Polygon Dissections and Standard Young Tableaux

Richard P. Stanley[1]
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
rstan@math.mit.edu

ABSTRACT

A simple bijection is given between dissections of a convex $(n+2)$-gon with d diagonals not intersecting in their interiors and standard Young tableaux of shape $(d+1, d+1, 1^{n-1-d})$.

[1]Partially supported by National Science Foundation grant #DMS-9206374.

For $0 \le d \le n-1$, let $f(n,d)$ be the number of ways to draw d diagonals in a convex $(n+2)$-gon, such that no two diagonals intersect in their interior. For instance, $f(n,n-1)$ is just the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. A result going back to Kirkman [3], Prouhet [4], and Cayley [1] (with Cayley giving the first complete proof) asserts that

$$f(n,d) = \frac{1}{n+d+2}\binom{n+d+2}{d+1}\binom{n-1}{d}. \tag{1}$$

K. O'Hara and A. Zelevinsky observed (unpublished) that the right-hand side of (1) is just the number of standard Young tableaux (as defined, e.g., in [5, p. 66]) of shape $(d+1, d+1, 1^{n-1-d})$, where 1^{n-1-d} denotes a sequence of $n-1-d$ 1's. It is natural to ask for a bijection between the polygon dissections and the standard Young tableaux. If one is willing to accept the formula for the number of standard Young tableaux of a fixed shape (either in the original form due to MacMahon or the hook-length formula of Frame-Robinson-Thrall), then one obtains a simple proof of equation (1). In this note we give a simple bijection of the desired type.

First we recall that there is a well-known bijection [2] between dissections D of an $(n+2)$-gon with d diagonals and integer sequences $\psi(D) = (a_1, a_2, \ldots, a_{n+d+1})$ such that (a) either $a_i = -1$ or $a_i \ge 1$, (b) exactly n terms are equal to -1, (c) $a_1 + a_2 + \cdots + a_i \ge 0$ for all i, and (d) $a_1 + a_2 + \cdots + a_{n+d+1} = 0$. This bijection may be defined recursively as follows. Fix an edge e of the dissected polygon D. When we remove e from D, we obtain a sequence of dissected polygons $D_1, D_2, \ldots, D_k$ (where $k+1$ is the number of sides of the region of D to which e belongs), arranged in clockwise order, with D_i and D_{i+1} intersecting at a single vertex. If D_i consists of a single edge, then define $\psi(D_i) = -1$, and set recursively $\psi(D) = (k-1, \psi(D_1)^*, \psi(D_2)^*, \ldots, \psi(D_{k-1})^*, \psi(D_k))$, where $\psi(D_k)^*$ denotes $\psi(D_k)$ with a -1 appended at the end.

Given a sequence $(a_1, a_2, \ldots, a_{n+d+1})$ as above, define a standard Young tableau T of shape $(d+1, d+1, 1^{n-1-d})$ as follows. We insert the elements $1, 2, \ldots, n+d+1$ successively into T. Once an element is inserted, it remains in place. (There is no "bumping" as in the Robinson-Schensted correspondence.) Suppose that the positive a_i's are given by $b_1, b_2, \ldots, b_{d+1}$, in that order. The insertion is then defined by the following three rules:

- If $a_i > 0$, then insert i at the end of the first row. (We write our tableaux in "English" style, so the longest row is at the top.)
- If $a_i = -1$ and the number of -1's preceding a_i is given by $b_1 + b_2 + \cdots + b_j$ for some $j \geq 0$, then insert i at the end of the second row.
- If $a_i = -1$ and the number of -1's preceding a_i in not of the form $b_1 + b_2 + \cdots + b_j$, then insert i at the bottom of the first column.

It is an easy exercise to check that the above procedure yields the desired bijection.

Example. Let the sequence corresponding to a dissection D (with $n = 14$, $d = 6$) be given by

$$(4,2,-\mathbf{1},1,-1,-1,3,-1,-\mathbf{1},1,1,-1,-\mathbf{1},-\mathbf{1},-1,-1,-\mathbf{1},2,-\mathbf{1},-\mathbf{1},-1).$$

We have $(b_1, \ldots, b_7) = (4,2,1,3,1,1,2)$. We have printed in boldface those -1's that are preceded by $b_1 + \cdots + b_j$ -1's for some j. The corresponding standard tableau $\psi(D)$ is given by

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 2 & 4 & 7 & 10 & 11 & 18 \\
\hline
3 & 9 & 13 & 14 & 17 & 19 & 20 \\
\hline
5 \\
\cline{1-1}
6 \\
\cline{1-1}
8 \\
\cline{1-1}
12 \\
\cline{1-1}
15 \\
\cline{1-1}
16 \\
\cline{1-1}
21 \\
\cline{1-1}
\end{array}$$

References

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