# Hyperplane Arrangements, Interval Orders and Trees

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## 1 Hyperplane arrangements

The main object of this paper is to survey some recently discovered connections between hyperplane arrangements, interval orders, and trees. We will only indicate the highlights of this development; further details and proofs will appear elsewhere. First we review some basic facts about hyperplane arrangements. A *hyperplane arrangement* is a finite collection $\mathcal{A}$ of affine hyperplanes in a (finite-dimensional) affine space $A$. We will consider here only the case $A = \mathbb{R}^n$ (regarded as an affine space). The theory of hyperplane arrangements has been extensively developed and has deep connections with many other areas of mathematics, such as algebraic geometry, algebraic topology, and the theory of hypergeometric functions; see for example [16][17]. We will be primarily concerned with the number $r(\mathcal{A})$ of regions of $\mathcal{A}$, i.e., the number of connected components of the space $\mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H$. Closely related to this number is the number $b(\mathcal{A})$ of *bounded* regions of $\mathcal{A}$.

A fundamental object associated with the arrangement $\mathcal{A}$ is its *intersection poset* $L_{\mathcal{A}}$ (actually a meet semilattice), defined as follows. The elements of $L_{\mathcal{A}}$ are the *nonempty* intersections of subsets of the hyperplanes in $\mathcal{A}$, including the empty intersection $A$. The elements of $L_{\mathcal{A}}$ are ordered by *reverse* inclusion, so in particular $L_{\mathcal{A}}$ has a unique minimal element $\hat{0} = A$. $L_{\mathcal{A}}$ will have a unique maximal element (and thus be a lattice) if and only if the intersection of all the hyperplanes in $\mathcal{A}$ is nonempty. For the basic facts about posets and lattices we are using here, see [27, Ch. 3]. The *characteristic*

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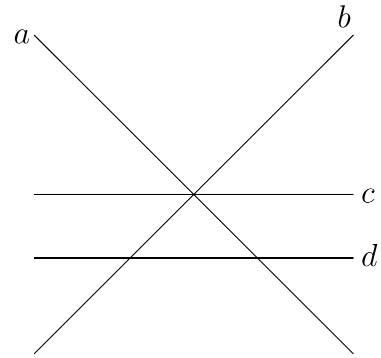


Figure 1: A hyperplane arrangement.

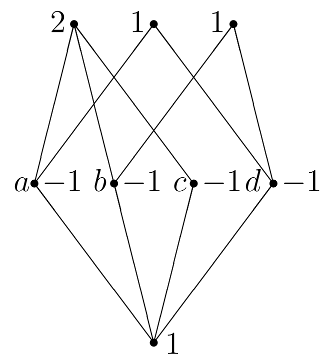


Figure 2: An intersection poset.

*polynomial* $\chi_{\mathcal{A}}(q)$ of $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A}}(q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{\dim x},$$

where $\mu$ denotes the Möbius function of $L_{\mathcal{A}}$ [27, Ch. 3]. Figure 1 illustrates a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^2$, while Figure 2 shows the intersection poset $L_{\mathcal{A}}$, with vertex $x$ labelled with the number $\mu(\hat{0}, x)$, and vertices corresponding to hyperplanes also labelled by the same letter as in Figure 1. From Figure 2 we see that $\chi_{\mathcal{A}}(q) = q^2 - 4q + 4$. The connection between the characteristic polynomial and the number of regions was discovered by Zaslavsky [32, §2].

**1.1 Theorem.** *With notation as above, we have*

$$\begin{aligned} r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) &= \sum_{x \in L_{\mathcal{A}}} |\mu(\hat{0}, x)| \\ b(\mathcal{A}) &= (-1)^{\rho(L_{\mathcal{A}})} \chi_{\mathcal{A}}(1) &= \left| \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) \right|, \end{aligned}$$

*where $\rho(L_{\mathcal{A}})$ denote the rank (one less than the number of levels) of the intersection poset $L_{\mathcal{A}}$.*

An important arrangement, known as the *braid arrangement* and denoted $\mathcal{B}_n$, consists of all hyperplanes $x_i = x_j$, where $1 \le i < j \le n$. (See [16, Example 1.10][17, Example 1.9].) It is easy to see that for the braid arrangement we have $r(\mathcal{B}_n) = n!$, since a region of the arrangement is specified by a linear ordering of the $n$ coordinates. (Moreover, $b(\mathcal{B}_n) = 0$ since the origin belongs to all the hyperplanes in $\mathcal{B}_n$.) With a little more work one can if fact show (see e.g. [16, Prop. 2.26][17, Prop. 2.54]) that

$$\chi_{\mathcal{B}_n}(q) = q(q-1)\cdots(q-n+1).$$

The hyperplane arrangements discussed in this paper are closely related to the braid arrangement and could be called modifications or *deformations* of the braid arrangement. Much of this work was done in collaboration with Christos Athanasiadis, Nati Linial, Igor Pak, Alexander Postnikov, and Shmulik Ravid, whose contributions will be noted in the appropriate places. I am also grateful to Persi Diaconis for helpful comments regarding exposition. Our primary concern will be with the following deformation of $\mathcal{B}_n$. Let $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n$, with $\ell_i > 0$, and define $\mathcal{A}_\ell$ to be the arrangement in $\mathbb{R}^n$ whose hyperplanes are given by

