THE SMITH NORMAL FORM OF A SPECIALIZED JACOBI-TRUDI MATRIX

RICHARD P. STANLEY

ABSTRACT. Let JT_λ be the Jacobi-Trudi matrix corresponding to the partition λ, so $\det \mathrm{JT}_\lambda$ is the Schur function s_λ in the variables $x_1, x_2, \dots$. Set $x_1 = \cdots = x_n = 1$ and all other $x_i = 0$. Then the entries of JT_λ become polynomials in n of the form $\binom{n+j-1}{j}$. We determine the Smith normal form over the ring $\mathbb{Q}[n]$ of this specialization of JT_λ . The proof carries over to the specialization $x_i = q^{i-1}$ for $1 \le i \le n$ and $x_i = 0$ for $i > n$, where we set $q^n = y$ and work over the ring $\mathbb{Q}(q)[y]$.

1. Introduction

Let M be an $r \times s$ matrix over a commutative ring R (with identity), which for convenience we assume has full rank r. In particular, we always have $r \le s$. If there exist invertible $r \times r$ and $s \times s$ matrices P and Q such that the product PMQ is a diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \dots, \alpha_r$ satisfying $\alpha_i \mid \alpha_{i+1}$ for all $1 \le i \le r-1$, then PMQ is called the *Smith normal form (SNF)* of M. In general, the SNF does not exist. It does exist when R is a principal ideal domain (PID) such as $\mathbb{Q}[n]$, the polynomial ring in the indeterminate n over the rationals (which is the case considered in this paper). Over a PID the SNF is unique up to multiplication of diagonal elements by units in R. Note that the units of the ring $\mathbb{Q}[n]$ are the nonzero rational numbers. Since the determinants of P and Q are units in R, we obtain when M is a nonsingular square matrix a canonical factorization $\det M = u\alpha_1\alpha_2\cdots\alpha_m$, where u is a unit. Thus whenever $\det M$ has a lot of factors, it suggests that it might be interesting to consider the SNF.

There has been a lot of recent work, such as [1][5], on the Smith normal form of specific matrices and random matrices, and on different situations in which SNF occurs. Here we will determine the SNF of certain matrices that arise naturally in the theory of symmetric functions. We will follow notation and terminology from [4, Chap. 7]. Namely, let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of some positive integer, and let h_i denote the complete homogeneous symmetric function of degree i in the variables $x_1, x_2, \dots$. Set $h_0 = 1$ and $h_m = 0$ for $m < 0$. Let t be an integer for which $\ell(\lambda) \le t$, where $\ell(\lambda)$ denotes the length (number of parts) of λ. Thus we may regard $\lambda = (\lambda_1, \dots, \lambda_t)$, where the last $t - \ell(\lambda)$ entries are 0. The *Jacobi-Trudi matrix* JT_λ is defined by

$$\mathrm{JT}_\lambda = [h_{\lambda_i+j-i}]_{i,j=1}^{t}\,.$$

The *Jacobi-Trudi identity* [4, §7.16] asserts that $\det \mathrm{JT}_\lambda = s_\lambda$, the Schur function indexed by λ.

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For a symmetric function f, let $\varphi_n f$ denote the specialization $f(1^n)$, that is, set $x_1 = \cdots = x_n = 1$ and all other $x_i = 0$ in f. It is easy to see [4, Prop. 7.8.3] that

$$\varphi_n h_i = \binom{n+i-1}{i}, \tag{1.1}$$

a polynomial in n of degree i. Identify λ with its (Young) diagram, so the squares of λ are indexed by pairs (i,j), $1 \le i \le \ell(\lambda)$, $1 \le j \le \lambda_i$. The *content* $c(u)$ of the square $u = (i,j)$ is defined to be $c(u) = j - i$. A standard result [4, Cor. 7.21.4] in the theory of symmetric functions states that

$$\varphi_n s_\lambda = \frac{1}{H_\lambda} \prod_{u \in \lambda} (n + c(u)), \tag{1.2}$$

where H_λ is a positive integer whose value is irrelevant here (since it is a unit in $\mathbb{Q}[n]$). Since this polynomial factors a lot (in fact, into linear factors) over $\mathbb{Q}[n]$, we are motivated to consider the SNF of the matrix

$$\varphi_n \mathrm{JT}_\lambda = \left[\binom{n+\lambda_i+j-i-1}{\lambda_i+j-i} \right]_{i,j=1}^{t}.$$

Let D_k denote the kth *diagonal hook* of λ, i.e., all squares $(i,j) \in \lambda$ such that either $i = k$ and $j \ge k$, or $j = k$ and $i \ge k$. Note that λ is a disjoint union of its diagonal hooks. If $r = \mathrm{rank}(\lambda) := \max\{i : \lambda_i \ge i\}$, then note also that $D_k = \emptyset$ for $k > r$. Our main result is the following.

