# Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley[1]
Department of Mathematics 2-375
Massachusetts Institute of Technology
Cambridge, MA 02139

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## 1 Introduction.

Algebraic combinatorics is concerned with the interaction between combinatorics and such other branches of mathematics as commutative algebra, algebraic geometry, algebraic topology, and representation theory. Many of the major open problems of algebraic combinatorics are related to positivity questions, i.e., showing that certain integers are nonnegative. The significance of positivity to algebraic combinatorics stems from the fact that a nonnegative integer can have both a combinatorial and an algebraic interpretation. The archetypal algebraic interpretation of a nonnegative integer is as the dimension of a vector space. Thus to show that a certain integer $m$ is nonnegative, it suffices to find a vector space $V_m$ of dimension $m$. Similarly to show that $m \leq n$, it suffices to find an injective map $V_m \to V_n$ or surjective map $V_n \to V_m$. Of course the inequality $m \leq n$ is equivalent to the positivity statement $n-m \geq 0$, while the injectivity of the map $\varphi : V_m \to V_n$ is equivalent to the statement that $n - m = \dim \operatorname{coker}(\varphi)$ (where $\operatorname{coker}(\varphi)$ denotes the cokernel $V_n/\varphi(V_m)$ of $\varphi$). However, it is often more natural to deal with the inequality $m \leq n$ rather than with $n - m \geq 0$.

We will attempt here an overview of the outstanding open problems in algebraic combinatorics related to positivity. Naturally our choice is subjective, and we do not claim to be comprehensive.

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# 2 $f$-vectors

Many geometric objects $\Gamma$ are defined in terms of simple objects, which we call *faces*, with well-defined dimensions. Examples (in increasing order of generality) include simplicial complexes, polyhedral complexes, regular CW complexes, and CW complexes. If $\Gamma$ has dimension $d-1$ and has $f_i$ $i$-dimensional faces, then the vector $f(\Gamma) = (f_0, f_1, \ldots, f_{d-1})$ is called the *$f$-vector* of $\Gamma$ and is a fundamental combinatorial invariant of $\Gamma$. (Unless $\Gamma = \emptyset$ one regards the empty set $\emptyset$ as a face of $\Gamma$ of dimension $-1$, so $f_{-1} = 1$.)

Much of algebraic combinatorics is concerned with obtaining a complete characterization, or at least significant information, about the $f$-vector of various classes of geometric objects. A good summary of this area, together with a list of open problems, is given by Billera and Björner [9]. Quite sophisticated tools such as commutative algebra, exterior algebra, homological algebra, and toric varieties can be used to investigate $f$-vectors. One important result in the area, known as the *g-theorem* for simplicial polytopes, gives a complete characterization of the $f$-vector of a simplicial polytope $\mathcal{P}$ (i.e., a convex polytope all of whose proper faces are simplices). This result was conjectured by McMullen [66] in 1971. The sufficiency of McMullen's conditions was proved by Billera and Lee [13][14], while the necessity was proved by Stanley [75] using the theory of toric varieties. Later McMullen [67][68] gave a new proof of necessity avoiding toric varieties. We will not state the full result here but will explain a corollary of it known as the *Generalized Lower Bound Theorem* (GLBT) for simplicial polytopes. Given any $(d-1)$-dimensional (abstract or geometric) simplicial complex $\Delta$ with $f$-vector $(f_0, \ldots, f_{d-1})$, define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of $\Delta$ by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

If $\Delta$ is the boundary complex of a simplicial polytope, then the *Dehn-Sommerville equations* assert that $h_i = h_{d-i}$. The GLBT consists of the inequalities $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$. The significance of the GLBT is that it gives the most general linear inequalities satisfied by $f$-vectors of simplicial polytopes.

It is natural to ask whether the Dehn-Sommerville equations, GLBT, and $g$-theorem can be extended to more general objects than simplicial polytopes. The most general objects that seem feasible for this purpose are the *Gorenstein* complexes*. These are simplicial complexes $\Delta$ such that for every face $F \in \Delta$ (including $F = \emptyset$) we have

$$\tilde{H}_i(\mathrm{link}(F); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } i = \dim \mathrm{link}(F) \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\mathrm{link}(F) = \{G \in \Delta : F \cap G = \emptyset,\ F \cup G \in \Delta\},$$

the *link* of $F$ in $\Delta$, and where $\tilde{H}$ denotes reduced simplicial homology. It is not hard to see that the Dehn-Sommerville equations $h_i = h_{d-i}$ continue to hold for Gorenstein* complexes.

**Problem 1.** *Does the GLBT (or more generally the $g$-theorem) hold for Gorenstein* complexes?*

Problem 1 is perhaps the main open problem in the subject of $f$-vectors. Special cases of Gorenstein* complexes for which the GLBT is also open include triangulations of spheres, PL-spheres, and complete simplicial fans. It was shown independently by Kalai [53] (using algebraic shifting) and Stanley [84, Cor. 2.4] (using Cohen-Macaulay rings) that the GLBT holds for the boundary $\Delta$ of a $(d-1)$-dimensional ball that is a subcomplex of the boundary complex of a simplicial $d$-polytope. It does not seem to be known exactly which simplicial complexes occur in this way. (The techniques of Kalai and Stanley only establish the GLBT, and not the $g$-theorem, for the above complexes $\Delta$.)

When $\mathcal{P}$ is an arbitrary (i.e., not necessarily simplicial) convex polytope, then the $h$-vector no longer has nice properties. For simplicial polytopes the number $h_i$ is the $2i$th Betti number of a certain toric variety $X_{\mathcal{P}}$ satisfying Poincaré duality, whence the Dehn-Sommerville equations $h_i = h_{d-i}$. For nonsimplicial polytopes, however, there is little connection between the homology of $X_{\mathcal{P}}$ and the $f$-vector of $\mathcal{P}$. Moreover, $X_{\mathcal{P}}$ fails to satisfy Poincaré duality, and the Dehn-Sommerville equations fail to hold for the $h$-vector of $\mathcal{P}$. This unfortunate state of affairs can be rectified by dealing with the (middle perversity) *intersection homology* $\mathrm{IH}_*(X_{\mathcal{P}};\mathbb{R})$ (or cohomology $\mathrm{IH}^*(X_{\mathcal{P}};\mathbb{R})$) of $X_{\mathcal{P}}$, which can be defined whenever $\mathcal{P}$ has *rational* vertices. We then have $\mathrm{IH}_{2i+1}(X_{\mathcal{P}};\mathbb{R}) = 0$. Define $h_i = \dim_{\mathbb{R}} \mathrm{IH}_{2i}(X_{\mathcal{P}};\mathbb{R})$. If $\dim \mathcal{P} = d$, then the vector $h(\mathcal{P}) = (h_0, h_1, \ldots, h_d)$ is called the *toric h-vector* (formerly the *generalized h-vector*) of $\mathcal{P}$. A purely combinatorial definition of the toric $h$-vector can be given that extends to nonrational polytopes [77] and even more general objects, the most general being Eulerian posets [86]. The toric $h$-vector depends only on the combinatorial type of $\mathcal{P}$ (not on how it is embedded into $\mathbb{R}^d$) and satisfies the Dehn-Sommerville equations $h_i = h_{d-i}$. However, the toric $h$-vector does *not* depend just on the $f$-vector of $\mathcal{P}$, nor can $f(\mathcal{P})$ be recovered from $h(\mathcal{P})$. When $\mathcal{P}$ is simplicial, then the toric $h$-vector coincides with the usual $h$-vector. When $\mathcal{P}$ has rational vertices, the connection with toric varieties and intersection homology leads to a number of results concerning the toric $h$-vector of $\mathcal{P}$, in particular the GLBT $h_0 \le h_1 \le \cdots \le h_{\lfloor d/2 \rfloor}$. For *nonrational* polytopes it is not known whether the GLBT and related results continue to hold. If $P$ is a (finite) poset, then the *order complex* of $P$ is the abstract simplicial complex $\Delta(P)$ whose faces are the chains of $P$. We say that $P$ has a certain topological property such as Gorenstein* if $\Delta(P)$ has that property. Then the GLBT (or even just $h_i \ge 0$) fails in general for Eulerian posets, but it remains open for Gorenstein* *lattices* (with bottom element $\hat{0}$ and top element $\hat{1}$ removed).