$$x_i - x_j = \ell_i, \quad i \neq j. \tag{1}$$

A classical theorem of Whitney [30] gives a formula for the characteristic polynomial of any subarrangement $\mathcal{G}$ of the braid arrangement $\mathcal{B}_n$. Such an arrangement is called a *graphical arrangement*, because its set of hyperplanes $x_i = x_j$ may be identified with the edges $ij$ of a graph $G$ with vertices $1, 2, \ldots, n$. Whitney's theorem for the arrangement $\mathcal{G}$ asserts that

$$\chi_{\mathcal{G}}(q) = \sum_{S \subseteq E(G)} (-1)^{\#S} q^{c(S)},$$

where $E(G)$ denotes the set of edges of $G$, and $c(S)$ is the number of connected components of the spanning subgraph $G_S$ of $G$ with edge set $S$. Postnikov [20] has generalized Whitney's theorem to subarrangements of arbitrary deformations of the braid arrangement. Rather than state Postnikov's theorem in its full generality here, we will just cite special cases as needed, calling the resulting formula the "Whitney formula" for that arrangement.

For many of the arrangements we will be considering, the characteristic polynomial is actually determined by the number of regions. More precisely, suppose that $\mathcal{A} = (\mathcal{A}_1,\ \mathcal{A}_2, \ldots)$ is a sequence of arrangements such that $\mathcal{A}_n$ is an arrangement in $\mathbb{R}^n$, and every hyperplane in $\mathcal{A}_n$ is parallel to some hyperplane of the braid arrangement $\mathcal{B}_n$. Let $S$ be a $k$-element subset of $\{1, 2, \ldots, n\}$. Let $\mathcal{A}_n^S$ denote the subarrangement of $\mathcal{A}_n$ consisting of all hyperplanes parallel to $x_i - x_j = 0$ for $i, j \in S$. We call the sequence $\mathcal{A}$ an *exponential sequence* of arrangements if $r(\mathcal{A}_n^S) = r(\mathcal{A}_j)$ for all $k$-element subsets $S$ of $\{1, 2, \ldots, n\}$, where $1 < k < n$. The following result is a simple consequence of Theorem 1.1 and the exponential formula of enumerative combinatorics (e.g., [25, Cor. 6.2]).

**1.2 Theorem.** *Let $\mathcal{A}$ be an exponential sequence of arrangements, and write $r_n = r(\mathcal{A}_n)$, $\chi_n(q) = \chi_{\mathcal{A}_n}(q)$. Then*

$$\sum_{n\geq 0} \chi_n(q) \frac{x^n}{n!} = \left( \sum_{n\geq 0} (-1)^n r_n \frac{x^n}{n!} \right)^{-q}.$$

*(Equivalently, the sequence $\chi_0(q),\ \chi_1(q),\ \ldots$ is a sequence of polynomials of binomial type in the sense of [21][22].) In particular, if $b_n = b(\mathcal{A}_n)$, then*

$$\sum_{n\geq 1} b_n \frac{x^n}{n!} = 1 - \left( \sum_{n\geq 0} r_n \frac{x^n}{n!} \right)^{-1}.$$

## 2 Interval orders

Let $P = \{I_1, \ldots, I_n\}$ be a collection of closed intervals of positive length on the real line. Partially order the set $P$ by defining $I_i < I_j$ if $I_i$ lies entirely to the left of $I_j$, i.e., if $I_i = [a, b]$ and $I_j = [c, d]$ then $b < c$. Any partially ordered set isomorphic to $P$ is known as an *interval order*. A basic reference for the theory of interval orders is [5], which gives references to the origins of this subject within economics and psychology. We will be considering *labelled* interval orders whose intervals have specified lengths. Thus given a sequence $\ell = (\ell_1, \ldots, \ell_n)$ of positive real numbers, let $\mathcal{I}_\ell$ be the set of all partial orderings of $1, 2, \ldots, n$ for which there exists a set of intervals $I_1, \ldots, I_n$ satisfying: (a) $I_i$ has length $\ell_i$, and (b) $i < j$ in $P$ if and only if $I_i$ lies entirely to the left of $I_j$. If each $\ell_i = 1$, then the corresponding interval orders are known as *unit interval orders* or *semiorders*, and have been subjected to considerable scrutiny. For other values of $\ell_i$ there has been considerably less work. The following result shows the main connection between interval orders and deformations of the braid arrangement.

