Irreducible Symmetric Group Characters of Rectangular Shape

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1 The main result.

The irreducible characters χ^λ of the symmetric group $\mathfrak{S}_n$ are indexed by partitions λ of n (denoted $\lambda \vdash n$ or $|\lambda| = n$), as discussed e.g. in [5, §1.7] or [8, §7.18]. If $w \in \mathfrak{S}_n$ has cycle type $\nu \vdash n$ then we write $\chi^\lambda(\nu)$ for $\chi^\lambda(w)$. If λ has exactly p parts, all equal to q, then we say that λ has *rectangular shape* and write $\lambda = p \times q$. In this paper we give a new formula for the values of the character $\chi^{p\times q}$.

Let μ be a partition of $k \leq n$, and let $(\mu, 1^{n-k})$ be the partition obtained by adding $n-k$ 1's to μ. Thus $(\mu, 1^{n-k}) \vdash n$. Define the *normalized character* $\widehat{\chi}^\lambda(\mu, 1^{n-k})$ by

$$\widehat{\chi}^\lambda(\mu, 1^{n-k}) = \frac{(n)_k \chi^\lambda(\mu, 1^{n-k})}{\chi^\lambda(1^n)},$$

where $\chi^\lambda(1^n)$ denotes the dimension of the character χ^λ and $(n)_k = n(n-1)\cdots(n-k+1)$. Thus [5, (7.6)(ii)][8, p. 349] $\chi^\lambda(1^n)$ is the number f^λ of standard Young tableaux of shape λ. Identify λ with its *diagram* $\{(i,j) : 1 \leq j \leq \lambda_i\}$, and regard the points $(i,j) \in \lambda$ as squares (forming the Young diagram of λ). We write diagrams in "English notation," with the first coordinate increasing from top to bottom and the second coordinate from left to right. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where λ' is the

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conjugate partition to λ. The *hook length* of the square $u = (i, j) \in \lambda$ is defined by

$$h(u) = \lambda_i + \lambda'_j - i - j + 1,$$

and the Frame-Robinson-Thrall *hook length formula* [5, Exam. I.5.2][8, Cor. 7.21.6] states that

$$f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)}.$$

For $w \in \mathfrak{S}_n$ let $\kappa(w)$ denote the number of cycles of w (in the disjoint cycle decomposition of w). The main result of this paper is the following.

Theorem 1. *Let $\mu \vdash k$ and fix a permutation $w_\mu \in \mathfrak{S}_k$ of cycle type μ. Then*

$$\widehat{\chi}^{p\times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{uv=w_\mu} p^{\kappa(u)}(-q)^{\kappa(v)},$$

where the sum ranges over all $k!$ pairs $(u, v) \in \mathfrak{S}_k \times \mathfrak{S}_k$ satisfying $uv = w_\mu$.

The proof of Theorem 1 hinges on a combinatorial identity involving hook lengths and contents. Recall [5, Exam. I.1.3][8, p. 373] that the *content* $c(u)$ of the square $u = (i, j) \in \lambda$ is defined by $c(u) = j - i$. We write $s_\lambda(1^p)$ for the Schur function s_λ evaluated at $x_1 = \cdots = x_p = 1$, $x_i = 0$ for $i > p$. A well known identity [5, Exam. I.3.4][8, Cor. 7.21.4] in the theory of symmetric functions asserts that

$$s_\lambda(1^p) = \prod_{u\in\lambda} \frac{p + c(u)}{h(u)}. \tag{1}$$

Since the right-hand side is a polynomial in p, it makes sense to define

$$s_\lambda(1^{-q}) = \prod_{u\in\lambda} \frac{-q + c(u)}{h(u)}. \tag{2}$$

Equivalently, $s_\lambda(1^{-q}) = (-1)^{|\lambda|} s_{\lambda'}(1^q)$. Regard p and q as fixed, and let $\lambda = (\lambda_1, \ldots, \lambda_p) \subseteq p \times q$ (containment of diagrams). Define the partition $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_p)$ by

$$\tilde{\lambda}_i = q - \lambda_{p+1-i}. \tag{3}$$

Thus the diagram of $\tilde{\lambda}$ is obtained by removing from the bottom-right corner of $p \times q$ the diagram of λ rotated 180°. Write

$$H_\lambda = \prod_{u\in\lambda} h(u),$$

the product of the hook lengths of λ.

