a letter of endorsement from an established mathematician in the field, an academic vita, a list of publications and of research in progress, and a statement indicating his objectives in this program.

Senior research mathematicians applying for subsistence or travel grants should write to the Director providing the same information requested from postdoctoral candidates. Dates of visit should be specified.

Figure 1: Bowdoin seminar announcement

Rota et al. in which powers like a^n are replaced by a_n.[8] The main shortcoming of finite operator calculus within EAC is its limited applicability. Rota was a little miffed when I relegated finite operator calculus to a five-part exercise [142, Exer. 5.37] in my two books on enumerative combinatorics.

Adriano Garsia followed in Rota's footsteps by beginning in analysis but converting to combinatorics (under Rota's influence). Garsia's first combinatorics paper was an exposition [48] of finite operator calculus entitled "An exposé of the Mullin-Rota theory of polynomials of binomial type." Rota was annoyed at the title since "exposé" can have a negative connotation.[9] However, Garsia simply intended for "exposé" to mean "exposition." Garsia went on to make many significant contributions to EAC, including (with Stephen Milne) a long sought-for combinatorial proof of the Rogers-Ramanujan identities [49]. Later he did much beautiful work related to symmetric functions, in particular, Macdonald polynomials, diagonal harmonics, and the $n!$ conjecture.

Rota's work (with many collaborators) on classical invariant theory never really caught on. One reason is that its algorithmic approach toward algebraic questions (such as finite generation) were superseded by the work of David Hilbert. Connections with algebraic geometry were better handled by modern techniques, viz., the geometric invariant theory pioneered by Mumford. More recently, however, there has been somewhat of a resurgence of interest in classical invariant theory, as attested by the book [107] of Peter Olver. Olver states that this resurgence is "driven by several factors: new theoretical developments; a revival of computational methods coupled with powerful new computer algebra packages; and a wealth of new applications, ranging from number theory to geometry, physics to computer vision." Nevertheless, classical invariant theory has had limited impact on modern EAC. There is scant mention of Rota in Olver's book.

My grades (using the common A–F grading system at most American academic institutions) for the significance of Rota's four main research topics within EAC are as follows:

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[8] Gessel [50] wrote a good paper for getting the flavor of umbral calculus.

[9] One dictionary definition is "a report of facts about something, especially a journalistic report that reveals something scandalous."

posets	A+
matroid theory	A
finite operator calculus	B–
classical invariant theory	C

Matroid theory gets a slightly lower grade than posets because posets are more ubiquitous objects both inside and outside EAC than matroids. However, matroids have made many surprising appearances in other areas that put them just below posets (in my opinion, of course).[10]

One further topic of interest to Rota was Hopf algebras. In particular, he anticipated the subject of combinatorial Hopf algebras, e.g., in his perceptive article [71] with Joni. I refrain from giving a grade to Rota's work on Hopf algebras since he did not develop the theory as intensively as the four topics above. For two recent surveys of Hopf algebras in combinatorics, see Grinberg-Reiner [61] and Loday-Ronco [90].

Getting back to the Bowdoin conference, let me mention three frivolous anecdotes. One day Rota organized a tandem lecture. He chose six people from the audience, including me. We had to leave the room and not communicate among ourselves. He called us in the room one at a time. Each person had to deliver a five-minute math lecture before the next person was called in. The first lecture could be arbitrary, but after that the lecture had to be a continuation of what was written on the chalkboard. You were not allowed to erase anything that you wrote, though you could erase what earlier speakers had written. My strategy (not being the first person) was to decide on my lecture topic in advance and to find some ridiculous way to link it to what appeared on the board. It would be interesting to hold some more tandem lectures.

At the end of the meeting Rota also organized a prize ceremony. The prizes were based on puns and other nonsense. I remember that I received the Marriage Theorem (discussed by Rota in one of his lectures) award because I arrived at the meeting two weeks late shortly after my honeymoon. In addition to the prizes, Rota also furnished several cases of champagne.

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[10]For information on the life and work of Rota, see [82][83][106].

The final anecdote concerns what might be the all-time greatest EAC pun. A famous theorem of Tutte characterizes graphic matroids in terms of five excluded minors. Planar graphic matroids can be characterized by two additional excluded minors (Kuratowski's theorem). Dan Kennedy suggested in his writeup of Greene's talks on combinatorial geometries that a *pornographic matroid* is one for which all minors are excluded.

A further indication of the growth of AEC was the occurrence of workshops on this topic at the Mathematische Forschungsinstitut Oberwolfach. It has held regular week-long workshops (eventually held almost every week of the year) on different mathematical topics since 1949. The first workshop on combinatorics *per se* was entitled Angewandte Kombinatorik (Applied Combinatorics) and held October 15–21, 1972.[11] All participants seem to have been Europeans. Only a few were connected with EAC, including Foata, Adalbert Kerber, and Volker Strehl.

The second Oberwolfach workshop on combinatorics, entitled Reine Kombinatorik (Pure Combinatorics), was held March 26–31, 1973 (my first trip overseas). It was the first of several workshops organized by Foata and Konrad Jacobs. The focus was almost entirely on EAC. See Figure 2 for the signatures of the participants appearing in the *Gästebuch III*.[12] One highlight for me was the opportunity to meet Herbert Foulkes, a pioneer of symmetric function combinatorics. I gave a talk on my paper [133].

I had an amusing experience after this meeting unrelated to mathematics. Martin Aigner invited me to visit him in Tübingen. I then needed to take the train from Tübingen to the Frankfurt airport for my return to the USA. It was necessary to change trains in Stuttgart. Being unfamiliar with German cities and trains, when the train arrived at Stuttgart-Bad Cannstatt I thought this was the main station for Stuttgart and got off the train. By the time I realized my error, the train had already departed. According to the posted schedules, there was no way to get to the Frankfurt airport in time for my flight. At the time I knew only a tiny bit of German and tried to find someone in the station to whom to explain in English my predicament. When that failed I

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[11]There were some previous workshops related to combinatorics, on such topics as discrete geometry, graph theory, and convex bodies.