**2.1 Theorem.** *Let $\ell = (\ell_1, \ldots, \ell_n)$ with $\ell_i > 0$. Then*

$$r(\mathcal{A}_\ell) = \#\mathcal{I}_\ell,$$

*the number of elements of $\mathcal{I}_\ell$.*

The proof of Theorem 2.1 is a straightforward consequence of the relevant definitions. Theorem 2.1 suggests several generalizations of the concept of interval order which may be worth further investigation. (Some work in this direction appears in [3] and [4], but the enumerative aspects are not considered.) Perhaps the most straightforward of these generalizations corresponds to the arrangement

$$x_i - x_j = \ell_i^{(1)}, \ldots, \ell_i^{(m_i)},\ i \neq j, \tag{2}$$

for positive integers $m_i$ and real numbers $0 < \ell_i^{(1)} < \ell_i^{(2)} < \cdots < \ell_i^{(m_i)}$. This arrangement corresponds to a collection of *marked* intervals $I_i$ of length $\ell_i^{(m_i)}$. The interval $I_i$ is marked with "dots" at distances $\ell_i^{(1)}, \ldots, \ell_i^{(m_i)}$ from the left endpoint (so in particular the right endpoint is marked). We want to count the number of different ways of placing these intervals on the real axis, where two placements $P_1$ and $P_2$ are considered the same if for every $i$ and $j$, the number of marked points of $I_i$ to the left of the left endpoint of $I_j$ is the same for $P_1$ as for $P_2$. The number of inequivalent placements ("generalized interval orders") is the number of regions of the arrangement (2). We could even allow $\ell_i^{(1)} = 0$, in which case we must require in the definition of placement that the left endpoint of $I_i$ not coincide with any marked point of another interval. (Thus the order type does not change under small perturbations of the interval placements.)

There is a special case of these generalized interval orders with a further connection with arrangements. It is clear what we mean for two placements $P_1$ and $P_2$ of marked intervals to be *isomorphic*, namely, there is a bijection $\varphi$ between the intervals of $P_1$ and those of $P_2$ such that for all intervals $I$ of $P_1$, $\varphi(I)$ has the same number of marks as $I$, and for all intervals $I,\ J$ of $P_1$, the number of marks of $I$ to the left of the left endpoint of $J$ is equal to the number of marks of $\varphi(I)$ to the left of the left endpoint of $\varphi(J)$.

**2.2 Theorem.** *Let $\ell_1, \ldots, \ell_m > 0$, and let $\mathcal{A}_n$ denote the arrangement in $\mathbb{R}^n$ given by*

$$x_i - x_j = \ell_1, \ldots, \ell_m,\ i \neq j. \tag{3}$$

*(Note that this is the special case of (2) when all the marked intervals are identical.) Let $\mathcal{A}_n^0$ denote the arrangement obtained from $\mathcal{A}_n$ by adjoining the hyperplanes $x_i = x_j$, i.e.,*

$$\mathcal{A}_n^0 = \mathcal{A}_n \cup \mathcal{B}_n.$$

*Then $r(\mathcal{A}_n^0) = n!\nu(\mathcal{A}_n)$, where $\nu(\mathcal{A}_n)$ is the number of* nonisomorphic *generalized interval orders corresponding to $\mathcal{A}_n$.*

There is a direct connection between the number of regions of the arrangements $\mathcal{A}_n$ and $\mathcal{A}_n^0$ of the previous theorem, obtained in collaboration with A. Postnikov. Regard $\ell = (\ell_1, \ldots, \ell_m)$ as fixed, and define the generating functions

$$\begin{aligned} F_\ell(x) &= \sum_{n\geq 0} r(\mathcal{A}_n)\frac{x^n}{n!} \\ F_\ell^0(x) &= \sum_{n\geq 0} r(\mathcal{A}_n^0)\frac{x^n}{n!} \\ &= \sum_{n\geq 0} \nu(\mathcal{A}_n)x^n. \end{aligned}$$

**2.3 Theorem.** *We have*

$$F_\ell(x) = F_\ell^0(1-e^{-x}).$$

In the special case $\ell = (1)$ (i.e., $m = 1$ and $\ell_1 = 1$), we have that $r(\mathcal{A}_n)$ is the number of labelled semiorders on $n$ points, while $\nu(\mathcal{A}_n)$ is the number of unlabelled (i.e., nonisomorphic) semiorders on $n$ points. It is a well-known result of Wine and Freund [31][5, p. 98][29, p. 195] that this latter number is just the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, so Theorem 2.3 in the case of semiorders may be regarded as determining the number of labelled semiorders. This result is equivalent to a result of Chandon, Lemaire, and Pouget [2]. It is not difficult to show that if $\ell = (1, 2, \ldots, k)$, then

$$\nu(\mathcal{A}_n) = \frac{1}{kn+1}\binom{(k+1)n}{n},$$

generalizing the result of Wine and Freund. For instance, if $n = 3$ and $k = 2$, then we get twelve nonisomorphic placements of three marked intervals, each of length two, with a mark in the center and at the right endpoint. These twelve placements are shown in Figure 3, where each of the three symbols $\bullet$, $\circ$, and $\star$ indicates the left endpoint, center, and right endpoint of an interval. More generally, we have the following result of Athanasiadis [1].