Lemma. *With notation as above we have*

$$H_{p\times q} = (-1)^{|\lambda|} H_\lambda H_{\tilde{\lambda}} s_\lambda(1^p) s_\lambda(1^{-q}).$$

Proof. Let $\lambda^\natural$ denote the shape λ rotated 180°. Let $\mathrm{SQ}(\lambda)$ denote the skew shape obtained by removing $\lambda^\natural$ from the lower right-hand corner of $p\times q$ and adjoining $\lambda^\natural$ at the right-hand end of the top edge of $p\times q$ and at the bottom end of the left edge. See Figure 1 for the case $p=4$, $q=6$, and $\lambda=(4,3,1)$. It follows immediately from [6, Thm. 1] that

$$\begin{aligned} H_{\mathrm{SQ}(\lambda)} &= H_{\tilde{\lambda}} \prod_{u\in\lambda}(p+c(u)) \prod_{v\in\lambda'}(q+c(u)) \\ &= (-1)^{|\lambda|} H_{\tilde{\lambda}} \prod_{u\in\lambda}(p+c(u))(-q+c(u)). \end{aligned} \tag{4}$$

It was proved in [1][3][7] that the multiset of hook lengths of the shape $\mathrm{SQ}(\lambda)$ is the union of those of the shapes $p\times q$ and λ, so

$$H_{\mathrm{SQ}(\lambda)} = H_{p\times q} H_\lambda. \tag{5}$$

The proof now follows from equations (1), (2), (4), and (5). $\square$

Proof of Theorem 1. Let $\ell=\ell(\mu)$. We first obtain an expression for $\chi^{p\times q}(\mu, 1^{pq-k})$ using the Murnaghan-Nakayama rule [5, Exam. I.7.5][8, Thm. 7.17.3]. According to this rule,

$$\chi^{p\times q}(\mu, 1^{pq-k}) = \sum_T (-1)^{\mathrm{ht}(T)},$$

where T ranges over all border-strip tableaux $(B_1, B_2, \ldots, B_{\ell+pq-k})$ of shape $p\times q$ and type $(\mu, 1^{n-k})$. Here we are regarding T as a sequence of border strips removed successively from the shape $p\times q$. (See [5] or [8] for further details.) The first ℓ border strips $B_1,\ldots,B_\ell$ will occupy some shape $\lambda\vdash k$, rotated 180°, in the lower right-hand corner of $p\times q$. If we fix this shape λ, then the number of choices for $B_1,\ldots,B_\ell$, weighted by $(-1)^{\mathrm{ht}(B_1)+\cdots+\mathrm{ht}(B_\ell)}$, is by the Murnaghan-Nakayama rule just $\chi^\lambda(\mu)$. The remaining border strips

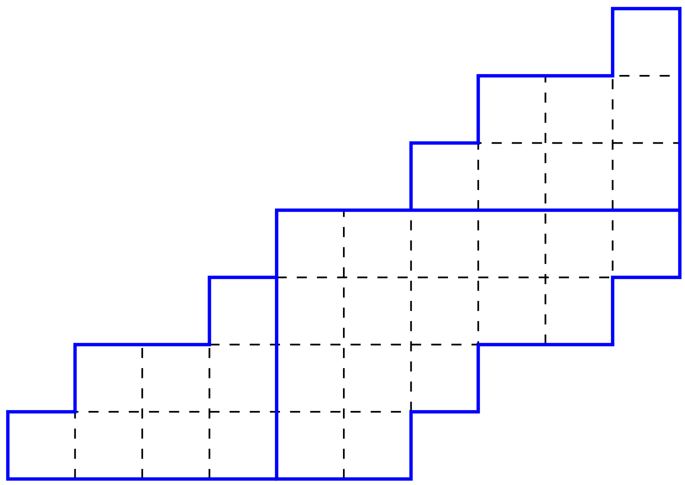

Figure 1: The shape SQ(4, 3, 1) for $p = 4$, $q = 6$

$B_{\ell+1}, \ldots, B_{\ell+pq-k}$ all have one square (and hence height 0) and can be added in $f^{\tilde{\lambda}}$ ways, where $\tilde{\lambda}$ has the same meaning as in (3). Hence