[12]The Vortragsbücher (Books of Abstracts) and Gästebücher (Guest Books) of the Oberwolfach workshops are available online at the Oberwolfach Digital Archive [105].
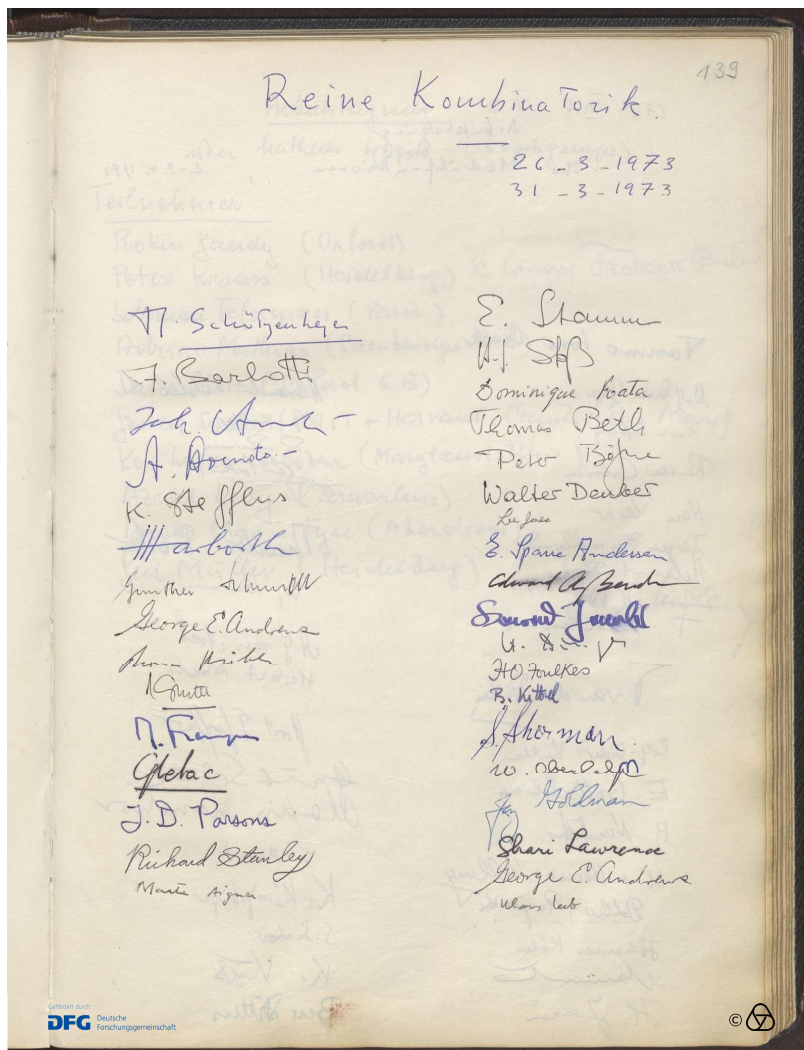
Figure 2: Oberwolfach participant list

went to a taxi stand and said that I wanted to go to the Frankfurt airport. The driver was incredulous but eventually said that the fare was around $100 (US dollars), which was a huge amount in 1973 for me to spend on a taxi. Nevertheless, I could think of no feasible alternative so I agreed. Given this sad situation, who could believe that I would arrive at the Frankfurt airport one hour before the train at a cost of $5.00?

I showed the driver some German currency Travelers Checks (credit cards were not in common use then) to show that I could pay. He was not familiar with Travelers Checks and took me to an office in the train station where they could be verified. Fortunately the woman in the office could speak English. She told me that it might be cheaper to fly from Stuttgart to Frankfurt. She called a travel agency and found out that there was a special connector flight to my Frankfurt flight, and that the cost was free! She got me a reservation, and the taxi driver ended up driving me only to the Stuttgart airport for around $5. I ended up arriving at the Frankfurt airport about one hour before the train! Had I known about this possibility in advance, I could have even saved money by purchasing my train ticket only to Stuttgart.

Conferences on EAC in the USA and Europe were becoming increasingly more common. Another one in Oberwolfach (this time entitled just Kombinatorik) in 1975 gave me the chance to meet for the first time such luminaries as Kerber, Gilbert de B. Robinson, Gordon James, Glanffrwd Thomas, Paul Stein, Louis Comtet, Ron King, Hanafi Farahat, and Michael Peel. Many of these persons worked on symmetric functions and the representation theory of the symmetric group, which was quickly becoming a major area of EAC. I remember that Comtet gave perfectly organized talks, with every square inch of the chalkboard planned in advance, similar to later talks I heard by Ian Macdonald.

One other conference in the mid-1970's was especially noteworthy for me. This was the NATO Advanced Study Institute on Higher Combinatorics, held in (West) Berlin on September 1–10, 1976 [1]. A nonmathematical highlight of the meeting was a highly structured and supervised visit to East Berlin. I remember that when we returned to West Berlin, the East German guards used mirrors to look under our bus to check for possible defectors. One of the participants of the Berlin meeting was a graduate student from Kungliga Tekniska högskolan (Royal Institute of Technology) in Stockholm

named Anders Björner. This meeting inspired him to work in mainstream EAC, especially the emerging area of topological combinatorics. He was a visiting graduate student at M.I.T. during the academic year 1977–78 and went on to become a leading researcher in EAC and a close collaborator of mine. In 1979 I was the opponent (somewhat like an external examiner) for his thesis defense in Stockholm.[13] A slightly earlier visitor to the Boston area was Louis Billera, who was at Brandeis University for the 1974–75 academic year. He started out in operations research and game theory but became interested in combinatorial commutative algebra. At that time Brandeis was a world center for commutative algebra, but Billera spent quite a bit of time at M.I.T. and also developed into an EAC leader.