**2.4 Theorem.** *Let $0 < \ell_1 < \cdots < \ell_m$ be integers such that the set $\{1, 2, 3, \ldots, \} - \{\ell_1, \ldots, \ell_m\}$ is closed under addition (so in particular $\ell_1 = 1$). Let $P(x) = \sum_{j=1}^m x^{\ell_j - 1}$. For $n > 0$ let $R_n(x)$ be the remainder upon dividing $(1 + (x-1)P(x))^n$ by $(1-x)^n$. Let $\mathcal{A}_n^0$ be as in Theorem 2.2. Then $r(\mathcal{A}_n^0)$*

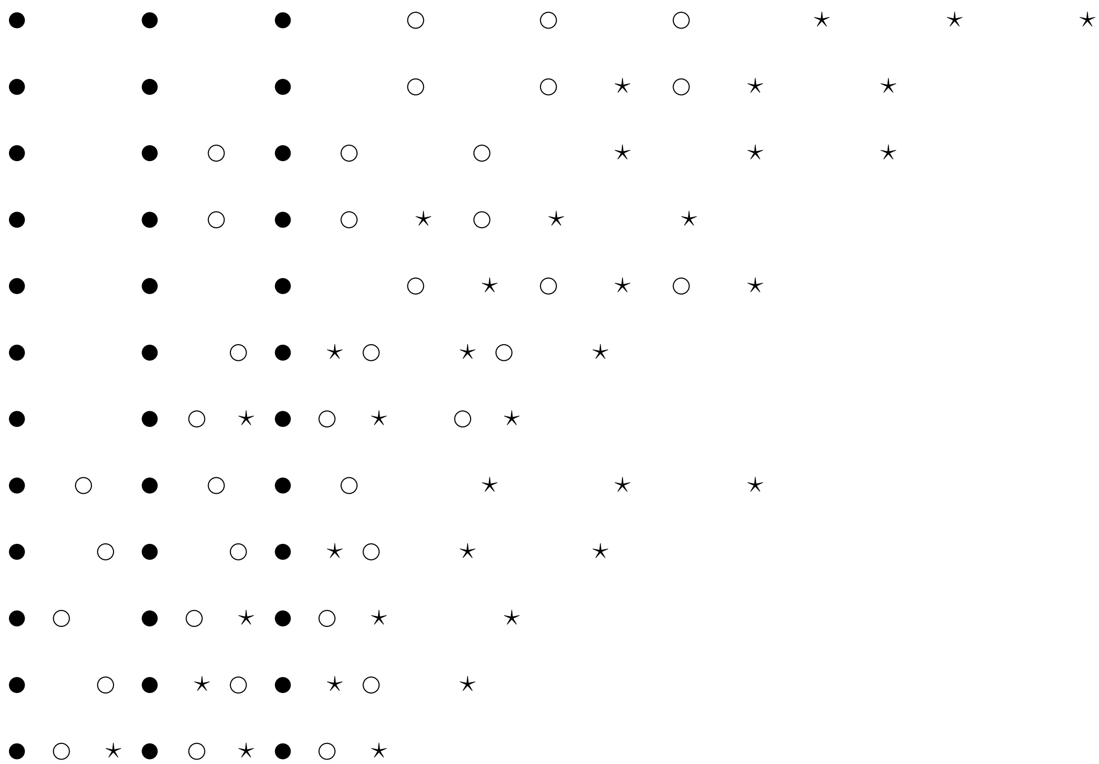

Figure 3: Nonisomorphic marked interval placements.

*is equal to the coefficient of $x^n$ in the Taylor series expansion about $x = 0$ of the rational function $(n-1)!x^{n-1}R_n(1/x)(1-x)^{-n}$.*

For any $\ell_1, \ldots, \ell_m > 0$ the symmetric group $\mathfrak{S}_n$ acts on the arrangements $\mathcal{A}_n$ and $\mathcal{A}_n^0$ (by permutation of coordinates), and therefore also acts on the intersection posets of these arrangements. Thus one can apply the representation-theoretic machinery of [26], as has been done by Robert Gill [9] in the case $\ell = (1, 2, \ldots, k)$. For structural properties of generalized interval orders corresponding to the arrangement $x_i - x_j = \ell_1,\ \ell_2$ for $i \neq j$, see [3] and [4]. These generalized interval orders are there called *double semiorders*.

# 3 Generic interval lengths

An interesting special case of the arrangement $\mathcal{A}_\ell$ occurs when the $\ell_i$ are *generic*. Intuitively this means that the hyperplanes (1) have as few intersections as possible. More precisely, we mean that the intersection poset of the arrangement $\mathcal{A}_\ell$ is the same as the case when $\ell_1, \ldots, \ell_n$ are linearly independent over the rationals. It is not difficult to determine the exact criterion on $\ell_1, \ldots, \ell_n$ necessary for this condition to hold, though we do not state this result here. We have in particular that $(\ell_1, \ldots, \ell_n)$ is generic if $\ell_1, \ldots, \ell_n$ are linearly independent over the rationals. Hence the set of generic interval lengths $(\ell_1, \ldots, \ell_n)$ is dense in the positive orthant of $\mathbb{R}^n$. (In fact, the set of nongeneric interval lengths has measure 0.) Moreover $(\ell_1, \ldots, \ell_n)$ is generic if $\ell_1, \ldots, \ell_n$ are *superincreasing*, i.e., $\ell_{i+1}$ is much larger than $\ell_i$. It might be interesting to find a characterization of interval orders whose interval lengths are superincreasing in terms of forbidden subposets, similar to the well-known characterizations of interval orders and semiorders (e.g., [5, pp. 28 and 30][29, pp. 86 and 193]).