**Problem 2.** *Let $\mathcal{P}$ be an arbitrary convex polytope, or even a Gorenstein* poset that is a lattice with $\hat{0}$ and $\hat{1}$ removed. Does the toric h-vector $(h_0, h_1, \ldots, h_d)$ of $\mathcal{P}$ satisfy the GLBT $h_0 \le h_1 \le \cdots \le h_{\lfloor d/2 \rfloor}$, or even just $h_i \ge 0$?*

Some related problems dealing with subdivisions of polytopes and other geometric objects appear in [83, Conjs. 4.11 and 5.4].

There are numerous additional conjectures concerning $f$-vectors. For instance, the $f$-vectors of *centrally-symmetric* simplicial polytopes are poorly understood (see [82] for some results in this area), and not even a conjecture analogous to the GLBT is known. Such a conjecture would give the most general linear inequalities satisfied by the $f$-vectors of centrally-symmetric simplicial $d$-polytopes. One simple problem for *arbitrary* centrally-symmetric polytopes, due to Kalai [52], is the following.

**Problem 3.** *Let $(f_0, f_1, \ldots, f_{d-1})$ be the $f$-vector of a centrally-symmetric $d$-polytope. Is it true that*

$$1 + f_0 + f_1 + \cdots + f_{d-1} \geq 3^d?$$

One feature that makes this problem difficult is that there is more than one polytope that achieves this bound. We can extend Problem 3 somewhat as follows.

**Problem 3$'$.** *Let $L$ be finite* lattice *of rank $d+1$ such that $L - \{\hat{0}, \hat{1}\}$ is Gorenstein*. Suppose that $L$ has a lattice automorphism $\sigma$ that is an involution and that fixes only $\hat{0}$ and $\hat{1}$. Is it true that $\#L \geq 3^d + 1$?*

A *missing face* of an abstract simplicial complex $\Delta$ is a set $S$ of vertices of $\Delta$ such that $S$ is not a face of $\Delta$, but every proper subset of $S$ is a face. A *flag complex* is a simplicial complex for which every missing face has two elements. For instance, order complexes of posets are flag complexes, as are Coxeter complexes of finite Coxeter groups. Flag complexes are the same as *clique complexes* or *stable set complexes* of graphs. There is a lot of interest in obtaining information about $f$-vectors of flag complexes. One of the most interesting open problems is known as the *Charney-Davis conjecture* [27][87, p. 100] and is a discrete analogue of a conjecture of H. Hopf on the Euler characteristic of a closed Riemannian manifold of nonpositive sectional curvature. (Charney and Davis made their conjecture originally for spherical flag complexes, but we have extended it to the Gorenstein* case.)

**Problem 4.** *Let $\Delta$ be a $2e-1$-dimensional Gorenstein* flag complex with $h$-vector $(h_0, h_1, \ldots, h_{2e})$. Is it true that*

$$(-1)^e(h_0 - h_1 + h_2 - \cdots + h_{2e}) \geq 0?$$

For certain classes of simplicial complexes $\Delta$ whose $f$-vector satisfies certain linear inequalities, these linear inequalities can be "displayed" by a suitable *decomposition* (or partition) of the set of faces of $\Delta$. For instance, suppose that $\Delta$ is acyclic (has vanishing reduced homology, say over $\mathbb{Q}$). It was shown in [79] that there exists a partition $\Pi$ of $\Delta$ into two-element sets $\{F, F'\}$ such that (a) $F \subset F'$ and $|F' - F| = 1$, and (b) the set

$\{F : \{F, F'\} \in \Pi,\ F \subset F'\}$ is a subcomplex $\Gamma$ of $\Delta$. If follows that

$$(1+x)\sum_i f_{i-1}(\Gamma)x^i = \sum_i f_{i-1}(\Delta)x^i.$$

If $C(\Gamma)$ denotes the cone over $\Gamma$, then clearly $C(\Gamma)$ is acyclic and

$$(1+x)\sum_i f_{i-1}(\Gamma)x^i = \sum_i f_{i-1}(C(\Gamma))x^i.$$

Hence $f$-vectors of acyclic simplicial complexes coincide with $f$-vectors of cones, which are relatively easy to characterize. Thus the existence of the decomposition $\Pi$ leads to a characterization of $f$-vectors of acyclic simplicial complexes. (Similar decompositions appear in the work of R. Forman [31][32] on discrete Morse theory.) What other classes of simplicial complexes lend themselves to this technique? Duval [28] has extended the above argument to give a "decomposition-theoretic" proof of the characterization of pairs $(f, \beta)$, where $f$ is the $f$-vector and $\beta$ the sequence of Betti numbers of a simplicial complex, originally due to Björner and Kalai [16][17]. A different generalization of the decomposition of acyclic complexes was conjectured by Stanley [79, Conj, 2.4] as follows.

**Problem 5.** *Suppose that $\Delta$ is a (finite) abstract simplicial complex such that the link of every face $F$ of $\Delta$ of dimension at most $j$ (including* $\mathrm{link}(\emptyset) = \Delta$*) is acyclic. Can $\Delta$ (regarded as a partially ordered set under set inclusion) be partitioned into intervals $[F, F']$ such that (a)* $\dim F' - \dim F = j + 2$*, and (b) the bottom elements $F$ of these intervals form a subcomplex of $\Delta$?*

A central role in the combinatorics of simplicial complexes is played by the *Cohen-Macaulay* complexes. A simplicial complex $\Delta$ is said to be Cohen-Macaulay (say over $\mathbb{Q}$) if for every $F \in \Delta$ (including as usual $F = \emptyset$) we have

$$\tilde{H}_i(\mathrm{link}(F); \mathbb{Q}) = 0, \text{ if } i \neq \dim \mathrm{link}(F).$$

It can be proved algebraically (e.g., [87, Cor. II.3.2]) that if $\Delta$ is Cohen-Macaulay with $h$-vector $(h_0, \ldots, h_d)$, then $h_i \geq 0$. On the other hand, a pure (i.e., all maximal faces have the same dimension) $(d-1)$-dimensional simplicial complex $\Delta$ is called *partitionable* if it can be partitioned into intervals

$[F, F']$ such that each $F'$ is a facet (maximal face) of $\Delta$. It is easy to see that if $\Pi$ is a partition of $\Delta$ into such intervals, then

$$\sum_{[F,F']\in\Pi} x^{1+\dim F} = \sum_i h_i x^i.$$

Hence $h_i \geq 0$ if $\Delta$ is partitionable, and it is natural to ask if there is a connection between the Cohen-Macaulay property and partitionability. The simplicial complex with facets $ab, ac, bc, de, df, ef$ has the partitioning

$$\Pi = \{[\emptyset, ab], [c, bc], [ac, ac], [d, de], [e, ef], [f, df]\}$$

but is not Cohen-Macaulay. However, the converse question remains a central open problem concerning the combinatorics of simplicial complexes.

**Problem 6.** *Is every Cohen-Macaulay complex partitionable?*

For further information related to partitionability, see [87, §III.2].