Books on EAC saw a rapid development beginning in the 1960's. Before then, we have Whitworth's treatise *Choice and Chance* on elementary enumeration and probability theory, first published in 1867. The 1901 fifth edition [152] includes 1000 exercises. Two further premodern books are the pioneering treatise [104] by Eugen Netto, and the fascinating opus [99] by MacMahon. Riordan's 1958 book [119] is a kind of bridge between the premodern and modern eras. Netto dealt primarily, and MacMahon and Riordan exclusively, with enumerative combinatorics.

An interesting book [37] by Florence Nightingale David and D. E. Barton on enumerative combinatorics aimed at statisticians was published in 1962. They followed up in 1966, joined by Maurice George Kendall, with tables of symmetric function data (but with nothing on Schur functions) preceded by a lengthy introduction [38]. Herbert Ryser in 1963 wrote an engaging monograph [124] on combinatorics containing some chapters on enumeration. In 1967 Marshall Hall (my undergraduate adviser) wrote a textbook [63] that gave a quite broad coverage of combinatorics, including such topics as permutations, Möbius functions of posets, generating functions, partitions, Ramsey theory, extremal problems, the simplex method, de Bruijn sequences, block designs, and difference sets. There appeared one year later a book [11] by Claude Berge[14] (with a very entertaining Foreword by Rota[15]) on EAC with

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[13]A nice Swedish tradition is that the student treats the thesis defense attendees to dinner after the defense.

[14]Berge was primarily a graph theorist, but he was amazingly prescient in his selection of topics for his book on combinatorics.

[15]This Foreword includes the famous statement, referring to the 1968 French edi-

many interesting topics, including the Möbius function of a poset and, for the first time in a book, a discussion of standard Young tableaux, RSK, and counting chains in Young's lattice. Also in 1969 there appeared a book [89] by Chung-Laung Liu based on a course taught in the Electrical Engineering Department of M.I.T. As in Hall's book there is a broad selection of topics including some enumerative combinatorics. In 1968 Riordan followed up his book [119] on combinatorial analysis with a book [120] on combinatorial identities.

In 1964 appeared a collection of articles by leading mathematicians, applied mathematicians, and physicists entitled *Applied Combinatorial Mathematics* [8], edited by Edwin F. Beckenbach. The book were divided into four parts: Computation and Evaluation, Counting and Enumeration, Control and Extremization, and Construction and Existence. The part on Counting and Enumeration had the articles "Generating Functions" by Riordan, "Lattice Statistics" by Elliot W. Montroll, "Pólya's Theory of Counting" by de Bruijn, and "Combinatorial Problems in Graphical Enumeration" by Frank Harary. The authors of articles in the other three parts include Derrick H. Lehmer, Richard Bellman, Albert W. Tucker, Marshall Hall, Jr., Jacob Wolfowitz, and George Gamow. There are four appendices reprinted from a 1949 English translation [150] (somewhat revised) of a 1926 article by Hermann Weyl. The first appendix is entitled "Ars Combinatoria."

A very interesting set of lecture notes was published in 1969 by the mathematical physicist Jerome Percus [110] (upgraded to a more polished version [111] in 1971). The first part deals with enumerative topics like generating functions, MacMahon's Master Theorem, partitions, permutations, and Redfield-Pólya theory. The second part concerns combinatorial problems arising from statistical mechanics, such as the dimer problem, square ice, and the Ising model. This deep topic is one of the most significant applications of EAC to other areas and remains of great interest today. Let me just mention the book [7] of Rodney Baxter as an example of this development.

A significant publication was Comtet's 1970 *Analyse combinatoire* [30], with an expanded English edition [31] published in 1974. Despite the small

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tion, "Soon after that reading, I would be one of many who unknotted themselves from the tentacles of the Continuum and joined the then Rebel Army of the Discrete."

physical size ($4\frac{1}{2}''\times 6\frac{15}{16}''$) of the 1970 French edition, it was packed with an incredible amount of interesting enumerative combinatorics. The expanded English edition contains even more information. Finally, mention should be made of Earl Glen Whitehead's 1972 lecture notes [151]. Though rather routine, it seems to be the first book or monograph with the title "Enumerative Combinatorics" or something similar.

According to MathSciNet, at the time of this writing (2021) there are around 35 journals devoted to combinatorics or to the connection between combinatorics and some other area. The first such journal did not appear until 1966 (the year I started graduate school). This was the *Journal of Combinatorial Theory*, later split into parts A (mainly EAC) and B (mainly graph theory). Frank Harary and Rota founded the journal, Tutte was the first Editor-in-Chief, and Ron Mullin became the de facto Managing Editor. The first issue had a foreword by Pólya, who called it "a stepping stone to further progress." Hans Rademacher, Robert Dickson, and Robert Plotkin had the honor of having the first paper in this journal, followed by Tutte. Further details on the founding of JCT and the subsequent split into A and B are recounted by Edwin Beschler [13] and the editorial article [6].

The first prize to be established in combinatorics was the George Pólya Prize in Applied Combinatorics, awarded by the Society for Industrial and Applied Mathematics (SIAM). The recipients prior to 1980 were Ronald Graham, Klaus Leeb, Bruce Rothschild, Alfred Hales, and Robert Jewett in 1971, Richard Stanley, Endre Szemerédi, and Richard Wilson in 1975, and László Lovász in 1979. The 1975 prize was awarded in San Francisco, so as a bonus Richard Wilson and I (Szemerédi was not present) were invited by Pólya, then aged 87, to visit his home in Palo Alto. We spent an unforgettable evening being shown by Pólya his scrapbooks and other memorabilia. A further combinatorics prize established prior to 1980 is the Delbert Ray Fulkerson Prize of the American Mathematical Society. It was first awarded in 1979, to Richard Karp, Kenneth Appel, Wolfgang Haken, and Paul Seymour. Appel and Haken received it for their computer assisted proof of the four-color conjecture.