Define a power series

$$y = 1 + x + 5\frac{x^2}{2!} + 46\frac{x^3}{3!} + 631\frac{x^4}{4!} + 9655\frac{x^5}{5!} + 267369\frac{x^6}{6!} + 7442758\frac{x^7}{7!} + \cdots$$

by the equation

$$1 = y(2 - e^{xy}).$$

Let

$$\begin{aligned} z &= \sum_{n\geq 0} c_n \frac{x^n}{n!} \\ &= 1 + x + 3\frac{x^2}{2!} + 19\frac{x^3}{3!} + 195\frac{x^4}{4!} + 2831\frac{x^5}{5!} + 53703\frac{x^6}{6!} + 1264467\frac{x^7}{7!} \\ &\quad +35661979\frac{x^8}{8!} + 1173865927\frac{x^9}{9!} + 44218244942\frac{x^{10}}{10!} + \cdots \end{aligned}$$

be the unique power series satisfying

$$\frac{z'}{z} = y^2, \quad z(0) = 1.$$

**3.1 Theorem.** *Let $c_n$ be as above. Then $c_n$ is equal to the number of regions of the arrangement (1), where $\ell_1, \ldots, \ell_n$ are generic.*

Theorem 3.1 shows that the number of labelled interval orders with $n$ generic interval lengths does not depend on the actual lengths (provided they are generic). On the other hand, the posets themselves (or even their isomorphism types) do depend on the choice of lengths.

The basic tool used to prove Theorem 3.1 is Whitney's formula (as discussed in Section 1) for the arrangement $\mathcal{A}_\ell$. The next theorem states this result in a somewhat simplified form. (The case $q = -1$ is all that is needed to prove Theorem 3.1.)

**3.2 Theorem.** *Let $\mathcal{A}_n$ be the arrangement (1), where $\ell_1, \ldots, \ell_n$ are generic. Then*

$$\chi_{\mathcal{A}_n}(q) = \sum_G (-1)^{e(G)} 2^{b(G)} q^{c(G)},$$

*where $G$ ranges over all bipartite graphs on the vertex set $1, 2 \ldots, n$, and where $e(G)$ denotes the number of edges of $G$, $b(G)$ the number of blocks (maximal doubly connected subgraphs), and $c(G)$ the number of connected components.*

Theorem 1.2 applies to the generic arrangement $\mathcal{A}_\ell$, so we obtain the following corollary.

**3.3 Corollary.** *If $\mathcal{A}_n$ is the arrangement of Theorem 3.2 then we have*

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(q)\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{x^n}{n!}\right)^{-q}.$$

*In particular,*

$$\sum_{n\geq 1} b(\mathcal{A}_n)\frac{x^n}{n!} = 1 - \left(\sum_{n\geq 0} r(\mathcal{A}_n)\frac{x^n}{n!}\right)^{-1}.$$

The first few polynomials $\chi_{\mathcal{A}_n}(q)$ are given by

$$\begin{aligned}
\chi_{\mathcal{A}_1}(q) &= q \\
\chi_{\mathcal{A}_2}(q) &= q^2 - 2q \\
\chi_{\mathcal{A}_3}(q) &= q^3 - 6q^2 + 12q \\
\chi_{\mathcal{A}_4}(q) &= q^4 - 12q^3 + 60q^2 - 122q \\
\chi_{\mathcal{A}_5}(q) &= q^5 - 20q^4 + 180q^3 - 850q^2 + 1780q \\
\chi_{\mathcal{A}_6}(q) &= q^6 - 30q^5 + 420q^4 - 3390q^3 + 15780q^2 - 34082q.
\end{aligned}$$

# 4 Alternating trees and local search trees

In this section we will be concerned with the arrangement in $\mathbb{R}^n$ given by

$$x_i - x_j = 1,\ 1 \leq i < j \leq n.$$

Denote this arrangement by $\mathcal{L}_n$, and set $r(\mathcal{L}_n) = g_n$. N. Linial conceived the idea of looking at this arrangement, and he and S. Ravid made some computations from which a fascinating conjecture about the value of $g_n$ was obtained. This conjecture was recently proved by Postnikov. We first make a number of relevant combinatorial definitions.

- An *alternating tree* or *intransitive tree* is a labelled tree, say with the $n+1$ vertices $0, 1, \ldots, n$, such that if $a_1, \ldots, a_k$ are the vertices of a

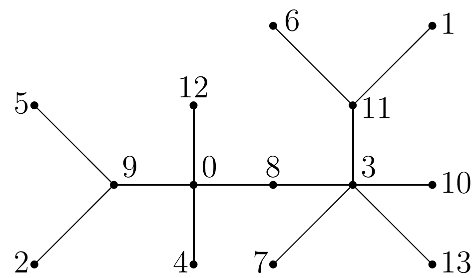


Figure 4: An alternating tree.

path in the tree (in the given order), then either $a_1 < a_2 > a_3 < a_4 > \cdots a_k$ or $a_1 > a_2 < a_3 > a_4 < \cdots a_k$. See Figure 4 for an example. Alternating trees first arose in the work of Gelfand, Graev, and Postnikov [6, §5] (where they are called *admissible* trees), and were further investigated by Postnikov [19]. He showed that if $f_n$ denotes the number of alternating trees on $n+1$ vertices and if

$$\begin{aligned} y &= \sum_{n\geq 0} f_n \frac{x^n}{n!} \\ &= 1 + x + 2\frac{x^2}{2!} + 7\frac{x^3}{3!} + 36\frac{x^4}{4!} + 246\frac{x^5}{5!} + 2104\frac{x^6}{6!} + \cdots, \end{aligned}$$

then

$$\begin{aligned} y &= e^{\frac{x}{2}(y+1)} \\ f_{n-1} &= \frac{1}{n2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}. \end{aligned}$$