$$\chi^{p\times q}(\mu, 1^{pq-k}) = \sum_{\substack{\lambda\subseteq p\times q\\ \lambda\vdash k}} \chi^{\lambda}(\mu) f^{\tilde{\lambda}},$$

so

$$\begin{aligned} \widehat{\chi}(\mu, 1^{pq-k}) &= \frac{(pq)_k}{f^{p\times q}} \sum_{\substack{\lambda\subseteq p\times q\\ \lambda\vdash k}} \chi^{\lambda}(\mu) f^{\tilde{\lambda}} \\ &= \frac{(pq)_k H_{p\times q}}{(pq)!} \sum_{\substack{\lambda\subseteq p\times q\\ \lambda\vdash k}} \chi^{\lambda}(\mu) \frac{(pq-k)!}{H_{\tilde{\lambda}}} \\ &= H_{p\times q} \sum_{\substack{\lambda\subseteq p\times q\\ \lambda\vdash k}} \chi^{\lambda}(\mu) H_{\tilde{\lambda}}^{-1}. \qquad (6) \end{aligned}$$

Now let $\rho(w)$ denote the cycle type of a permutation $w \in \mathfrak{S}_k$. The following identity appears in [2, Prop. 2.2] and [8, Exer. 7.70]:

$$\sum_{\lambda\vdash k} H_{\lambda} s_{\lambda}(x) s_{\lambda}(y) s_{\lambda}(z) = \frac{1}{k!} \sum_{\substack{uvw=1\\ \text{in } \mathfrak{S}_k}} p_{\rho(u)}(x) p_{\rho(v)}(y) p_{\rho(w)}(z),$$

where $p_\nu(x)$ is a power sum symmetric function in the variables $x = (x_1, x_2, \ldots)$. Set $x = 1^p$, $y = 1^{-q}$, take the scalar product (as defined in [5, §I.4] or [8, §7.9]) of both sides with p_μ, and multiply by $(-1)^k$. Since (in standard symmetric function notation) the number of permutations in $\mathfrak{S}_k$ of cycle type μ is $k!/z_\mu$, and since $\langle p_\mu, p_\mu \rangle = z_\mu$ and $\langle s_\lambda, p_\mu \rangle = \chi^\lambda(\mu)$, we get

$$(-1)^k \sum_{\lambda \vdash k} H_\lambda s_\lambda(1^p) s_\lambda(1^{-q}) \chi^\lambda(\mu) = (-1)^k \sum_{uv = w_\mu} p^{\kappa(u)} (-q)^{\kappa(v)}. \qquad (7)$$

Note that $s_\lambda(1^p)s_\lambda(1^{-q}) = 0$ unless $\lambda \subseteq p \times q$. Hence we can assume that $\lambda \subseteq p \times q$ in the sum on the left-hand side of (7).

Now the coefficient of $\chi^\lambda(\mu)$ in (6) is $H_{p\times q} H_{\tilde{\lambda}}^{-1}$, while the coefficient of $\chi^\lambda(\mu)$ on the left-hand side of (7) is $(-1)^k H_\lambda s_\lambda(1^p) s_\lambda(1^{-q})$. By the lemma these two coefficients are equal, and the proof follows. $\square$

2 Generalizations.

The next step after rectangular shapes would be shapes that are the union of two rectangles, then three rectangles, etc. Figure 2 shows a shape $\sigma \vdash \sum_{i=1}^m p_i q_i$ that is a union of m rectangles of sizes $p_i \times q_i$, where $q_1 > q_2 > \cdots > q_m$.