One of my main mathematical interests, from graduate school to the present day, is the theory of symmetric functions. I will briefly discuss its rise to prominence in the 1960's and 70's. I already mentioned the remarkable

paper of Schensted [125] from 1961, which was one of the main sources of stimulation for further development of combinatorics related to symmetric functions. An even more important paper for this purpose was published in 1959 in an obscure conference proceedings by Philip Hall (1904–1982) [64].[16] The subject might have developed sooner if this publication had been more widely known.[17] It develops the now-familiar linear algebraic approach toward symmetric functions, presenting known results in terms of five bases for symmetric functions (monomial, elementary, complete, power sums, Schur), the transition matrices between them, the involution ω, the scalar product making the Schur functions an orthonormal basis, etc. The MacTutor History of Mathematics [65] gives the following quote by Philip Hall:

> ... whenever in mathematics you meet with partitions, you have only to turn over the stone or lift up the bark, and you will, almost infallibly, find symmetric functions underneath. More precisely, if we have a class of mathematical objects which in a natural and significant way can be placed in one-to-one correspondence with the partitions, we must expect the internal structure of these objects and their relations to one another to involve sooner or later ... the algebra of symmetric functions.

I learned of the paper of Philip Hall from Robert McEliece (1942–2019), who was my colleague at the Caltech Jet Propulsion Laboratory when I spent the summers of 1965–1970 there. McEliece had studied for a year with Hall and had a copy of his paper [64]. McEliece knew that I was becoming interested in tableaux-like objects and showed me Hall's paper. I was immediately transfixed by this beautiful connection between algebra and combinatorics, and I realized that it would have many applications to plane partitions. This resulted in my survey paper [132] entitled "Theory and application of plane partitions." As I explain in [68], my original title for this paper was "Symmetric functions and plane partitions." Rota had a keen political sense and told me to change the title since he was angling for me to eventually receive tenure in the Applied Mathematics Group of the M.I.T. Department of

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[16]Marshall Hall and Philip Hall are not related.

[17]Around 1969 I gave a talk to my fellow graduate students at Harvard on Hall's paper. This was perhaps the first presentation of this material since Hall himself in 1959.

Mathematics. When the Department of Mathematics split into two groups in 1964 (discussed by Harvey Greenspan in [129, pp. 309–314]), the Pure Group was not interested in combinatorics (represented by Rota and Daniel Kleitman). The Applied Group was more accommodating, so combinatorics ended up there. Any tenured faculty could choose whichever group they wanted. Since I started in Applied and the other senior combinatorialists were there, I stayed there throughout my career. However, it would have been more interesting for me to be involved with hiring and promotion decisions for algebraists and number theorists rather than numerical analysts, PDE specialists, etc. I should mention that the Pure Group now has a lot more respect for combinatorics than it did in 1964.

There is one further underrecognized researcher on symmetric function theory worthy of mention here. This is Dudley Ernest Littlewood[18] (1903–1979), who made many significant contributions. These include some Schur function expansions of infinite products, a product formula for the principal specialization $s_\lambda(1, q, \ldots, q^{n-1})$ which easily implies the "hook-content formula" [132, Thm. 15.3], formulas for super-Schur functions in a restricted number of variables, the Jacobi-Trudi theorem for the orthogonal and symplectic groups, and the expansion of the orthogonal and symplectic analogue of Schur functions in terms of Schur functions. Much of his work is collected in his book [88]. One reason his work was not adequately recognized during his lifetime is his use of old-fashioned notation and terminology.

Other mathematicians were becoming interested in symmetric functions in the 1960's. Ronald Read wrote a mainly expository paper [117] on Redfield-Pólya theory based on symmetric functions, in which Schur functions play a prominent role. We have already mentioned the work of Schützenberger on RSK and his subsequent collaboration with Lascoux. Schützenberger told Donald Knuth (1938–) about Schensted's paper and its connection with the representation theory of the symmetric group, thereby inspiring Knuth in 1970 to come up with a generalization of RSK (and the notion of dual RSK, which for permutations is the same as RSK) to permutations of multisets [76]. Lascoux pooh-poohed this work as straightforward and unoriginal since it can be deduced easily from the original RSK (see [142, Lemma 7.11.6]).

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[18]Not to be confused with the better known John Edensor Littlewood, who in fact was D. E. Littlewood's tutor at Trinity College, Cambridge.

In a strict sense Lascoux was correct, but Knuth's version is necessary for a host of applications. Knuth himself shows that it gives a combinatorial proof of the fundamental Cauchy and dual Cauchy identities (though Knuth did not use this terminology). Knuth also establishes the far-reaching "Knuth relations" which determine when two permutations in $\mathfrak{S}_n$ have the same insertion tableau under RSK. This result led to many further developments, including Curtis Greene's important and subtle characterization [60] of the shape of P or Q when $w \overset{\text{RSK}}{\longrightarrow} (P, Q)$ in terms of increasing and decreasing subsequences of w, and the theory of the plactic monoid [85] due to Lascoux and Schützenberger. Knuth's combinatorial proofs of the Cauchy and dual Cauchy identities were extended by William H. Burge [26] to prove four similar identities due to D. E. Littlewood [88, second ed., p. 238].

Two years after Knuth's paper, Edward Bender and Knuth showed that Knuth's generalized RSK is the perfect tool for enumerating certain classes of plane partitions, including plane partitions with at most r rows and with largest part at most m, and symmetric plane partitions.[19] Numerous other contributions to enumerative combinatorics are scattered throughout Knuth's monumental opus *The Art of Computer Programming*, with the first volume [75] appearing in 1968.