- A *local binary search tree* (LBST) is a labelled (plane) binary tree, such that every left child has a smaller label than its parent, and every right child has a larger label than its parent. (Compare with the notion of a binary search tree, in which *all* the nonroot vertices of the left subtree of a vertex $v$ have lower labels than $v$, and similarly for right subtrees.) See Figure 5 for an example. LBST's were first considered by Gessel [7], though not with that terminology. Postnikov [20] found a bijection between alternating trees with $n+1$ vertices and LBST's with $n$ vertices labelled $1, 2, \ldots, n$.

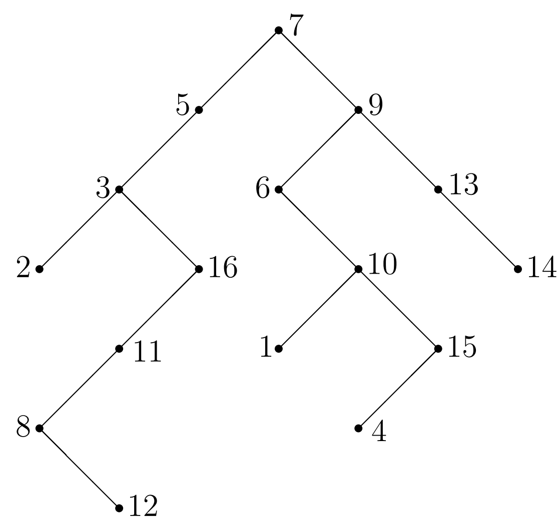


Figure 5: A local binary search tree.

- An easy bijection, obtained independently by A. Postnikov and S. Ravid, shows that $g_n$ is equal to the number of tournaments $T$ on the vertex set $\{1, 2, \ldots, n\}$ such that in every directed cycle of $T$, there are more edges $(i, j)$ with $i < j$ than with $i > j$.

- It is also easy to see that $g_n$ is equal to the number of partially ordered sets on the vertex set $\{1, 2, \ldots, n\}$ that are the intersection of a semiorder (as defined in Section 2) with the chain $1 < 2 < \cdots < n$. (More generally, if $\ell = (\ell_1, \ldots, \ell_n)$ with $\ell_i > 0$, then the number of regions of the arrangement $x_i - x_j = \ell_i,\ 1 \leq i < j \leq n$, is equal to the number of posets obtained by intersecting an element of $\mathcal{I}_\ell$ with the chain $1 < 2 < \cdots < n$.) Let us call the intersection of a semiorder on the vertex set $\{1, 2, \ldots, n\}$ with the chain $1 < 2 < \cdots < n$ a *sleek poset*. For instance, the poset with cover relations $1 < 2,\ 3 < 2,\ 3 < 4$ is a semiorder. When we intersect it with the chain $1 < 2 < 3 < 4$ we obtain the poset $1 < 2,\ 3 < 4$, which is not a semiorder (or even an interval order) but is sleek. It was shown in collaboration with A. Postnikov that a poset on the vertex set $\{1, 2, \ldots, n\}$ is sleek if and only it contains no induced subposet of the four types shown in Figure 6, where $a < b < c < d$. This is the analogue for sleek posets of the characterization of Scott and Suppes [5, p. 30][28][29, p. 193] of

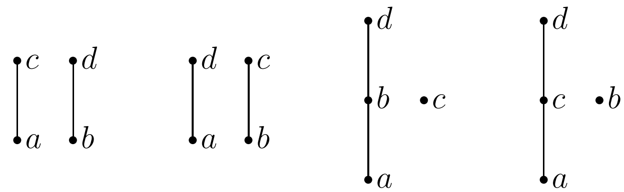


Figure 6: Obstructions to sleekness.

semiorders in terms of two forbidden induced subposets.

- Whitney's formula for the arrangement $\mathcal{L}_n$ yields that

$$g_n = \sum_G (-1)^{\kappa(G)},$$

where $G$ ranges over all bipartite graphs on the vertex set $1, 2, \ldots, n$ such that if $i_1, i_2, \ldots, i_{2k}$ are the vertices of a cycle (in that order), then exactly $k$ indices $1 \leq j \leq 2k$ satisfy $i_j > i_{j+1}$ (where we take subscripts modulo $2k$), and where $\kappa(G)$ denotes the cyclomatic number (number of linearly independent cycles in the mod 2 cycle space) of $G$.

- Athanasiadis [1] has shown, based on a combinatorial interpretation [17, Thm. 2.3.22] of the characteristic polynomial of an arrangement defined over a finite field, that the characteristic polynomial $\chi_n(q)$ of $\mathcal{L}_n$ is given by

$$\chi_n(q) = q \sum_{k=1}^{n} (k-1)! S(n,k) \sum_{i=0}^{n-k} \binom{n-k}{i} \binom{q-k-i-1}{k-1},$$

where $S(n,k)$ denotes a Stirling number of the second kind.