We have mentioned the toric $h$-vector as an extension of the $h$-vector of a simplicial polytope. A different extension, which is much more natural from the combinatorial point of view and which conveys much more information, is the flag $f$-vector. It is most conveniently defined for graded posets, i.e., posets for which every maximal chain has the same length. Let $P$ be a (finite) graded poset of rank $n-1$, so every maximal chain of $P$ has $n$ elements. Define the *rank function* $\rho : P \to \mathbb{Z}$ of $P$ by letting $k = \rho(t)$ be the number of elements in the longest chain $t_1 < t_2 < \cdots < t_k = t$. If $S$ is any subset of $\{1, 2, \ldots, n\}$, then define the *rank-selected subposet*

$$P_S = \{t \in P \,:\, \rho(t) \in S\},$$

and let $\alpha_P(S)$ be the number of maximal chains of $P_S$. The function $\alpha_P$ is called the *flag $f$-vector* of $P$. Thus $\alpha_P$ counts the number of chains $C$ in $P$ according to the ranks of the elements of $C$. The "flag-analogue" of the $h$-vector is the *flag $h$-vector* $\beta_P$, defined by either of the equivalent conditions

$$\begin{aligned}
\alpha_P(S) &= \sum_{T\subseteq S} \beta_P(T) \\
\beta_P(S) &= \sum_{T\subseteq S} (-1)^{\#(S-T)} \alpha_P(T).
\end{aligned}$$

If $\Delta(P)$ denotes the order complex of $P$, then the $f$-vector and $h$-vector of $\Delta(P)$ are related to the flag $f$-vector $\alpha_P$ and flag $h$-vector $\beta_P$ by

$$\begin{aligned} f_i(\Delta(P)) &= \sum_{\substack{S\subseteq\{1,\dots,n\} \\ \#S=i+1}} \alpha_P(S) \\ h_i(\Delta(P)) &= \sum_{\substack{S\subseteq\{1,\dots,n\} \\ \#S=i}} \beta_P(S). \end{aligned}$$

We will mention here one central open problem in the theory of flag $f$-vectors, viz., the problem of determining all linear inequalities satisfied by the flag $f$-vector of a Gorenstein* poset $P$. (Note for experts: According to the definitions given here, Gorenstein* posets do not have a $\hat{0}$ and $\hat{1}$. Thus for instance the face lattice of a convex polytope $\mathcal{P}$, *with the improper faces* $\emptyset$ *and* $\mathcal{P}$ *removed*, is a Gorenstein* poset.) The corresponding problem for arbitrary graded posets was solved by Billera and Hetyei [12], while partial results were obtained for Eulerian posets by Bayer and Hetyei [5]. Let $a$ and $b$ be noncommuting variables. Given $S \subseteq \{1, 2, \dots, n\}$, define a noncommutative monomial $u^S = u_1 u_2 \cdots u_n$ by setting $u_i = a$ if $i \notin S$ and $u_i = b$ if $i \in S$. If $P$ is a graded poset of rank $n-1$, then define a noncommutative polynomial

$$\Psi_P(a, b) = \sum_{S\subseteq\{1,\dots,n\}} \beta_P(S) u^S.$$

Thus $\Psi_P(a, b)$ is a noncommutative generating function for the flag $h$-vector $\beta_P$. Moreover, it follows immediately from the definitions of $\alpha_P$ and $\beta_P$ that

$$\Psi_P(a, a+b) = \sum_{S\subseteq\{1,\dots,n\}} \alpha_P(S) u^S.$$

In the case when $P$ is Gorenstein* (or even Eulerian), Bayer and Billera [4] determined the most general linear equalities that can hold among the numbers $\alpha_S(P)$ (or equivalently $\beta_S(P)$). Fine (see [6, Prop. 2]) discovered an exceptionally elegant way to state these Bayer-Billera relations. Namely, there exists a polynomial $\Phi_P(c, d)$ in noncommutative variables $c$ and $d$ such that

$$\Psi_P(a, b) = \Phi_P(a + b, ab + ba).$$

The polynomial $\Phi(c, d)$ is called the *cd-index* of $P$. The main open problem concerning this polynomial is the following.

**Problem 7.** *If $P$ is a Gorenstein* poset, then are all coefficients of the cd-index $\Phi_P(c, d)$ nonnegative?*

It was conjectured in [85, Conj. 2.1] that the answer to Problem 7 is affirmative. Moreover, it was shown in Theorem 2.1 of this reference that this result, if true, is best possible in the sense that any linear inequality satisfied by all flag $f$-vectors of Gorenstein* posets of rank $n-1$ is a consequence of the nonnegativity of the coefficients of the $cd$-index. A special case of Problem 7, which includes face lattices of convex polytopes, was proved in [85, Thm. 2.2].

E. Babson observed (stated incorrectly in [87, p. 103] without the factor of $2^{-m}$) that if $P$ is a Gorenstein* poset of even rank $2m$, then the coefficient of $d^m$ in $\Phi_P(c, d)$ is given by

$$[d^m]\,\Phi_P(c, d) = (-2)^{-m}(h_0 - h_1 + h_2 - \cdots + h_{2m}),$$

where $(h_0, h_1, \ldots, h_{2m})$ is the $h$-vector of the order complex of $P$. It follows that the Charney-Davis conjecture (Problem 4) for the special case of order complexes is a consequence of an affirmative answer to Problem 7.

In general the $cd$-index is a highly intractable object. It would be of great interest to find a natural algebraic or geometric description of the $cd$-index. For some further work related to the $cd$-index, see for instance [10][11] and the references given there.

The $f$-vectors of *cubical complexes* are much less well understood than those of simplicial complexes. We may regard a (finite) abstract cubical complex as a finite meet-semilattice such that every interval $[\hat{0}, t]$ is isomorphic to the face lattice of a cube (whose dimension depends on $t$). An analogue of the $h$-vector of a simplicial complex was defined by R. Adin [1] for *pure* cubical complexes, i.e., cubical complexes such that every maximal face has the same dimension. Let $L$ be a pure cubical complex of rank $d$ (or dimension $d-1$). Adin's *cubical $h$-vector* $h(L) = (h_0(L), h_1(L), \ldots, h_d(L))$ may be defined as follows. Let $s$ be a vertex (element covering $\hat{0}$) of $L$. The subposet $\{t \in L : t \geq s\}$ is the face poset of a simplicial complex $\Delta_s = \mathrm{link}(s)$. Let $h(\Delta_s, x) = \sum_{i=0}^{d-1} h_i(\Delta_s)x^i$, where $(h_0(\Delta_s), \ldots, h_{d-1}(\Delta_s))$ is the usual

$h$-vector of $\Delta_s$. We can now define $h(L)$ by the equation

$$\sum_{i=0}^{d} h_i(L)x^i = \frac{1}{1+x}\left(2^{d-1} + x\sum_s h(\Delta_s, x) + (-2)^{d-1}\widetilde{\chi}(L)x^{d+1}\right), \qquad (1)$$

where $s$ ranges over all vertices of $L$ and

$$\widetilde{\chi}(L) = \sum_{t\in L}(-1)^{\mathrm{rank}(t)-1},$$

the reduced Euler characteristic of $L$. It is not difficult to see that the right-hand side of equation (1) is indeed a polynomial in $x$.

**Problem 8.** (a) *Let $L$ be a pure cubical complex of rank $d$. If $L$ is a Cohen-Macaulay poset, then is $h_i(L) \geq 0$ for all $i$?*

(b) *If $L$ is in addition a Gorenstein* poset, then is it true that $h_0(L) \leq h_1(L) \leq \cdots \leq h_{\lfloor d/2\rfloor}(L)$? (Adin [1, §3] shows that $h_i = h_{d-i}$.)*

Problem 8(a) was raised by Adin and solved by him when $L$ is shellable [1, §5], while Problem 8(b) was raised by Adin in the special case that $L$ is the face poset of the boundary of a cubical polytope [1, §5, Question 2]. It was shown in [3] that an affirmative answer to Problem 8(b) would be best possible, i.e., would give the tightest possible set of linear inequalities for the Adin $h$-vector of a cubical Gorenstein* poset (or even a cubical sphere).