Proposition 1. *Let σ be the shape in Figure 2, and fix $k \geq 1$. Set $n = |\sigma|$ and*

$$F_k(p_1, \ldots, p_m; q_1, \ldots, q_m) = \widehat{\chi}^\sigma(k, 1^{n-k}).$$

Then $F_k(p_1, \ldots, p_m; q_1, \ldots, q_m)$ is a polynomial function of the p_i's and q_i's with integer coefficients, satisfying

$$(-1)^k F_k(1, \ldots, 1; -1, \ldots, -1) = (k + m - 1)_k.$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and

$$\mu = (\mu_1, \ldots, \mu_r) = (\lambda_1 + r - 1, \lambda_2 + r - 2, \ldots, \lambda_r).$$

Define $\varphi(x) = \prod_{i=1}^r (x - \mu_i)$. A theorem of Frobenius (see [5, Exam. I.7.7]) asserts that

$$\widehat{\chi}^\lambda(k, 1^{n-k}) = -\frac{1}{k}[x^{-1}] \frac{(x)_k \varphi(x-k)}{\varphi(x)}, \qquad (8)$$

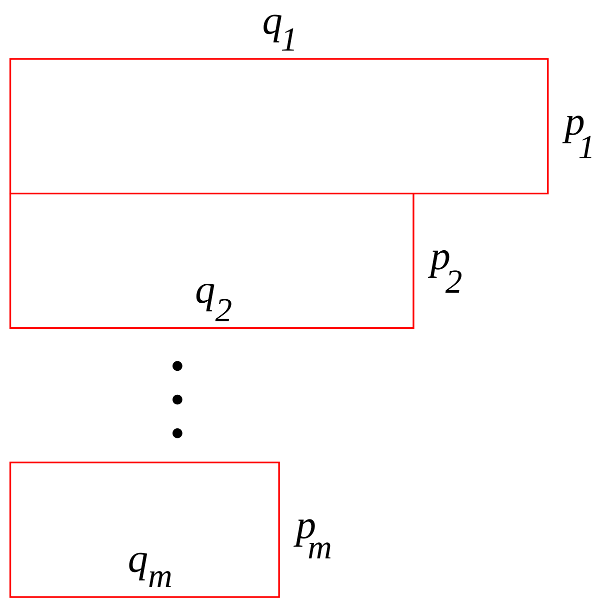

Figure 2: A union of m rectangles

where $[x^{-1}]f(x)$ denotes the coefficient of x^{-1} in the expansion of $f(x)$ in *descending* powers of x (i.e., as a Taylor series at $x = \infty$).

If we let $\lambda = \sigma$ in (8) and cancel common factors from the numerator and denominator, we obtain

$$\begin{aligned} \widehat{\chi}^{\sigma}(k, 1^{n-k}) &= -\frac{1}{k}[x^{-1}] \frac{(x)_k \prod_{i=1}^{m} (x - (q_i + p_i + p_{i+1} + \cdots + p_m))_k}{\prod_{i=1}^{m} (x - (q_i + p_{i+1} + p_{i+2} + \cdots + p_m))_k} \qquad (9) \\ &= -\frac{1}{k}[x^{-1}] H_k(x), \end{aligned}$$

say. Since

$$\frac{1}{x-a} = \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \cdots,$$

it is clear that $[x^{-1}]H_k(x)$ will be a polynomial $F_k(p_1, \ldots, p_m; q_1, \ldots, q_m)$ in the p_i's and q_i's with integer coefficients. If we put $p_i = 1$ and $q_i = -1$ then we obtain (after cancelling common factors)

$$F_k(1,\ldots,1;-1,\ldots,-1) = -\frac{1}{k}[x^{-1}]\frac{(x-k+1)(x-m+1)_k}{x+1}.$$

Since the sum of the residues of a rational function $R(x)$ in the extended complex plane is 0, it follows that

$$\begin{aligned}
-\frac{1}{k}[x^{-1}]\frac{(x-k+1)(x-m+1)_k}{x+1} &= -\frac{1}{k}\mathrm{Res}_{x=-1}\left(\frac{(x-k+1)(x-m+1)_k}{x+1}\right) \\
&= (-m)_k \\
&= (-1)^k(k+m-1)_k.
\end{aligned}$$