The primary result on the Linial arrangement $\mathcal{L}_n$ is the following. It was conjectured by this writer on the basis of data supplied by Linial and Ravid, and recently proved by Postnikov.

**4.1 Theorem.** *For all $n \geq 0$ we have $f_n = g_n$.*

Theorem 1.2 applies to the arrangement $\mathcal{L}_n$ (see Corollary 4.2(a) below), so Theorem 4.1 in fact determines the characteristic polynomial of $\mathcal{L}_n$.

**4.2 Corollary.** *The characteristic polynomial $\chi_n(q)$ of $\mathcal{L}_n$ is given by*

$$\sum_{n\geq 0}\chi_n(q)\frac{x^n}{n!}=\left(\sum_{n\geq 0}(-1)^n g_n\frac{x^n}{n!}\right)^{-q}.$$

*In particular*

$$\sum_{n\geq 1}b(\mathcal{L}_n)\frac{x^n}{n!}=1-\left(\sum_{n\geq 0}g_n\frac{x^n}{n!}\right)^{-1}.$$

*Moreover,*

$$\chi_n(q)=\sum_{i=1}^{n}(-1)^{n-i}f_{i,n}q^i,$$

*where $f_{i,n}$ is the number of alternating trees on the vertices $0,1,\ldots n$ such that vertex 0 has degree $i$.*

# 5 The Shi arrangement and parking functions.

An arrangement closely related to those discussed above is given by $x_i-x_j=0,1$ for $1\leq i<j\leq n$ and will be called the *Shi arrangement* (called by Headley [12, Ch. VI] the *sandwich arrangement* associated with the symmetric group $\mathfrak{S}_n$), denoted $\mathcal{S}_n$. It was first considered by J.-Y. Shi [23] in his investigation of the affine Weyl group $\tilde{A}_n$, so we will call it the *Shi arrangement*. Shi showed the surprising result [23, Cor. 7.3.10]

$$r(\mathcal{S}_n)=(n+1)^{n-1} \tag{4}$$

using group-theoretic techniques. Later Shi [24] generalized his result to other Weyl groups. Headley [11][12, Ch. VI] gave a proof of equation (4) based on Zaslavsky's theorem (Theorem 1.1), and in fact computed the characteristic polynomial $\chi_{\mathcal{S}_n}(q)$, viz.,

$$\chi_{\mathcal{S}_n}(q)=q(q-n)^{n-1}. \tag{5}$$

One can also deduce (5) from (4) using Theorem 1.2; and an elegant "coloring" proof has been given by Athanasiadis [1]. We will give a bijective proof of a refinement of equation (4) related to inversions of trees. This work was done in collaboration with Igor Pak.

In order to motivate our result, first consider the case of the braid arrangement $\mathcal{B}_n$. If $w$ is a permutation of $1, 2 \ldots, n$, then define its *code* or *inversion table* to be the sequence $C(w) = (a_1, \ldots, a_n)$ where $a_i$ is the number of elements $j$ for which $j > i$ and $w^{-1}(j) < w^{-1}(i)$. (The definition of $C(w)$ is sometimes given as a minor variation of our definition here.) In particular, $\sum_i a_i = \ell(w)$, the number of *inversions* of $w$ (or the *length* of $w$ in the sense of Coxeter groups). It is clear that $0 \leq a_i \leq n - i$, and it is easy to see that $C$ is a bijection from the symmetric group $\mathfrak{S}_n$ to the set of sequences $(a_1, \ldots, a_n)$ with $0 \leq a_i \leq n - i$. There follows the well-known result [27, Cor. 1.3.10]

$$\sum_{w \in \mathfrak{S}_n} q^{\ell(w)} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}). \tag{6}$$

Now let $R_0$ be the region of $\mathcal{B}_n$ defined by $x_1 > x_2 > \cdots > x_n$, which we call the *base region.* We will assign an $n$-tuple $\kappa(R)$ of nonnegative integers to every region $R$ of $\mathcal{B}_n$ as follows. First define $\kappa(R_0) = (0, 0, \ldots, 0)$. Suppose that $\kappa(R)$ has been defined, and that $R'$ is a region such that (a) $\kappa(R')$ has not yet been defined, (b) some hyperplane $x_i = x_j$ (with $i < j$) is a boundary facet of both $R$ and $R'$, and (c) $R_0$ and $R$ lie on the same side of the hyperplane $x_i = x_j$. Define $\kappa(R') = \kappa(R) + \varepsilon_i$, where $\varepsilon_i$ is the $i$th unit coordinate vector. It is then easy to see that $\kappa(R)$ is well-defined, and that $\kappa(R)$ is the code of some permutation $w \in \mathfrak{S}_n$. Moreover, for any code $C(w)$ with $w \in \mathfrak{S}_n$ there is a unique region $R$ of $\mathcal{B}_n$ with $\kappa(R) = C(w)$. Thus the map $C^{-1}\kappa$ defines a bijection between the regions of $\mathcal{B}_n$ and the symmetric group $\mathfrak{S}_n$, with the property that if $C^{-1}\kappa(R) = w$, then the number of hyperplanes of $\mathcal{B}_n$ separating $R$ from $R_0$ is $\ell(w)$, the number of inversions of $w$. We now describe a completely analogous construction for the arrangement $\mathcal{S}_n$.