# 3 Representation theory and symmetric functions.

The theory of symmetric functions is rife with positivity results and problems, stemming from the possibility of expanding a symmetric function in terms of a number of possible bases. If the coefficients in such an expansion are real numbers (respectively, polynomials with real coefficients), then we can ask whether they are nonnegative (respectively, have nonnegative coefficients).

Often these coefficients will have a representation-theoretic interpretation, such as the multiplicity of an irreducible representation within some larger representation. Sometimes the only known proof of positivity will be such an interpretation, and the problem will be to find a combinatorial proof. Occasionally the reverse situation will hold. Finally it may happen that positivity is only a conjecture, and we can try to find either an algebraic or combinatorial proof. We will assume for the remainder of this section a basic knowledge of the theory of symmetric functions as developed in [65, Ch. 1] or [89, Ch. 7]. In particular, we will be dealing with the following bases, indexed by partitions $\lambda$ of $n$ (denoted $\lambda \vdash n$), for the $\mathbb{Q}$-vector space $\Lambda^n_{\mathbb{Q}}$ of homogeneous symmetric functions of degree $n$ in the variables $x = (x_1, x_2, \ldots)$:

- $m_\lambda$: monomial symmetric functions
- $h_\lambda$: complete symmetric functions
- $e_\lambda$: elementary symmetric functions
- $p_\lambda$: power sum symmetric functions
- $s_\lambda$: Schur functions.

If $\{u_\lambda\}$ is a basis for the space $\Lambda_{\mathbb{Q}}$ of symmetric functions, then we say that $f \in \Lambda$ is *u-positive* if the expansion of $f$ as a linear combination of $u_\lambda$'s has nonnegative coefficients. If $\lambda$ is a partition with a single part $n$, then we write $u_n$ for $u_\lambda$.

The archetypal example of a successful positivity result in the theory of symmetric functions is the Littlewood-Richardson rule [65, §I.9][89, §A1.3] for multiplying Schur functions. The *Littlewood-Richardson coefficient* $c^\lambda_{\mu\nu}$ is defined by the expansion

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

The representation-theoretic interpretations of $s_\lambda$ as an irreducible character of $\mathrm{GL}(n, \mathbb{C})$ or as the (Frobenius) characteristic of an irreducible character of the symmetric group $S_n$ make it clear that $c^\lambda_{\mu\nu} \geq 0$. The celebrated *Littlewood-Richardson rule* gives a combinatorial interpretation of $c^\lambda_{\mu\nu}$, thereby giving a combinatorial proof of nonnegativity.

There are some variations of the multiplication of Schur functions that are not as well understood. The three most well-known are the following.

- Let $f[g]$ denote the plethysm [65, §1.8][89, Def. A.2.6] (denoted $f \circ g$ in [65]) of the symmetric functions $f$ and $g$. A simple representation-theoretic argument shows that the symmetric function $s_m[s_n]$ (or more generally $s_\lambda[s_\mu]$) is $s$-positive, namely, if $S^k$ denotes the $k$th symmetric power, then the coefficient $\langle s_m[s_n], s_\lambda\rangle$ of $s_\lambda$ in $s_m[s_n]$ (when expanded in terms of Schur functions) is equal to the multiplicity of the irreducible character $s_\lambda$ of $\mathrm{GL}(V)$ (where $V$ is a complex vector space of sufficiently large dimension) in $S^n(S^m V)$.

**Problem 9.** *Find a combinatorial interpretation of the "plethysm coefficients" $\langle s_m[s_n], s_\lambda\rangle$, thereby combinatorially reproving that they are nonnegative.*

A minor but still interesting conjecture related to the plethysm $s_m[s_n]$ is the *Foulkes plethysm conjecture* [33, p. 206]: if $n \geq m$ then $s_n[s_m] - s_m[s_n]$ is $s$-positive. The strongest results on this conjecture are due to Brion [23].

The Foulkes plethysm conjecture can be generalized as follows. Let $X$ be an $r$-element set, and let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of $r$ into $l$ parts (so $\lambda_1 \geq \cdots \geq \lambda_l > 0$ and $\sum \lambda_i = r$). Let $\Pi_\lambda$ denote the set of all partitions $\pi = \{B_1, \dots, B_l\}$ of $X$ whose block sizes are $\lambda_1, \dots, \lambda_l$. A partition $\sigma = \{B'_1, \dots, B'_m\}$ of $X$ is *orthogonal* to $\pi$ (denoted $\sigma \perp \pi$) if the block sizes of $\sigma$ are $\lambda'_1, \dots, \lambda'_m$ where $(\lambda'_1, \dots, \lambda'_m)$ is the conjugate partition to $\lambda$, and if $\#(B_i \cap B'_j) \leq 1$ for all $i, j$. For any set $S$, let $\mathbb{Q}S$ be the $\mathbb{Q}$-vector space with basis $S$. Define a linear transformation $\varphi_\lambda : \mathbb{Q}\Pi_\lambda \to \mathbb{Q}\Pi_{\lambda'}$ by

$$\varphi_\lambda(\pi) = \sum_{\sigma \perp \pi} \sigma, \quad \pi \in \Pi_\lambda.$$

**Conjecture.** *If $\lambda \geq \lambda'$ in dominance order (i.e., $\lambda_1 + \cdots + \lambda_i \geq \lambda'_1 + \cdots + \lambda'_i$ for all $i$), then $\varphi_\lambda$ is injective.*

Using the well-known connection between plethysm and the representation theory of the symmetric group [65, p. 158][89, Thm. A2.8], the Foulkes plethysm conjecture follows from the case where $\lambda$ has $m$ parts equal to $n$ [18]. Moreover, it can be shown that the above conjecture is true when $\lambda$ is the "hook" with one part equal to $k$ and $j$ parts equal to 1, with $k \geq j+1$.

A further intriguing conjecture concerning the positivity of a plethysm is due to S. Sundaram [93, Conj. 2.7][94, Conj. 2.2]. She defines an action of $S_n$ on a certain homology group (the top homology of the poset of all partitions of $\{1, 2, \ldots, n\}$ with an even number of blocks, with $\hat{0}$ removed when $n$ is even). Let $R_n$ denote the Frobenius characteristic of this action. Sundaram shows [93, Thm. 2.1] that $R_n$ is determined by the plethystic recurrence

$$\sum_{n\geq 0}(-1)^n R_{2n+1} = (h_1 - R_2 + R_4 - \cdots)\,[h_1 + h_2 + \cdots]\,,$$

and she shows [93, Thm. 2.5] that $R_{2n}$ is a polynomial in the symmetric functions $h_1$ and $h_2$. Sundaram's conjecture is that when $R_{2n}$ is written as a polynomial in $h_1$ and $h_2$, the coefficients are nonnegative. In other words, $R_{2n}$ is $h$-positive.

- Let $\chi^\lambda$ and $\chi^\mu$ be the irreducible characters of $S_n$ indexed by the partitions $\lambda$ and $\mu$ of $n$. The *Kronecker product* $\chi^\lambda\chi^\mu$ is defined by $(\chi^\lambda\chi^\mu)(w) = \chi^\lambda(w)\chi^\mu(w)$. Let

$$\chi^\lambda\chi^\mu = \sum_\nu g_{\lambda\mu\nu}\chi^\nu, \tag{2}$$

the decomposition of $\chi^\lambda\chi^\mu$ in terms of irreducible characters $\chi^\nu$. If $M^\lambda$ is an $S_n$-module with character $\chi^\lambda$, then the natural action of $S_n$ on $M^\lambda \otimes M^\mu$ has character $\chi^\lambda\chi^\mu$. It follows that $g_{\lambda\mu\nu} \geq 0$.