It remains to show that the coefficients of $F_k(p_1,\ldots,p_m;q_1,\ldots,q_m)$ are integers. Equivalently, the coefficients of the polynomial

$$[x^{-1}]\frac{(x)_k\varphi(x-k)}{\varphi(x)}$$

are divisible by k. But

$$\frac{(x)_k\varphi(x-k)}{\varphi(x)} \equiv (x)_k \pmod{k}$$

and

$$[x^{-1}](x)_k = 0,$$

so the proof follows. $\square$

NOTE. For any fixed $\mu \vdash k$, J. Katriel has shown (private communication), based on a method [4] for expressing $\widehat{\chi}^\lambda(\mu, 1^{n-k})$ in terms of the values $\widehat{\chi}^\lambda(j, 1^{n-j})$, that $\widehat{\chi}^\sigma(\mu, 1^{n-k})$ is a polynomial $F_\mu(p_1,\ldots,p_m;q_1,\ldots,q_m)$ with rational coefficients satisfying

$$(-1)^k F_\mu(1,\ldots,1;-1,\ldots,-1) = (k+m-1)_k.$$

It can be deduced from the Murnaghan-Nakayama rule that in fact the function $F_\mu(p_1,\ldots,p_m;q_1,\ldots,q_m)$ is a polynomial with *integer* coefficients. We conjecture that in fact the coefficients of $F_\mu(p_1,\ldots,p_m;q_1,\ldots,q_m)$ are nonnegative:

Conjecture 1. *For fixed $\mu \vdash k$, $\widehat{\chi}^\sigma(\mu, 1^{n-k})$ is a polynomial $F_\mu(p_1, \ldots, p_m; q_1, \ldots, q_m)$ with integer coefficients such that $(-1)^k F_\mu(p_1, \ldots, p_m; -q_1, \ldots, -q_m)$ has nonnegative coefficients summing to $(k+m-1)_k$.*

We do not have a conjectured combinatorial interpretation of the coefficients of $(-1)^k F_\mu(p_1, \ldots, p_m; -q_1, \ldots, -q_m)$. When $m = 2$ we have the following data, where we write $a = p_1$, $p = p_2$, $b = q_1$, $q = q_2$:

$$\begin{aligned}
-F_1(a,p;-b,-q) &= ab + pq \\
F_2(a,p;-b,-q) &= a^2b + ab^2 + 2apq + p^2q + pq^2 \\
-F_3(a,p;-b,-q) &= a^3b + 3a^2b^2 + 3a^2pq + ab^3 + 3abpq + 3ap^2q + 3apq^2 \\
&\quad + p^3q + 3p^2q^2 + pq^3 + ab + pq \\
F_4(a,p;-b-q) &= a^4b + 6a^3b^2 + 4a^3pq + 6a^2b^3 + 12a^2bpq + 6a^2p^2q \\
&\quad + 6a^2pq^2 + ab^4 + 4ab^2pq + 4abp^2q + 4abpq^2 + 4ap^3q \\
&\quad + 14ap^2q^2 + 4apq^3 + p^4q + 6p^3q^2 + 6p^2q^3 + pq^4 + 5a^2b \\
&\quad + 5ab^2 + 10apq + 5p^2q + 5pq^2.
\end{aligned}$$

We can say something more specific about the leading terms of $F_k(p_1, \ldots, p_m; q_1, \ldots, q_m)$. Let $G_k(p_1, \ldots, p_m; q_1, \ldots, q_m)$ denote these leading terms, viz., the terms of total degree $k+1$.

Proposition 2. *We have*

$$\frac{1}{x} + \sum_{k \geq 0} G_k(p_1, \ldots, p_m; q_1, \ldots, q_m)x^k =$$

$$\frac{1}{\left(\dfrac{x \displaystyle\prod_{i=1}^m (1 - (q_i + p_{i+1} + p_{i+2} + \cdots + p_m)x)}{\displaystyle\prod_{i=1}^m (1 - (q_i + p_i + p_{i+1} + \cdots + p_m)x)} \right)^{\langle -1 \rangle}}, \tag{10}$$

where ${}^{\langle -1 \rangle}$ denotes compositional inverse [8, §5.4] with respect to x. In particular, the generating function $\sum G_k x^k$ is algebraic over $\mathbb{Q}(p_1, \ldots, p_m, q_1, \ldots, q_m, x)$.