Let $T$ be a tree with vertices $0, 1, \ldots, n$. An *inversion* of $T$ is a pair $1 \leq i < j$ such that vertex $j$ lies on the unique path in $T$ from 0 to $i$. Write $\ell(T)$ for the number of inversions of $T$. The polynomial

$$I_{n+1}(q) = \sum_T q^{\ell(T)},$$

summed over all trees with vertices $0, 1, \ldots, n$, is known as the *inversion enumerator* for trees. Analogously to equation (6) (but more difficult to prove [8][15]) we have

$$\sum_{n \geq 0} (1+q)^{\binom{n}{2}} \frac{x^n}{n!} = \exp \sum_{n \geq 1} q^{n-1} I_n(1+q) \frac{x^n}{n!}.$$

Consider now $n$ cars $C_1, \ldots, C_n$ that want to park on a one-way street with parking places $0, 1, \ldots, n-1$ in that order. Each car $C_i$ has a preferred space $a_i$. The cars enter the street one at a time in the order $C_1, \ldots, C_n$. A car tries to park in its preferred space. If that space is occupied, then it parks in the next available space. If there is no space then the car leaves the street. The sequence $(a_1, \ldots, a_n)$ is called a *parking function* if all the cars can park, i.e., no car leaves the street. (Our definition is a slight variant of the usual definition.) It is not difficult to see that the sequence $(a_1, \ldots, a_n)$ is a parking function if and only if it has at most $i$ terms greater than or equal to $n-i$, for $1 \leq i \leq n$. Equivalently, a parking function is a permutation of the code of a permutation. For further information on parking functions, see [10, §2.6][13]. The number of parking functions of length $n$ is $(n+1)^{n-1}$, and Kreweras [14] in fact gave a bijection $C$ between trees with vertices $0, 1, \ldots, n$ and parking functions such that if $C(T) = (a_1, \ldots, a_n)$ then $a_1 + \cdots + a_n = \binom{n}{2} - \ell(T)$. Thus the number of parking functions $(a_1, \ldots, a_n)$ of length $n$ such that $\sum a_i = k$ is equal to the number of labelled trees with vertices $0, 1, \ldots, n$ and with $\binom{n}{2} - k$ inversions. It follows that the parking function $C(T)$ is a good analogue of the code of a permutation. Theorem 5.1 below makes this analogy even stronger.

Let $R_0$ be the region of $\mathcal{S}_n$ defined by $x_1 > x_2 > \cdots > x_n$ and $x_1 - x_n < 1$, which we call the *base region*. Equivalently, $R_0$ is the unique region contained between all pairs of parallel hyperplanes of $\mathcal{S}_n$. We will assign an $n$-tuple $\lambda(R)$ of nonnegative integers to every region $R$ of $\mathcal{S}_n$ as follows. First define $\lambda(R_0) = (0, 0, \ldots, 0)$. Suppose that $\lambda(R)$ has been defined, and that $R'$ is a region such that (a) $\lambda(R')$ has not yet been defined, (b) some hyperplane $H$ of $\mathcal{S}_n$ is a boundary facet of both $R$ and $R'$, and (c) $R_0$ and $R$ lie on the same side of the hyperplane $H$. Define

$$\lambda(R') = \left\{ \begin{array}{ll} \lambda(R) + \varepsilon_i, & \text{if } H \text{ is given by } x_i - x_j = 0 \text{ with } i < j \\ \lambda(R) + \varepsilon_j, & \text{if } H \text{ is given by } x_i - x_j = 1 \text{ with } i < j. \end{array} \right\}$$

It is then easy to see that $\lambda(R)$ is well-defined, and that $\lambda(R)$ is a parking function. Moreover, if $\lambda(R) = (a_1, \ldots, a_n)$, then $a_1 + \cdots + a_n$ is equal to the number of hyperplanes in $\mathcal{S}_n$ separating $R$ from $R_0$. Not so evident is the following result obtained in collaboration with Igor Pak [18].

**5.1 Theorem.** *The map $\lambda$ defined above is a bijection from the regions of $\mathcal{S}_n$ to the set of all parking functions of length $n$. Consequently, the number of regions $R$ for which $i$ hyperplanes separate $R$ from $R_0$ is equal to the number of trees on the vertices $0, 1, \ldots, n$ with $\binom{n}{2} - i$ inversions.*

Theorem 5.1 can be reformulated in terms of posets. Given a permutation $w \in \mathfrak{S}_n$, let $P_w = \{(i,j) : 1 \le i < j \le n,\ w(i) < w(j)\}$. Partially order $P_w$ by the rule $(i,j) \le (k,l)$ if $k \le i < j \le l$. Let $F(J(P_w), q)$ denote the rank-generating function of the lattice $J(P_w)$ of order ideals of $P_w$, as defined in [27, pp. 99 and 106]. Then it is not difficult to show that Theorem 5.1 is equivalent to the formula

$$\sum_{w \in \mathfrak{S}_n} F(J(P_w), q) = I_{n+1}(q).$$

Theorem 5.1 suggests a host of additional problems dealing with the Shi arrangement and related arrangements. Many of these problems are currently under investigation.

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