We can define $g_{\lambda\mu\nu}$ purely in terms of symmetric functions by

$$\frac{1}{\prod_{i,j,k}(1 - x_i y_j z_k)} = \sum_{\lambda,\mu,\nu} g_{\lambda\mu\nu} s_\lambda(x) s_\mu(y) s_\nu(z). \tag{3}$$

The product $*$ on symmetric functions defined by

$$s_\lambda * s_\mu = \sum_\nu g_{\lambda\mu\nu} s_\nu \tag{4}$$

is called the *internal product*. For further information see [65, pp. 115–116][89, Exer. 7.78–7.87].

**Problem 10.** *Find a combinatorial interpretation of the "Kronecker product coefficients" $g_{\lambda\mu\nu}$, thereby combinatorially reproving that they are nonnegative.*

For Problem 10 we should take equation (3) as the definition of $g_{\lambda\mu\nu}$, or equivalently equation (2) with $\chi^\lambda$ defined combinatorially by the Murnaghan-Nakayama rule [65, Exam. I.7.5][89, §7.17], so that $g_{\lambda\mu\nu}$ is not *a priori* nonnegative. For some work related to Problem 10, see [96] and the references given there.

- Let $w$ be a permutation of $\mathbb{P} = \{1, 2, \ldots\}$ that fixes all but finitely many elements of $\mathbb{P}$. Let $\mathfrak{S}_w$ denote the *Schubert polynomial* indexed by $w$ [64]. The Schubert polynomials $\mathfrak{S}_w$ form a $\mathbb{Z}$-basis for the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$. If $w$ fixes $n+1, n+2, \ldots$ (which we denote as $w \in S_n$) then in fact $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$. If $e_i$ denotes the $i$th elementary symmetric function in the variables $x_1, \ldots, x_n$, then the quotient ring $R_n = \mathbb{Z}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$ is isomorphic to the cohomology ring $H^*(X; \mathbb{Z})$ of the flag variety $X = \mathrm{GL}(n, \mathbb{C})/B$. Under this isomorphism the images of the Schubert polynomials $\mathfrak{S}_w$ for $w \in S_n$ form a $\mathbb{Z}$-basis which correspond to the closed Schubert cells in $X$. It then follows from basic intersection theory [34] that the intersection coefficients $c_{uv}^w$ defined by

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w$$

are nonnegative.

**Problem 11.** *Find a combinatorial interpretation of the "Schubert intersection coefficients" $c_{uv}^w$, thereby combinatorially reproving that they are nonnegative.*

For some work on Problem 11, see [8]. A "quantum generalization" of Problem 11 is due to Fomin and Kirillov [29, Conj. 11][30, Conj. 8.1].

A further interesting positivity problem whose only known solution uses representation theory concerns the action of $S_n$ on itself by conjugation, i.e., $w \in S_n$ acts on $S_n$ by $w \cdot u = wuw^{-1}$. Let $G$ be any finite group, and let $\psi_G$ denote the character of the action of $G$ on itself by conjugation. It is well-known (e.g., [89, Exer. 7.71]) that the multiplicity of an irreducible character $\chi$ of $G$ in $\psi_G$ is given by

$$\langle \psi_G, \chi \rangle = \sum_K \chi(K),$$

where $K$ ranges over all conjugacy classes of $G$, and where $\chi(K)$ denotes $\chi(w)$ for any $w \in K$. Hence the "row sums" $\sum_K \chi(K)$ of the character table of $G$ are nonnegative.

**Problem 12.** *Give a combinatorial interpretation of the row sums of the character table of $S_n$, thereby combinatorially reproving that they are nonnegative.*

For further information on this problem, see [89, Exer. 7.71]. For the column sums of the character table of $S_n$, see [65, Exam. 11, p. 120][89, Exer. 7.69(b)].

Let us turn to some positivity problems involving symmetric functions for which *no* proof is known. Undoubtedly the most important such problem concerns the $(q,t)$-Kostka polynomials $K_{\lambda\mu}(q,t)$. The definition of these polynomials is at first sight rather obscure and will be omitted here. For readers familiar with the Macdonald symmetric functions $P_\mu(X;q,t)$ and plethystic notation, they can be defined by

$$J_\mu(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda[X(1-t)],$$

where $J_\mu$ is a certain normalization of $P_\mu$ known as the *integral form.* For further details see [65, Ch. VI]. A priori the above definition of $K_{\lambda\mu}(q,t)$ only shows that they are rational functions of $q$ and $t$. It was not until recently that several separate proofs [39][56][57][59][70] were given that $K_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t]$.

**Problem 13.** *Show that the $(q,t)$-Kostka polynomial $K_{\lambda\mu}(q,t)$ has nonnegative coefficients.*

It follows readily from the definition of $K_{\lambda\mu}(q,t)$ that $K_{\lambda\mu}(1,1) = f^\lambda$, the number of standard Young tableaux of shape $\lambda$ (see [65, Ch. VI, (8.16)]). Hence the coefficients of $K_{\lambda\mu}(q,t)$ should count the number of standard Young tableaux of shape $\lambda$ with some property, but not even a conjecture is known about what this property might be. When $q = 0$, we have that $K_{\lambda\mu}(0,t)$ is the "ordinary" Kostka polynomial introduced by Foulkes (see [65, Ch. VI, (8.12)]). Here the coefficients have a combinatorial meaning discovered by Lascoux and Schützenberger [61][65, p. 242].

One reason why Problem 13 is so intriguing is that it has led to surprising properties of the Macdonald symmetric functions and to deep connections with representation theory and algebraic geometry. For some remarkable positivity conjectures related to Problem 13, see [7][37][38][39]. Garsia-Haiman [36] have a simple conjectured representation-theoretical interpretation of the coefficients of $K_{\lambda\mu}(q,t)$ as the dimensions of certain vector spaces. Haiman [49] has given many equivalent forms of this conjecture and has shown (based on a suggestion of C. Procesi) that it is intimately connected with the Hilbert scheme of points in the plane and with the variety of commuting matrices.

A special case of Problem 13 is of particular interest. Let $*$ denote the internal product of symmetric functions, as defined in equation (4). It is known (see [89, Exer. 7.86]) that

$$s_\lambda * s_\mu(1, q, q^2, \ldots) = \frac{T_{\lambda\mu}(q)}{H_\lambda(q)},$$

where $T_{\lambda\mu}(q) \in \mathbb{Z}[q]$, $T_{\lambda\mu}(1) = f^\mu$, and $H_\lambda(q) = \prod_{u\in\lambda} \left(1 - q^{h(u)}\right)$. Here $u$ ranges over all squares of (the diagram of) $\lambda$ and $h(u)$ is the hook length of $\lambda$ at $u$. It was conjectured by R. Brylinski and R. Stanley (see [76, Conj. 8.3]) that $T_{\lambda\mu}(q)$ has nonnegative coefficients. It is shown in [65, Exam. VI.8.3, pp. 362–363] that

$$T_{\lambda\mu}(q) = K_{\mu\lambda}(q,q).$$

Hence the conjecture of Brylinski-Stanley follows from a positive answer to Problem 13. For some algebraic aspects of the Brylinski-Stanley conjecture, see [24][25].