Proof. From (9) we have

$$\begin{aligned} G_k(p_1,\dots,p_k;q_1,\dots,q_k) &= -\frac{1}{k}[x^{-1}]\frac{x^k\prod_{i=1}^m (x-(q_i+p_i+p_{i+1}+\cdots+p_m))^k}{\prod_{i=1}^m (x-(q_i+p_{i+1}+p_{i+2}+\cdots+p_m))^k} \\ &= -\frac{1}{k}[x^{-1}]L(x)^k, \end{aligned}$$

say. Let $L(1/x)=M(x)/x$, so $M(0)=1$. Regard $M(x)$ as a power series in ascending powers of x, i.e., an ordinary Taylor series at $x=0$. Then by the Lagrange inversion formula [8, Thm. 5.4.2] we have

$$[x^{-1}]L(x)^k=[x^{k+1}]M(x)^k=-k[x^k]\frac{1}{(x/M(x))^{\langle -1\rangle}},$$

so equation (10) follows. □

Proposition 2 was also proved by Philippe Biane (private communication) in the same way as here, though using the language of free probability theory.

It follows from Proposition 1 or Proposition 2 that $(-1)^kG_k(p_1,\dots,p_m;-q_1,\dots,-q_m)$ is a polynomial with integer coefficients summing to

$$S_k:=(-1)^kG_k(1,\dots,1;-1,\dots,-1).$$

From Proposition 2 we have

$$-\frac{1}{x}+\sum_{k\ge 0}S_kx^k=\frac{-1}{\left(\frac{x(1-x)}{1-(m-1)x}\right)^{\langle -1\rangle}},$$

an algebraic function of degree two. When $m=1$ we have $S_k=C_k$, the kth Catalan number. Hence by Theorem 1 C_k is equal to the number of pairs $(u,v)\in\mathfrak{S}_k\times\mathfrak{S}_k$ such that $\kappa(u)+\kappa(v)=k+1$ and $uv=(1,2,\dots,k)$, a known result (e.g., [8, Exer. 6.19(hh)]). Moreover, it follows easily from Proposition 2 that

$$(-1)^kG_k(p;-q)=\sum_{i=1}^k N(k,i)p^{k+1-i}q^i,$$

where $N(k,i) = \frac{1}{k}\binom{k}{i}\binom{k}{i-1}$, a *Narayana number* [8, Exer 6.36]. Hence $N(k,i)$ is equal to the number of pairs $(u,v) \in \mathfrak{S}_k \times \mathfrak{S}_k$ such that $\kappa(u) = i$, $\kappa(v) = k+1-i$, and $uv = (1,2,\ldots,k)$. When $m = 2$ we have $S_k = r_k$, a (big) *Schröder number* [8, p. 178].

It would follow from Conjecture 1 that the polynomial $(-1)^k G_k(p_1,\ldots,p_m; -q_1,\ldots,-q_m)$ has nonnegative coefficients. In fact, Sergi Elizalde has shown (private communication of May, 2002) that

$$\begin{aligned}
(-1)^k & G_k(p_1,\ldots,p_m;-q_1,\ldots,-q_m) \\
&= \frac{1}{k} \sum_{i_1+\cdots+i_m+j_1+\cdots+j_m=k+1} \binom{k}{i_1} \left(\!\!\binom{i_1}{j_1}\!\!\right) \\
\prod_{s=2}^{m} & \left(\sum_{r=0}^{\min(i_s,j_s)} \binom{k}{r} \left(\!\!\binom{r}{j_s-r}\!\!\right) \binom{k-r-i_1-\cdots-i_{s-1}-j_1-\cdots-j_{s-1}}{i_s-r_s} \right) \\
& p_1^{i_1}\cdots p_m^{i_m} q_1^{j_1}\cdots q_m^{j_m},
\end{aligned}$$

where $\left(\!\binom{a}{b}\!\right) = \binom{a+b-1}{b}$. Thus in particular $(-1)^k G_k(p_1,\ldots,p_m;-q_1,\ldots,-q_m)$ indeed does have nonnegative coefficients. Do they have a simple combinatorial interpretation?

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