Another person bitten by the symmetric function bug was Ian Macdonald. He was originally a leading algebraist. Some of his algebraic results involved both combinatorics and representation theory, such as his famous paper [96] on affine root systems, so he was in a good position to work on symmetric functions. His book [97] was the first comprehensive treatment of the theory of symmetric functions. In addition to the "basic" theory of [142, Ch. 7], it included such topics as polynomial representations of $\mathrm{GL}(n, \mathbb{C})$ (only outlined in [142] without proofs) in the setting of polynomial functors, characters of the wreath product $G \wr \mathfrak{S}_n$ (where G is any finite group), Hall polynomials, Hall-Littlewood symmetric functions, and the characters of $\mathrm{GL}(n, \mathbb{F}_q)$. One highlight for combinatorialists are some "examples" [97, 98, Exam. I.5.13–19] which use root systems to unify many results and conjectures on plane partitions. (A vastly expanded second edition [98], with contributions from

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[19]The argument for symmetric plane partitions easily extends to symmetric plane partitions with at most r rows (and therefore contained in an $r \times r$ square), though Bender and Knuth do not mention this.

Andrey Zelevinsky, contains, among other things, a discussion of what are now called Macdonald polynomials.) For a combinatorialist, the biggest defect of this book (either edition) is the omission of RSK. In fact, Macdonald once told me that his main regret regarding his book was this omission.

A major area of EAC that developed in the 1960's and 70's was its connections with geometry. The three main topics (all interrelated) discussed here are (1) simplicial complexes, (2) hyperplane arrangements, and (3) Ehrhart theory. Of course there was already lots of work on connections between combinatorics and geometry prior to 1960. Much of this work (such as Helly's theorem) belongs more to extremal combinatorics than EAC so will not be considered here. In regard to simplicial complexes, the main question of interest here is what can be said about the number of i-dimensional faces of a simplicial complex[20] satisfying certain properties, such as being pure (all maximal faces have the same dimension), triangulating a sphere, having vanishing reduced homology, etc. If a $(d-1)$-dimensional simplicial complex Δ has f_i faces of dimension i (or cardinality $i+1$), then the f-vector of Δ is defined by $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$. The first significant combinatorial result concerning f-vectors is the famous Kruskal-Katona theorem, which characterizes f-vectors of arbitrary simplicial complexes. This result was first stated without proof by Schützenberger [126] in 1959, and then rediscovered independently by Joseph Kruskal [80] and Gyula Katona [73]. It turns out that Francis Macaulay [92][93] had previously given a multiset analogue of the Kruskal-Katona theorem which he used to characterize Hilbert functions of graded algebras. This result played an important role in later EAC developments. A common generalization of the Kruskal-Katona theorem and the Macaulay theorem is due to George Clements and Bernd Lindström [29]. Further progress on f-vectors of simplicial complexes was greatly facilitated by the introduction of tools from commutative algebra, and later, algebraic geometry.

Around 1975 Melvin Hochster and I independently defined a ring (in fact, a graded algebra over a field K) $K[\Delta]$ associated with a simplicial complex Δ. Hochster was interested in algebraic and homological properties of this ring and gave it to his student Gerald Reisner for a thesis topic. Hochster was motivated by his paper [69], a pioneering work in the interaction between

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[20]All simplicial complexes considered here are finite abstract simplicial complexes.

commutative algebra and convex polytopes, while I was interested in the connection between $K[\Delta]$ and the f-vector of Δ. The paper [118] of Reisner (based on his Ph.D. thesis) was just what was needed to obtain information on f-vectors, beginning with [134][135].

As I explain in [145], I was led to define the ring $K[\Delta]$ (which I denoted R_Δ) by my previous work [133] on "magic squares." Specifically, let $H_n(r)$ denote the number of $n \times n$ matrices of nonnegative integers for which every row and column sums to r. Anand, Dumir, and Gupta [2] conjectured that $H_n(r)$ is a polynomial in r of degree $(n-1)^2$ with certain additional properties. I proved polynomiality by showing that $H_n(r)$ is the Hilbert polynomial of a certain ring. This was the beginning of the subject of *combinatorial commutative algebra* [101][139], now a well-established constituent of EAC. Just after the paper [133] was written, I became aware that $H_n(r)$ is also the Ehrhart polynomial of the Birkhoff polytope of $n \times n$ doubly stochastic matrices, leading to a more elementary proof of the polynomiality of $H_n(r)$ (and some other properties). Ehrhart theory is discussed below.

Algebraic geometry had a long history of connections with combinatorics, such as the Schubert calculus and determinantal varieties. In the 1970's the theory of toric varieties [36][74] established connections with convex polytopes and related objects. Applications of algebraic geometry to combinatorics first appeared in 1980 [136][137], a bit too late for this paper. Subsequently there developed the flourishing subject *combinatorial algebraic geometry*, with many related papers, books, and conferences.

An important subject within EAC is the theory of hyperplane arrangements, or just *arrangements*, i.e., a discrete (usually finite) collection of hyperplanes in a finite-dimensional vector space over some field. If we remove a finite collection $\mathcal{A}$ of hyperplanes in $\mathbb{R}^d$ from $\mathbb{R}^d$, we obtain a finite number of open connected sets called *regions*. The closure of each region is a convex polyhedron $\mathcal{P}$ (the intersection, possibly unbounded, of finitely many closed half-spaces). A *face* of $\mathcal{A}$ is a relatively open face of one of the polyhedra $\mathcal{P}$. In particular, a region is a d-dimensional face. The primary combinatorial problem concerning $\mathcal{A}$ is the counting of its faces of each dimension k, and especially the number of regions. There was some early work on special classes of arrangements, such as those in $\mathbb{R}^3$ and those whose hyperplanes are in general position. Robert Winder [154] in 1966 was the first to give a

general result—a formula for the number of regions for any $\mathcal{A}$ when all the hyperplanes contained the origin.