The Macdonald symmetric functions $P_\lambda(x;q,t)$ may be regarded as a generalization of the Hall-Littlewood symmetric functions $P_\lambda(x;0,t)$ [65, p. 324]. A different kind of generalization of the Hall-Littlewood symmetric functions is afforded by the "ribbon polynomials" $H^{(k)}_{\lambda/\mu}(x;q)$ of Lascoux, Leclerc, and Thibon [60]. They are defined when $\lambda/\mu$ is a skew shape that admits a tiling by $k$-ribbons (or $k$-border strips) by

$$H^{(k)}_{\lambda/\mu}(x;q) = \sum_T q^{s(T)} x^T,$$

summed over all ribbon tableaux $T$ (also called border strip tableaux), as defined e.g. in [89, p. 346], of shape $\lambda/\mu$ with all ribbons (border strips) of size $k$. Here $s(T)$ denotes the *spin* of $T$, defined by

$$s(T) = \sum_R \frac{h(R)-1}{2},$$

summed over all ribbons $R$ appearing in $T$, where $h(R)$ is the height (number of rows) of $R$. When $\mu = \emptyset$ then $\lambda$ has a $k$-quotient (as defined e.g. in [89, p. 517]) $(\mu^0, \ldots, \mu^{k-1})$. The coefficients of $H^{(k)}_\lambda(q)$ are then $q$-analogues of the Littlewood-Richardson coefficients obtained from the product $s_{\mu^0} \cdots s_{\mu^{k-1}}$. Lascoux *et al.* give a number of fascinating theorems and conjectures about $H^{(k)}_{\lambda/\mu}(x;q)$ and related polynomials. These theorems and (subsequently) conjectures have for the most part been proved using deep results from representation theory (in particular, quantum affine algebras). In particular, $H^{(k)}_\lambda(x;q)$ is a symmetric function [60, Thm. VI.1]. It would nevertheless be desirable to have a combinatorial proof of this fundamental result. It is also shown in [60] that the coefficients of $H^{(k)}_\lambda(q)$ (suitable normalized) are certain parabolic Kazhdan-Lusztig polynomials of affine type $A$. One of the conjectures of Lascoux *et al.* (Conjecture VI.3) asserts that when $H^{(k)}_\lambda(x;q)$ is expanded in terms of Schur functions, the coefficients are polynomials in $q$ with *nonnegative* integer coefficients. In [62, p. 2], Leclerc and Thibon mention that this conjecture would follow from some results in representation theory that experts think are probable (an analogue of the positivity of the coefficients of Kahzdan-Lusztig polynomials for affine Weyl groups). Positivity is proven combinatorially in the case $k = 2$ by Carré and Leclerc [26].

**Problem 14.** (a) *Give a combinatorial proof that the polynomials* $H_\lambda^{(k)}(x;q)$ *are symmetric functions.*

(b) *When* $H_\lambda^{(k)}(x;q)$ *is expanded in terms of Schur functions, are the coefficients polynomials in* $q$ *with nonnegative integer coefficients?*

No discussion of positivity problems would be complete without mention of the Kazhdan-Lusztig polynomials $P_{u,v}(q)$ associated with a Coxeter group $W$ and a pair $u \leq v$ of elements in the Bruhat order of $W$. For a short exposition of the basic properties of these polynomials, see [51, Ch. 7]. When $W$ is a finite or affine Weyl group then a deep result of Kazhdan and Lusztig [55] shows that the coefficients of $P_{u,v}(q)$ are dimensions of certain intersection homology spaces and are therefore nonnegative. (A further class of Coxeter groups for which $P_{u,v}(q)$ has nonnegative coefficients was considered by Haddad [45].) For the remaining finite Coxeter groups (types $H_3$, $H_4$, and $I_p$) it has been checked that $P_{u,v}(q)$ always has nonnegative coefficients. Kazhdan and Lusztig [54, p. 166] conjectured an affirmative answer to the following problem.

**Problem 15.** *Are the coefficients of* $P_{u,v}(q)$ *nonnegative for any Coxeter group* $W$ *and any* $u \leq v$ *in the Bruhat order of* $W$*?*

It was shown by M. Dyer (unpublished) and H. Tagawa [95] that the coefficient of $q$ in $P_{u,v}(q)$ is nonnegative. Even though the answer to Problem 15 is known to be positive for finite and affine Weyl groups, we can still ask for a combinatorial proof avoiding intersection homology theory. This problem is especially interesting for the symmetric group $S_n$ since it is the Weyl group with the simplest and most tractable combinatorial properties.

**Problem 16.** *Give a combinatorial interpretation of the coefficients of* $P_{u,v}(q)$ *when* $W$ *is a finite or affine Weyl group, especially* $W = S_n$*, thereby combinatorially reproving that they are nonnegative.*

For some work related to Problem 16, see [20][22] and the references given there. There is a formal similarity between the previous problem and

Problem 2, stemming from the connection of both problems with intersection homology. For further details, see [83, Part II].

A host of open problems deal with the connection between symmetric functions and immanants. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix (with entries in some commutative ring $R$), and let $f : S_n \to R$. Define the *$f$-immanant* of $A$ by

$$\mathrm{Imm}_f(A) = \sum_{w \in S_n} f(w) a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)}.$$

Hence if $f(w) = \mathrm{sgn}(w)$ then $\mathrm{Imm}_f(A) = \det A$, while if $f(w) = 1$ then $\mathrm{Imm}_f(A) = \mathrm{per}\, A$, the *permanent* of $A$. Immanants were probably first considered by Schur [71] when $f$ is an irreducible character of $S_n$, although the term "immanant" was coined by D. E. Littlewood. A paper of Goulden and Jackson [43] initiated a flurry of activity related to combinatorial properties of immanants and has led to many conjectures. We do not have the space to discuss all these conjectures here but will give a few typical ones.

If $f$ is a function on $S_n$ (usually assumed to be a class function), then we define the *characteristic* $\mathrm{ch}\, f$ of $f$ [65, p. 113][89, §7.18] by

$$\mathrm{ch}\, f = \frac{1}{n!} \sum_{w \in S_n} f(w) p_{\rho(w)},$$

where $p_{\rho(w)}$ is the power sum symmetric function indexed by the cycle type $\rho(w)$ of $w$. If $f = \chi^\lambda$, the irreducible character of $S_n$ indexed by the partition $\lambda$ of $n$, then $\mathrm{ch}\, \chi^\lambda = s_\lambda$ (a Schur function). We abbreviate $\mathrm{Imm}_{\chi^\lambda}$ by $\mathrm{Imm}_\lambda$. Let $\phi^\lambda$ denote the unique class function on $S_n$ for which $\mathrm{ch}\, \phi^\lambda = m_\lambda$ (a monomial symmetric function). Define a real matrix $A$ to be *totally nonnegative* (sometimes called *totally positive*) if every minor of $A$ is nonnegative. Stembridge [91] showed that if $A$ is a totally nonnegative $n \times n$ matrix and $\lambda \vdash n$, then $\mathrm{Imm}_\lambda(A) \geq 0$. (A different proof was later given by Kostant [58].) Stembridge conjectured the even stronger result that the answer to the following problem is affirmative.

**Problem 17.** *If $A$ is a totally nonnegative $n \times n$ matrix and $\lambda \vdash n$, then is $\mathrm{Imm}_{\phi^\lambda}(A) \geq 0$?*

Problem 17 has the following equivalent formulation (see [89, Exer. 7.92]).

**Problem 17′.** *Let $A = (a_{ij})$ be a totally nonnegative $n \times n$ real matrix. Define a symmetric function*

$$F_A = \sum_{w \in S_n} a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)} p_{\rho(w)}.$$

*Is $F_A$ h-positive, where $\{h_\lambda\}$ is the basis of complete symmetric functions?*

Now let $\mu$ and $\nu$ be partitions where $\nu \subseteq \mu$, with $\ell(\mu) \leq n$. Here $\ell(\mu)$ denotes the length (number of nonzero parts) of $\mu$. Define the *Jacobi-Trudi matrix*

$$H_{\mu/\nu} = \left[h_{\mu_i - \nu_j + j - i}\right]_{i,j=1}^{n}.$$

The *Jacobi-Trudi identity* [65, Ch. I, (3.4)][89, §7.16] states that $\det H_{\mu/\nu} = s_{\mu/\nu}$, a skew Schur function. Goulden and Jackson [43] were the first to consider other immanants of $H_{\mu/\nu}$. They conjectured that $\mathrm{Imm}_\lambda(H_{\mu/\nu})$ is $m$-positive (where $\{m_\lambda\}$ is the basis of monomial symmetric functions), which was proved by Greene [44]. Stembridge [92] made a series of conjectured strengthenings of this result. His Conjecture 4.2(a) asserts that $\mathrm{Imm}_\lambda(H_{\mu/\nu})$ is $s$-positive. A remarkable proof was given by Haiman [48] based on the theory of Kazhdan-Lusztig polynomials. Stembridge's strongest conjecture (Conjecture 4.1) asserts that the answer to the following problem is affirmative.