Theorem 1.1. *Let the SNF of* $\varphi_n \mathrm{JT}_\lambda$ *have main diagonal* $(\alpha_1, \alpha_2, \ldots, \alpha_t)$*, where* $t \ge \ell(\lambda)$*. Then we can take*

$$\alpha_i = \prod_{u \in D_{t-i+1}} (n + c(u)).$$

It follows from equation (1.2) that an altenative statement of Theorem 1.1 is that the α_i's are squarefree (as polynomials in n), since α_t is the largest squarefree factor of $\varphi_n s_\lambda$, α_{t-1} is the largest squarefree factor of $(\varphi_n s_\lambda)/\alpha_t$, etc.

Example 1.2. Let $\lambda = (7,5,5,2)$. Figure 1 shows the diagram of λ with the content of each square. Let $t = \ell(\lambda) = 4$. We see that

$$\begin{aligned} \alpha_4 &= (n-3)(n-2)\cdots(n+6) \\ \alpha_3 &= (n-2)(n-1)n(n+1)(n+2)(n+3) \\ \alpha_2 &= n(n+1)(n+2) \\ \alpha_1 &= 1. \end{aligned}$$

The problem of computing the SNF of a suitably specialized Jacobi-Trudi matrix was raised by Kuperberg [2]. His Theorem 14 has some overlap with our Theorem 1.1. Propp [3, Problem 5] mentions a two-part question of Kuperberg. The first part is equivalent to our Theorem 1.1 for rectangular shapes. (The second part asks for an interpretation in terms of tilings, which we do not consider.)

0	1	2	3	4	5	6
−1	0	1	2	3		
−2	−1	0	1	2		
−3	−2					

FIGURE 1. The contents of the partition $(7, 5, 5, 2)$

2. PROOF OF THE MAIN THEOREM

To prove Theorem 1.1 we use the following well-known description of SNF over a PID.

Lemma 2.1. *For $m \le n$, let $\operatorname{diag}(\alpha_1, \dots, \alpha_m)$ be the SNF of an $m \times n$ matrix M over a PID. Then $\alpha_1\alpha_2 \cdots \alpha_k$ is the greatest common divisor (gcd) of the $k \times k$ minors of M.*

Let λ be a partition of length at most t and with diagonal hooks $D_1, \dots, D_t$. Given the $t \times t$ matrix $\varphi_n \mathrm{JT}_\lambda$ and $1 \le k \le t$, let M_k be the square submatrix consisting of the last k rows and first k columns of $\varphi_n \mathrm{JT}_\lambda$. We claim the following.

C1. If $\det M_k = 0$ then $\varphi_n \mathrm{JT}_\lambda$ has a $k \times k$ minor equal to 1. Otherwise,

$$\det M_k = c_k \prod_{i=1}^{k} \prod_{i \in D_{t-i+1}} (n + c(u)), \tag{2.1}$$

where c_k is a nonzero rational number.

C2. If $\det M_k \neq 0$, then every $k \times k$ minor of $\varphi_n \mathrm{JT}_\lambda$ is divisible (in the ring $\mathbb{Q}[n]$) by $\det M_k$.

Proof of C1. It is well known and follows immediately from the Jacobi-Trudi identity for skew Schur functions that every minor of JT_λ is either 0 or a skew Schur function $s_{\rho/\sigma}$ for some skew shape ρ/σ. Let N be a $k \times k$ submatrix of JT_λ with determinant zero. This can only happen if N is strictly upper triangular, since otherwise the determinant is a nonzero $s_{\rho/\sigma}$. Each row of JT_λ that intersects N consists of a string of 0's, followed by a 1, and possibly followed by other terms. The 1's in these rows appear strictly from left-to-right as we move down JT_λ. Hence the $k \times k$ submatrix of JT_λ with the same rows as N and with each column containing 1 is upper unitriangular and hence has determinant 1. Since $\varphi_n s_{\rho/\sigma} \neq 0$, the same reasoning applies to $\varphi_n \mathrm{JT}_\lambda$, so the first assertion of (C1) is proved.

If on the other hand $\det M_k \neq 0$, then M_k is just the Jacobi-Trudi matrix for the subshape $\bigcup_{i=1}^{k} D_{t-i+1}$ of λ, so (C1) follows from equation (1.2).

Proof of C2. Suppose that $\det M_k \neq 0$. Thus M_k is the Jacobi-Trudi matrix for the partition $\mu = \bigcup_{i=1}^{k} D_{t-i+1}$. We now claim that any $k \times k$ submatrix of JT_λ is the Jacobi-Trudi matrix of a skew shape ρ/σ such that (the diagram of) ρ/σ has the following property:

(P) There is a subdiagram ν (an ordinary partition) of ρ/σ containing μ, and all other squares of ρ/σ are to the left of ν.