Arrangements did not get established as a separate subject until the pioneering 1974 thesis of Thomas Zaslavsky [156][157], one of the most influential Ph.D. theses in EAC. He gave formulas for the number of k-faces and bounded k-faces in terms of the Möbius function of the *intersection poset* $L_{\mathcal{A}}$ of all nonempty intersections of the hyperplanes in $\mathcal{A}$. Not only did this work initiate the subject of hyperplane arrangements within EAC, but also it cemented the role of the Möbius function as a fundamental EAC tool. If the hyperplanes of $\mathcal{A}$ all contain the origin, then $L_{\mathcal{A}}$ is a *geometric lattice*. Geometric lattices are equivalent to simple matroids, so Zaslavsky's work also elucidated the connection between arrangements and matroids. This connection led to the theory of oriented matroids, an abstraction of real arrangements (where one can consider on which side of a hyperplane a face lies) analogous to how matroids are an abstraction of linear independence over any field. The original development (foreseen by Ralph Rockafellar [121] in 1967) goes back to Robert Bland [17] in 1974, Jim Lawrence [87] in 1975, and Michel Las Vergnas [86] in 1975. Published details of this work appear in Robert Bland and Las Vergnas [18] and Jon Folkman and Jim Lawrence [46], both in 1978.

The last area of EAC to be discussed here is the combinatorics of integral (or more generally, rational) convex polytopes in $\mathbb{R}^n$, i.e., convex polytopes whose vertices lie in $\mathbb{Z}^n$ (or $\mathbb{Q}^n$). The first inkling of the general theory is the famous formula of Georg Pick (1859–1942) [112] in 1899 for the area A enclosed by a simple (i.e., not self-intersecting) polygon $\mathcal{P}$ in $\mathbb{R}^2$ with integer vertices in terms of the number of the number i of integer points in the interior of $\mathcal{P}$ and the number b of integer points on the boundary (that is, on the polygon itself), namely, $A = i + \frac{b}{2} - 1$. The question naturally arises of extending this result to higher dimensions. John Reeve did some work for three dimensions in the 1950's. The first general result is due to Macdonald [94] in 1963.

In the meantime Eugène Ehrhart was slowly developing since the 1950's his theory of integer points in polyhedra and their dilations, including his famous "loi de reciprocité," in a long series of papers published mostly in the journal *Comptes Rendus Mathématique*. He gave an exposition of this

Figure 3: Eugène Ehrhart

work in a monograph [41] of 1974.[21] Ehrhart's trailblazing work was not completely accurate. A rigorous treatment was first given by Macdonald [95] in 1971. Ehrhart theory is now a central area of EAC with many applications and connections to other subjects.

Eugène Ehrhart (1906-2000) had an interesting career [28]. In particular, he was a teacher in various lycées (French high schools) and did not receive his Ph.D. degree until the age of 59 or 60. He spent the latter part of his life in Strasbourg. Most of the professors in the Mathematics Department of the University of Strasbourg regarded him as somewhat of an amateur crackpot. When I gave a talk related to Ehrhart theory at this Department, the audience members were talking incredulously about whether the Ehrhart of Ehrhart theory was the same Ehrhart they knew. Ehrhart also had some artistic talent; Figure 3 is a self-portrait.

I had the pleasure of meeting Ehrhart once, at a 1976 conference "Com-

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[21]Ehrhart and his followers deal with *convex* polytopes, while Pick's theorem holds for arbitrary simple polygons. The polytopal Ehrhart theory can be extended to integral (or rational) polyhedral complexes Γ, e.g., [141, §4].

binatoire et représentation du group symétrique" organized by Foata in Strasbourg [45]. Some highlights of this conference are a survey by Robinson of the work of Alfred Young, Viennot's geometric version of RSK, and Schützenberger's presentation of the first proof of the validity of *jeu de taquin*. Macdonald was in attendance but did not speak, certainly the last time such a state of affairs occurred at a combinatorics conference!

At this conference Curtis Greene and I also had the pleasure of meeting Bernard Morin (1931–2018). Morin was a member of the first group to exhibit explicitly a crease-free eversion (turning inside-out) of the 2-sphere, and he also discovered the *Morin surface*, a half-way point for the sphere eversion.[22] Morin explained the sphere eversion in his office to Curtis Greene and me with the use of some models. The remarkable aspect of this story is that Morin was blind since the age of six!

I will conclude this paper with a discussion of EAC at M.I.T. in the 1960's and 1970's. Thanks to the influence of Rota, especially after he returned to M.I.T. from Rockefeller University in 1967, M.I.T. became a leading center for EAC research. His return to M.I.T. conveniently coincided with the time when I was getting interested in some combinatorial problems as a Harvard graduate student, so I could experience the development of EAC at M.I.T. almost from the beginning.

It was certainly a thrilling time to be at M.I.T. and interact with a plethora of graduate students, postdocs, visitors, seminar speakers, and the two senior combinatorialists, Kleitman and Rota. Rota's first graduate student in combinatorics at M.I.T. was Henry Crapo (1932–2019), who received his degree related to matroid theory in 1964.[23] Rota's other combinatorics students in the 1960's and 1970's included Thomas Brylawski (from Dartmouth), Thorkell Helgason (later holding several prominent government po-

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[22]Stephen Smale was the first to prove that a crease-free eversion exists, but he did not give an explicit description. When Smale, as a graduate student at the University of Michigan, told his thesis adviser Raoul Bott that he (Smale) had proved that the 2-sphere can be everted, Bott replied that Smale must have made a mistake! Bott had in mind a simple argument showing the impossiblility of eversion, but Bott's reasoning was faulty.

[23]Among Rota's pre-combinatorics graduate students was Peter Duren. Peter Duren was a secondary thesis adviser of Theodore Kaczynski, the notorious Unibomber. Thus I am a "secondary academic uncle" of Kaczynski.

sitions in Iceland), Walter Whiteley, Stephen Fisk (Harvard), Neil White (Harvard), Peter Doubilet, Stephen Tanny, Kenneth Holladay, Hien Nguyen, Joseph Kung, and Joel Stein (Harvard). The combinatorics instructors during the late 1960's consisted of Curtis Greene, Michael Krieger, and Bruce Rothschild.