**Problem 18.** *If $\ell(\mu) \leq n$, $\nu \subseteq \mu$, and $\lambda \vdash n$, then is $\mathrm{Imm}_{\phi^\lambda}(H_{\mu/\nu})$ s-postive?*

Haiman has given in his paper mentioned above some intriguing conjectures relating the virtual characters $\phi^\lambda$ to the Kazhdan-Lusztig basis $C'_w$ of the Hecke algebra $H_n(q)$ (of type $A_{n-1}$). We assume knowledge of Kazhdan-Lusztig theory in order to state Haiman's conjecture. Since the irreducible characters of $H_n(q)$ correspond to irreducible characters of $S_n$, it follows that the monomial (virtual) characters $\phi^\lambda$ of $S_n$ have unique analogues for $H_n(q)$. Haiman then conjectures an affirmative answer to the following question.

**Problem 19.** *For every monomial character $\phi^\lambda$ of $H_n(q)$ and every Kazhdan-Lusztig basis element $C'_w$, is it true that $\phi^\lambda(q^{\ell(w)/2} C'_w)$ is a polynomial with nonnegative integer coefficients, and moreover that these coefficients are unimodal and symmetric about $q^{\ell(w)/2}$?*

Haiman also gives in his paper some interesting refinements of the above problem.

# 4 Real zeros and total positivity.

There are a number of open problems in algebraic combinatorics concerning whether certain polynomials have (only) real zeros. The connection between real zeros and positivity is given by the theory of *total positivity*, in particular the following fundamental result of Aissen, Schoenberg, and Whitney [2]. (A famous conjectured generalization due to Schoenberg was proved independently by Edrei and Thoma, and more recently Olshanskii and Okounkov. See [89, Exer. 7.91] for further discussion and references.)

**Theorem 1.** *Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Set $a_i = 0$ if $i < 0$ or $i > n$. Then every zero of the polynomial $P(x) = a_0 + a_1x + \cdots + a_nx^n$ is a nonpositive real number if and only if the infinite Toeplitz matrix $A = (a_{j-i})_{i,j\geq 0}$ is totally nonnegative.*

NOTE. The above theorem gives *infinitely* many conditions that have to be checked in order for $P(x)$ to have only real zeros. Even for quadratic polynomials it does not suffice to check some finite subset of the minors of $A$. Nevertheless this theorem is a useful tool in showing that certain polynomials have only real zeros. Sometimes, for instance, it is possible to interpret the necessary determinants combinatorially by using the Gessel-Viennot non-intersecting lattice path method [42][81, §2.7]. See for instance Theorem 3 below. Gantmacher [35, Cor. on p. 203 of Vol. 2] was the first to explicitly state a set of $n-1$ inequalities among the coefficients of a real polynomial $P(x)$ of degree $n$ that are necessary and sufficient for every zero of $P(x)$ to be real. However, Gantmacher's conditions have not (yet) been applied to any combinatorially defined polynomials.

The above theorem has an interesting formulation in terms of symmetric functions. This result was first explicitly stated in [88, Thm. 2.11], but it is easily seen to be equivalent to the Aissen-Schoenberg-Whitney theorem.

**Theorem 2.** *Let $P(t) \in \mathbb{R}[t]$ with $P(0) = 1$. Let $F_P(x) = \prod_i P(x_i)$, a symmetric formal power series in the variables $x_i$. The following three conditions are equivalent:*

*(a) Every zero of $P(t)$ is real and negative.*

*(b) $F_P(x)$ is s-positive.*

*(c) $F_P(x)$ is e-positive, where $\{e_\lambda\}$ is the basis of elementary symmetric functions.*

The previous theorem opens up the possibility of giving combinatorial proofs that certain polynomials $P(t)$ (normalized so $P(0) = 1$) have real zeros. Namely, one can try to find a combinatorial interpretation of the coefficients in the expansion of $F_P(x)$ in terms of Schur functions or elementary symmetric functions, thereby combinatorially proving that these coefficients are nonnegative. We will mention below (see Theorem 4) a situation for which this technique is successful, but in general it seems difficult to apply. For instance, it is well-known (e.g., [72]) that the Eulerian polynomial $A_n(t)$ (defined below) has real zeros, but a combinatorial proof along the lines just mentioned is not known. (We should work with $P_n(t) = A_n(t)/t$, so $P_n(0) = 1$.)

An intriguing conjecture concerning real zeros is known as the *Poset Conjecture*. Let $P$ be a partial ordering of $1, 2, \ldots, n$, with the order relation denoted $\overset{P}{\le}$. We say that $P$ is *natural* if $i < j$ (as integers) whenever $i \overset{P}{<} j$. Let $\mathcal{L}(P)$ denote the set of all permutations $a_1 \cdots a_n$ of $1, \ldots, n$ such that $i < j$ if $a_i \overset{P}{<} a_j$. Such permutations are in an obvious one-to-one correspondence with the *linear extensions* of $P$. For a permutation $w = a_1 \cdots a_n$ of $1, \ldots, n$, define the *number of descents* $d(w)$ of $w$ by

$$d(w) = \#\{i \,:\, a_i > a_{i+1},\ 1 \le i \le n-1\}.$$

Set

$$W(P, x) = \sum_{w \in \mathcal{L}(P)} x^{d(w)}.$$

The polynomial $W(P, x)$ plays an important role in the combinatorics of $P$, and many special cases have been considered independently. For instance, if $P$ is an antichain (i.e., no two distinct elements are comparable) then $\mathcal{L}_P = S_n$ and $xW(P, x)$ is the *Eulerian polynomial* $A_n(x)$. For further information, see e.g. [74][81, §4.5].

The following problem (stated in a somewhat different form) was conjectured to have an affirmative answer by J. Neggers [69, p. 114] for natural posets, and was extended to arbitrary posets by Stanley. For some work on this conjecture, see [19][41][72][97]. It has been verified by Stembridge for $\#P \leq 8$.

**Problem 20.** *Is every zero of $W(P, x)$ real?*

When $P$ is natural it can be shown [19, Conj. 3] that Problem 20 is equivalent to the following. Let $L$ be a finite distributive lattice, i.e., a finite lattice $L$ whose lattice operations $\wedge$ and $\vee$ satisfy the distributive laws

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Equivalently , $L$ is isomorphic to a finite collection of sets, ordered by inclusion, and closed under the operations of union and intersection (see e.g. [81, Thm. 3.4.1]). For any finite poset $P$, let $c_k = c_k(P)$ denote the number of $k$-element chains in $P$ (so in particular $c_0 = 1$, corresponding to the empty chain, and $c_1 = \#P$). Hence the sequence $(c_1, c_2, \ldots)$ is just the $f$-vector of the order complex of $P$. Define the *chain polynomial* $C(P, t) = \sum_k c_k t^k$.

**Problem 20′.** *When $L$ is a distributive lattice, is every zero of $C(L, t)$ real?*

The zeros of $C(P, t)$ are of interest for posets other than distributive lattices. For instance, it is not known whether $C(L, t)$ has only real zeros when $L$ is a *modular* lattice.