Proof of (P). A $k \times k$ submatrix of JT_λ is indexed by a pair $(\boldsymbol{i}, \boldsymbol{j})$ of row indices $\boldsymbol{i} = (i_1, \dots, i_k)$ and column indices $\boldsymbol{j} = (j_1, \dots, j_k)$, where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. The

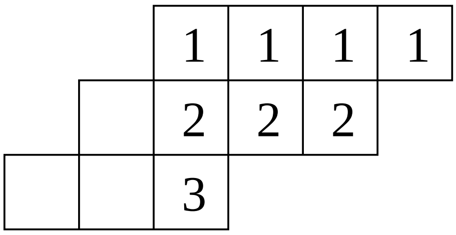

FIGURE 2. A partial Littlewood-Richardson filling

submatrix M_k corresponds to $\boldsymbol{i} = (t-k+1, t-k+2, \dots, t)$ and $\boldsymbol{j} = (1, 2, \dots, k)$. We can get from M_k to any other $k \times k$ submatrix M by a sequence of operations consisting of decreasing a row index i_r by 1 (when $i_{r-1} < i_r - 1$, where $i_0 = 0$) or increasing a column index j_s by 1 (when $j_{s+1} > j_s + 1$, where $j_{k+1} = t+1$). If $M = \mathrm{JT}_{\rho/\sigma}$, then decreasing the rth row index by 1 corresponds to adding a square to the right-hand end of the rth row of ρ/σ. This will be a valid skew shape γ/δ since $i_{r-1} < i_r - 1$. Clearly if ρ/σ has property (P) then so does γ/δ. Similarly, increasing the sth column index by 1 corresponds to adding a square to the left-hand end of the sth row of ρ/σ. This will be a valid skew shape α/β since $j_{s+1} > j_s + 1$. Clearly if ρ/σ has property (P) then so does α/β. We have shown that if $\mathrm{JT}_{\rho/\sigma}$ is any $k \times k$ submatrix of JT_λ, then we can get from μ to ρ/σ by a sequence of steps, where each step consists of adjoining a square at the beginning or the end of a row, always maintaining the shape of a skew partition. Clearly μ itself satisfies (P) and each step preserves (P), so the proof follows.

Suppose now that $\langle s_{\rho/\sigma}, s_\tau \rangle \neq 0$. We claim that $\mu \subseteq \tau$. This will complete the proof, since then $\det M_k = H_\mu^{-1} \prod_{u \in \mu}(n + c(u))$, and the contents of μ form a submultiset of the contents of τ.

The statement that $\langle s_{\rho/\sigma}, s_\tau \rangle \neq 0$ is equivalent to $c^\rho_{\sigma\tau} \neq 0$, where $c^\rho_{\sigma\tau}$ is a Littlewood-Richardson coefficient [4, eqn. (7.64)]. By the Littlewood-Richardson rule as formulated e.g. in [4, Thm. A1.3.3], $c^\rho_{\sigma\tau}$ is the number of semistandard Young tableaux (SSYT) of shape ρ/σ and content τ whose reverse reading word is a lattice permutation. By Property (P) such an SSYT must have the last μ_i entries in row i equal to i. Hence $\tau_i \geq \mu_i$ for all i, as desired. This completes the proof of (C2).

As an illustration of the proof of (C2), suppose that $\lambda = (7, 6, 6, 5, 3)$ and we take $k = 3$. Then $\mu = (4, 3, 1)$. The 3×3 minor with rows 3,4,5 and columns 1,3,5 (say) is given by

$$\begin{bmatrix} h_4 & h_6 & h_8 \\ h_2 & h_4 & h_6 \\ 0 & h_1 & h_3 \end{bmatrix},$$

which is the Jacobi-Trudi matrix for the skew shape $(6, 5, 3)/(2, 1)$. Any Littlewood-Richardson filling of this shape has to have the entries indicated in Figure 2, so the type τ of this filling satisfies $\tau \supseteq (4, 3, 1) = \mu$.

Proof of Theorem 1.1. If the kth diagonal hook is empty, then (C1) shows that JT_λ contains a $k \times k$ minor equal to 1. Hence the gcd of the $k \times k$ minors is also 1, and therefore the gcd of the $j \times j$ minors for each $j < k$ is 1. Thus by Lemma 2.1, we have $\alpha_k = 1$ as desired.