Intersecting my time as a graduate student at Harvard were Edward Bender and Jay Goldman. Bender was a Benjamin Peirce Instructor, and Jay Goldman was a junior faculty member in the Department of Statistics. Bender worked primarily in what today is called *analytic combinatorics*, though I mentioned above his role in the enumeration of plane partitions. Jay Goldman was trained as a statistician but was lured to combinatorics by Rota's spell. In the 1970's he wrote five influential papers with James Joichi and Dennis White which established the subject of *rook theory*.

During the academic year 1967–68 I took from Bender and Goldman a graduate course in combinatorics, my first course on this subject.[24] Officially, the Bender-Goldman course was entitled "Statistics 210: Combinatorial Analysis" in the fall of 1967, taught by Goldman, and "Mathematics 240: Combinatorial Analysis" in the spring of 1968, taught by Bender. This course was the second course in combinatorics offered at Harvard, the first being "Mathematics 240: Combinatorial Analysis" taught by Alfred Hales from Ryser's book [124] in the fall of 1965.[25] There was some interest around the time of the Bender-Goldman course in giving a unified development of generating functions. How to explain why generating functions like $\sum f(n)x^n$ and $\sum f(n)\frac{x^n}{n!}$ occurred frequently, while one never saw $\sum f(n)\frac{x^n}{n^2+1}$, for instance? Three theories soon emerged: (1) binomial posets, due to Rota and me [40, §8][144, §3.18], (2) dissects, due to Michael Henle [67], and (3) prefabs, due to Bender and Goldman [9]. The theory of prefabs was part of Bender and Goldman's course. None of these theories have played much of a role in subsequent EAC developments because of their limited applicability. Later, André Joyal developed the theory of species [72], based on category theory, which is probably the definitive way to unify generating functions.

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[24]Although Caltech was a center for combinatorics when I was an undergraduate there, at that time I did not think that combinatorics was a serious subject and declined to take any courses in this area!

[25]There were earlier undergraduate courses on Applied Discrete Mathematics in the Applied Mathematics Department, but these were not really courses in combinatorics.

After graduating from Harvard in 1970 I was an Instructor at M.I.T. for one year before becoming a Miller Research Fellow at Berkeley for two years. Although Berkeley did not have the EAC ambience of M.I.T., there were nevertheless many interesting persons with whom I could interact, including Elwyn Berlekamp, David Gale, Derrick and Emma Lehmer, and Raphael and Julia Robinson.[26] I also played duplicate bridge with Edwin Spanier. A mathematical highlight of my stay at Berkeley was regular visits, frequently with David Gale, to Stanford University in order to attend a combinatorics/computer science seminar held by Donald Knuth at his home on the Stanford campus. Figure 4 shows the participants for the December 6, 1971, talk of Richard Karp. (The person holding the book is Knuth.) This was the first public talk that discussed the P vs. NP problem.

In 1973 I returned to the exciting EAC atmosphere at M.I.T. Curtis Greene was an Assistant Professor 1971–1976, as was Joel Spencer 1972–1975. The combinatorics Instructors who were there during some subset of the period 1973–1979 were Stephen Fisk, Karanbir Sarkaria, Kenneth Baclawski, Thomas Zaslavsky, Joni Shapiro (later Saj-nicole Joni), Dennis Stanton, Ira Gessel, and Jeff Kahn. More M.I.T. graduate students were becoming interested in EAC. The first person to become my student was Emden Gansner, followed shortly thereafter by Ira Gessel, though Gessel was my first student to graduate. I had three students who graduated in the 1970's: Ira Gessel (1977), Emden Gansner (1978), and Bruce Sagan (1979). Paul Edelman, Robert Proctor, and Jim Walker were also my students mostly in the late 1970's, though they graduated after 1979. Walker was a dream student with regard to how much effort I needed to put in. I knew him as a graduate student for several years, but he talked to me about his research only occasionally and never asked about becoming my student. One day he walked into my office and asked whether some work he had written up was sufficient for a thesis. I looked it over for a few days and saw that it would make a fine thesis! Today such a scenario is not possible since the M.I.T. Math Department has instituted some strict rules for keeping track of graduate student progress.