There is a class of posets $P$ for which $C(P, t)$ can be shown to have real zeros using Theorem 2. Define a poset $P$ to be $(3 + 1)$-*free* if $P$ contains no induced subset isomorphic to the disjoint union of a three-element chain and a one-element chain. (A subposet $Q$ of $P$ is *induced* if it is obtained by choosing some subset of the elements of $P$ and all relations on these elements that hold in $P$. An $n$-element poset has $2^n$ induced subposets.) The following result appears in [88, Cor. 2.9]. A proof avoiding symmetric functions was later found by M. Skandera [73].

**Theorem 3.** *Let $P$ be a (finite) $(3+1)$-free poset. Then all zeros of $C(P,t)$ are real.*

The proof consists of applying Theorem 2 to a result of Gasharov [40]. Gasharov's result easily implies a combinatorial interpretation of the coefficients of the Schur function expansion of $F_{C(P)}(x) = \prod_i C(P, x_i)$, thereby establishing their nonnegativity. Gasharov's result suggests two further open problems, which we now discuss. Let $G$ be a graph on the vertex set $V = \{v_1, \ldots, v_n\}$, with no loops or multiple edges. A *proper coloring* of $G$ is a map $\kappa : V \to \{1, 2, \ldots\}$ such that $\kappa(u) \neq \kappa(v)$ whenever $uv$ is an edge of $G$. Think of $\kappa(v_j)$ as the "color" of vertex $v_j$. Define the *chromatic symmetric function* $X_G(x) = X_G(x_1, x_2, \ldots)$ of $G$ by

$$X_G(x) = \sum_{\kappa} x_1^{a_1(\kappa)} x_2^{a_2(\kappa)} \cdots,$$

summed over all proper colorings $\kappa$ of $G$, where $a_i(\kappa) = \#\kappa^{-1}(i)$, the number of vertices of $G$ colored $i$. Clearly $X_G(x)$ is a homogeneous symmetric function of degree $n$ in the variables $x_1, x_2, \ldots$. Its basic properties are developed in [80][88]. If $P$ is a finite poset, then let $\mathrm{inc}(P)$ denote its *incomparability graph*, i.e., the vertices of $\mathrm{inc}(P)$ are the elements of $P$, and $uv$ is an edge of $\mathrm{inc}(P)$ if and only if $u$ and $v$ are incomparable in $P$. The theorem of Gasharov mentioned above is the following.

**Theorem 4.** *If $P$ is a $(3+1)$-free poset and $G = \mathrm{inc}(P)$, then $X_G$ is s-positive.*

The following two problems are both strengthenings of Gasharov's theorem. The first strengthens the conclusion, while the second weakens the hypothesis. The first problem is due (in an equivalent form) to Stanley and

Stembridge [90, Conj. 5.5] and is related to Problems 18 and 19, while the second is due to Gasharov and is stated in [88, Conj. 1.4].

**Problem 21.** *If $P$ is a $(3+1)$-free poset and $G = \mathrm{inc}(P)$, then is $X_G$ e-positive?*

**Problem 22.** *Let $G$ be a finite* clawfree *graph, i.e., $G$ has no induced subgraph consisting of one vertex connected to three other vertices (and no further edges). Is $X_G$ s-positive?*

A special class of $(3+1)$-free posets for which Problem 21 is still open is the *semiorders* or *unit interval orders*, defined e.g. in [89, Exer. 6.30]. It is not hard to deduce from [90] that an affirmative answer to Problem 17 implies that Problem 21 has an affirmative answer for semiorders. Even the following very special case of Problem 21 is open (see [80, pp. 190–191]). Define

$$F_n = \sum_{i_1,\ldots,i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $i_1, \ldots, i_n$ ranges over all sequences of positive integers of length $n$ for which any three consecutive terms are distinct. Is $F_n$ $e$-positive?

Now let $G$ be a graph and $c_k$ the number of stable (or independent) $k$-element subsets $S$ of vertices, i.e., no two vertices in $S$ are adjacent in $G$. The *stable set polynomial* of $G$ is defined by $D(G,t) = \sum_k c_k t^k$. In particular, if $P$ is a poset then $D(\mathrm{inc}(P), t) = C(P,t)$. Just as Theorems 2 and 4 imply Theorem 3, in exactly the same way an affirmative answer to Problem 22 implies an affirmative answer to the following problem, which was first raised by Hamidoune [50, p. 242].

**Problem 23.** *Does the stable set polynomial $D(G,t)$ of a clawfree graph $G$ have only real zeros?*

There is one further class of problems we wish to mention concerning polynomials with real zeros. These problems concern some polynomials that arise in the combinatorial subject of *rook theory*. We will simply state the most central of these problems, which is due to Haglund, Ono, and Wagner [47]. For further problems of this nature, see [46]. Recall from the previous section that the *permanent* of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{w \in S_n} a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)}.$$

Let $J$ denote the $n \times n$ matrix of all 1's.

**Problem 24.** *Let $A$ be an $n \times n$ real matrix for which every column is weakly increasing. Does the polynomial $\operatorname{per}(A + xJ)$ have only real zeros?*

There are a number of conditions on a sequence $a_0, a_1, \ldots, a_n$ of real numbers that are weaker than being the coefficients of a polynomial with only real zeros, and that appear throughout combinatorics. The two best known of these conditions are unimodality and logarithmic concavity (or log-concavity for short). The sequence $a_0, a_1, \ldots, a_n$ is *unimodal* if $a_1 \le a_2 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n$ for some $j$, and *log-concave* if $a_i^2 \ge a_{i-1}a_{i+1}$ for $1 \le i \le n-1$. Isaac Newton showed that if the polynomial $\sum a_i x^i$ has only real zeros, then the sequence $a_0, \ldots, a_n$ is log-concave. (Even more strongly, the sequence $a_0/\binom{n}{0}, a_1/\binom{n}{1}, \ldots, a_n/\binom{n}{n}$ is log-concave.) Moreover, it is easy to see that if $a_0, \ldots, a_n$ is a log-concave sequence of positive real numbers, then it is also unimodal. The subject of log-concave and unimodal sequences arising in algebra, combinatorics, and geometry is surveyed in [78], with a sequel in [21]. In particular, a vast number of sequences have been conjectured to be log-concave or unimodal. We will state here only a few of the most intriguing such sequences, to give a flavor for this subject.

**Problem 25.** *Are the sequences below unimodal or log-concave?*

(a) *The absolute value of the coefficients of the chromatic polynomial of a graph, or more generally, the characteristic polynomial of a matroid.*

(b) *The number of $i$-edge spanning forests of a graph, or more generally, the number of $i$-element independent sets of a matroid.*

(c) *The number of elements of rank $i$ of a geometric lattice.*

Our own feeling is that these questions have negative answers, but that the counterexamples will be huge and difficult to construct. A similar phenomenon concerns the problem raised by T. S. Motzkin in 1961 and D. Welsh in 1972 whether the $f$-vector $(f_0, f_1, \ldots, f_{d-1})$ of a $d$-dimensional convex polytope is unimodal. The first counterexamples were obtained by Björner [15] and Lee [63] (see also [14, §8]). Subsequently it was shown that the smallest $d$ for which there exists a *simplicial* $d$-polytope with a nonunimodal $f$-vector is $d = 20$; the smallest such example known has $f_0 \approx 4.2 \times 10^{12}$ vertices (see [98, p. 272]). All known examples are far too large to be found by any kind of search; they are instead obtained by the use of the $g$-theorem for simplicial polytopes or by the use of techniques for constructing large polytopes from smaller ones (see [98, Example 8.41]). (Only the sufficiency of the conditions characterizing $f$-vectors of simplicial polytopes is needed here to construct examples of $f$-vectors.) Unfortunately there are no analogous sufficient conditions or construction techniques known for the sequences appearing in Problem 25.

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