If the kth diagonal hook is nonempty, then (C2) shows that every $k \times k$ minor is divisible by $\det M_k$. Hence the gcd of the $k \times k$ minors is equal to $\det M_k$, and the proof follows from equation (2.1) and Lemma 2.1. □

3. A q-analogue

There is a standard q-analogue $\varphi_n(q)s_\lambda$ of $\varphi_n s_\lambda$ [4, Thm. 7.21.2], namely,

$$\begin{aligned} \varphi_n(q)s_\lambda &= s_\lambda(1, q, q^2, \ldots, q^{n-1}) \\ &= \frac{q^{b(\lambda)}}{H_\lambda(q)} \prod_{u \in \lambda} (1 - q^{n+c(u)}), \end{aligned}$$

where $H_\lambda(q)$ is a polynomial in q (the q-analogue of H_λ) and $b(\lambda)$ is a nonnegative integer. What is the SNF of $\varphi_n(q)\mathrm{JT}_\lambda$? The problem arises of choosing the ring over which we compute the SNF. The most natural choice might seem to be to fix n and then work over the ring $\mathbb{Q}[q]$ (or even $\mathbb{Z}[q]$, assuming that the SNF exists). This question, however, is not really a q-analogue of what was done above, since we considered n to be *variable* while here it is a constant. In fact, it seems quite difficult to compute the SNF this way. Its form seems to depend on n is in a very delicate way. Instead we can set $y = q^n$. For instance,

$$\begin{aligned} \varphi_n(q)h_3 &= \frac{(1-q^{n+2})(1-q^{n+1})(1-q^n)}{(1-q^3)(1-q^2)(1-q)} \\ &= \frac{(1-q^2y)(1-qy)(1-y)}{(1-q^3)(1-q^2)(1-q)}. \end{aligned}$$

Since the entries of $\varphi_n(q)\mathrm{JT}_\lambda$ become polynomials in y with coefficients in the field $F = \mathbb{Q}(q)$, we can ask for the SNF over the PID $F[y]$. The proof of Theorem 1.1 carries over, mutatis mudandis, to this q-version.

Theorem 3.1. *Let M_λ denote the matrix obtained from $\varphi_n(q)\mathrm{JT}_\lambda$ by substituting $q^n = y$. Let the SNF of M_λ over the ring $\mathbb{Q}(q)[y]$ have main diagonal $(\beta_1, \beta_2, \ldots, \beta_t)$, where $t \geq \ell(\lambda)$. Then we can take*

$$\beta_i = \prod_{u \in D_{t-i+1}} (1 - q^{c(u)}y).$$

Perhaps this result still seems to be an unsatisfactory q-analogue (or in this case, a y-analogue) since we cannot substitute $y = 1$ to reduce to $\varphi_n\mathrm{JT}_\lambda$. Instead, however, make the substitution

$$y \to \frac{1}{(1-q)y+1}. \tag{3.1}$$

For any $k \in \mathbb{Z}$ write $\boldsymbol{(k)} = (1-q^k)/(1-q)$. For instance, $\boldsymbol{(-3)} = -q^{-1} - q^{-2} - q^{-3}$ and $\boldsymbol{(0)} = 0$. Under the substitution (3.1) we have for any $k \in \mathbb{Z}$,

$$1 - q^k y \to \frac{(1-q)(y + \boldsymbol{(k)})}{(1-q)y+1}.$$

For any symmetric function f let $\varphi^* f$ denote the substitution $q^n \to 1/((1-q)y+1)$ after writing $f(1, q, \ldots, q^{n-1})$ as a polynomial in q and q^n. Let A be a square submatix of JT_λ.

Since $\det A$ is a homogeneous symmetric function, say of degree d, the specialization $\varphi^* \det M$ will equal $\left(\frac{1-q}{(1-q)y+1}\right)^d$ times the result of substituting

$$1 - q^k y \to y + (\boldsymbol{k}) \qquad (3.2)$$

in M and then taking the determinant. It follows that the proof of Theorem 1.1 also carries over for the substitution (3.2). We obtain the following variant of Theorem 3.1, which is clearly a satisfactory q-analogue of Theorem 1.1.

Theorem 3.2. *For $k \geq 1$ let*

$$f(k) = \frac{y(y + (\mathbf{1}))(y + (\mathbf{2})) \cdots (y + (\boldsymbol{k-1}))}{(\mathbf{1})(\mathbf{2}) \cdots (\boldsymbol{k})}.$$

Set $f(0) = 1$ and $f(k) = 0$ for $k < 0$. Define

$$\mathrm{JT}(q)_\lambda = [f(\lambda_i - i + j)]_{i,j=1}^t,$$

where $\ell(\lambda) \leq t$. Let the SNF of $\mathrm{JT}(q)_\lambda$ over the ring $\mathbb{Q}(q)[y]$ have main diagonal $(\gamma_1, \gamma_2, \ldots, \gamma_t)$. Then we can take

$$\gamma_i = \prod_{u \in D_{t-i+1}} (y + \boldsymbol{c}(\boldsymbol{u})).$$

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E-mail address: `rstan@math.mit.edu`

Department of Mathematics, University of Miami, Coral Gables, FL 33124