There was a combinatorics seminar run by Rota (and later me) that met

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[26]Emma Lehmer and Julia Robinson were not officially affiliated with U.C. Berkeley, primarily due to anti-nepotism rules in effect at the time.
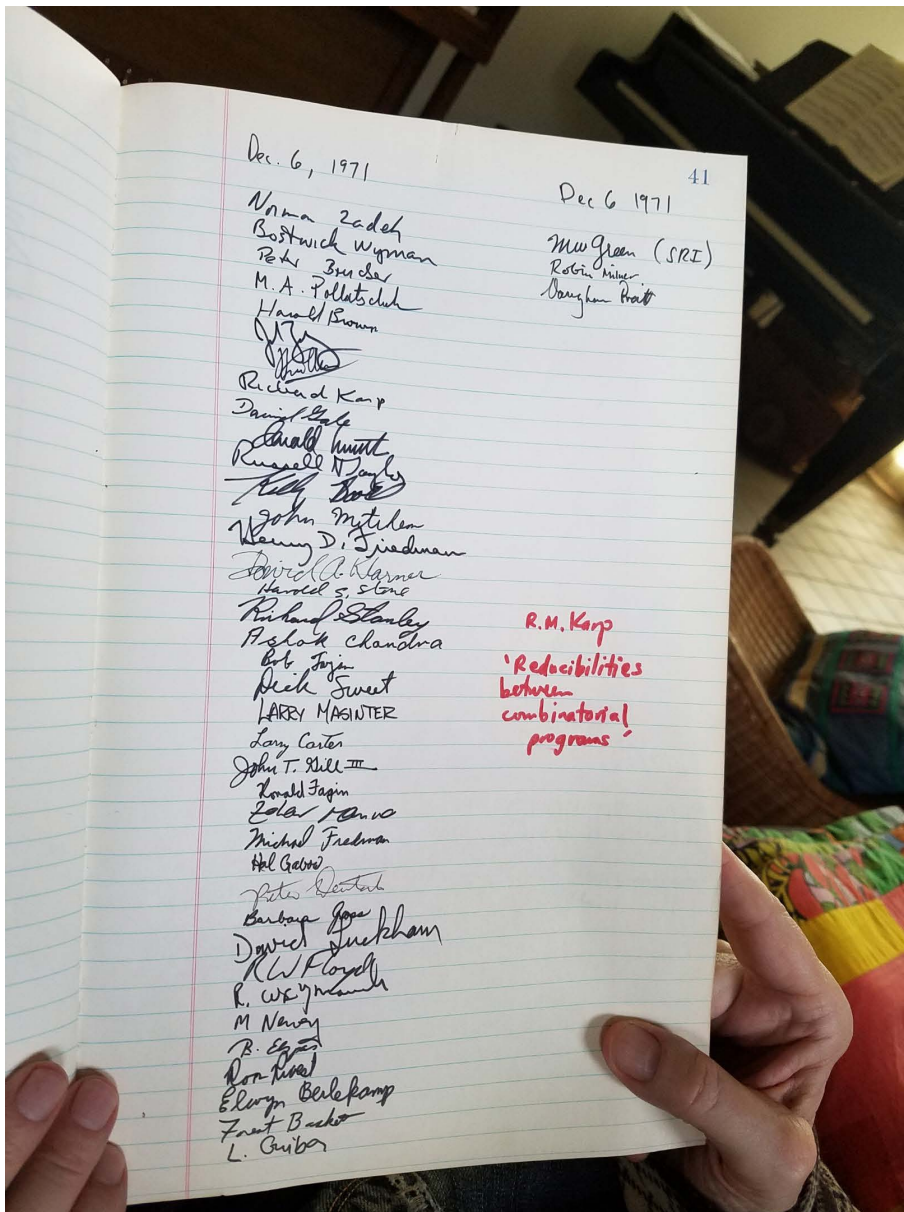
Figure 4: Knuth seminar participants

Wednesday afternoons. Later it was expanded to Wednesdays and Fridays. Since M.I.T. was a combinatorial magnet we had no problem attracting good speakers. Figure 5 shows a list of some seminar speakers for the period fall 1968–fall 1970. (George Andrews was a Visiting Professor at the M.I.T. Department of Mathematics for the 1970–71 academic year, thus explaining his many talks.) After the seminar we would frequently go to a student-run pub in a nearby building (Walker Memorial Hall), and then often to dinner. For a while (I don't recall the precise dates) Rota held a seminar on classical invariant theory and other topics entitled "Syzygy Street."[27]

Rota taught the course "18.17 Combinatorial Analysis" at M.I.T. in the fall of 1962, probably the first course on combinatorics at M.I.T.[28] The listed textbooks for this course in the M.I.T. course catalog were Ore [108] and Riordan [119], though much, if not all, of the material was prepared by Rota and written up as course notes [42]. These notes have a curious feature. They were written up by G. Feldman, J. Levinger, and Richard Stanley. However, that Richard Stanley was not me! In fact, I was a freshman at Caltech at the time. This other Richard Stanley (whose middle name unfortunately was John, not Peter) received a Ph.D. in linguistics from M.I.T. in 1969. His thesis [131] did have some combinatorial flavor.

ACKNOWLEDGEMENT. I am grateful to Adriano Garsia, Ming-chang Kang, Igor Pak, Victor Reiner, Christophe Reutenauer and especially Curtis Greene for many helpful suggestions.

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[27]A *syzygy* in this context is a certain relation among invariant polynomials. For readers not so familiar with American culture, the name "Syzygy Street" was inspired by the television program for preschool children called "Sesame Street."

[28]It was certainly the first combinatorics course that was taught during or after the fall of 1958. Rota began teaching at M.I.T. in the fall of 1959.

speaker	title	date
B. Rothschild	Ramsey-type theorems	October 2, 1968
J. Goldman	Finite vector spaces	October 30
D. Kleitman	Combinatorics and statistical mechanics	unknown
E. Lieb	Ice is nice	January 8, 1969
P. O'Neil	Random 0-1 matrices	February 12
D. Kleitman and		
B. Rothschild	Asymptotic enumeration of finite topologies	March 13
D. Kleitman	Asymptotic enumeration of tournaments	April 22
E. Berlekamp	Finite Riemann Hypothesis and	
	error-correcting codes	April 28
G. Katona	Sperner-type theorems	April 29
E. Bender	Plane partitions and Young tableaux	April 30
(unknown)	Tournaments	October 6
S. Sherman	Monotonicity and ferromagnetism	October 20
G.-C. Rota	Exterior algebra I	December 3
M. Aigner	Segments of ordered sets	December 8
G.-C. Rota	Exterior algebra II	December 10
A. Gleason	Segments of ordered sets	December 15
J. Spencer	Scrambling sets	February 9, 1970
M. Schützenberger	Planar graphs and symmetric groups	March 4
D. Kleitman	Antichains in ordered sets	March 16
G. Szekeres	Skew block designs	April 8
E. Lieb	Ising and dimer problems	April 27
P. Erdős	Problems of combinatorial analysis	June 3
D. Kleitman	(unknown)	September 29
G. Andrews	A partition problem of Adler	October 6
G. Andrews	Partitions from Euler to Gauss	October 9
G. Gallavotti	Some graphical enumeration problems	
	motivated by statistical mechanics	October 13
G. Andrews	The Rogers-Ramanujan identities	October 16
R. Reid	Tutte representability and Segre arcs	
	and caps	October 20
G. Andrews	Proof of Gordon's theorem	November 6
G. Andrews	Schur's partition theorem	November 13
G. Andrews	Extensions of Schur's theorem	November 20
M. Krieger	Some problems and conjectures	(unknown)
S. Fisk	Triangulations of spheres	December 1
D. Kleitman	Some network problems	December 8

Figure 5: M.I.T. Combinatorics Seminar, 1968–1970

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