SOME SCHUBERT SHENANIGANS

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Abstract. We give a conjectured evaluation of the determinant of a certain matrix $\tilde{D}(n,k)$. The entries of $\tilde{D}(n,k)$ are either 0 or specializations $\mathfrak{S}_w(1,\dots,1)$ of Schubert polynomials. The conjecture implies that the weak order of the symmetric group S_n has the strong Sperner property. A number of peripheral results and problems are also discussed.

1. Introduction

The primary purpose of this paper is to present a conjectured evaluation of the determinant of a certain matrix $\tilde{D}(n,k)$, where $0 \leq k < \frac{1}{2}\binom{n}{2}$, related to the weak order W_n of the symmetric group S_n. The entries of $\tilde{D}(n,k)$ are either 0 or $\mathfrak{S}_w(1,\dots,1)$, the sum of the coefficients of the Schubert polynomial $\mathfrak{S}_w$, for certain permutations w. If $\det \tilde{D}(n,k) \neq 0$ for all $0 \leq k < \frac{1}{2}\binom{n}{2}$, then W_n has the strong Sperner property. It is currently unknown whether W_n has just the ordinary Sperner property. We have an explicit conjecture for the value $\det \tilde{D}(n,k)$ which shows in particular that it is nonzero, but we have no idea how to prove this conjecture.

The above problem suggests some other questions related to the numbers

$$\nu_w := \mathfrak{S}_w(1,1,\dots,1). \tag{1.1}$$

In particular, it is well-known that $\nu_w = 1$ if and only if w is 132-avoiding (or *dominant*). We have a conjectured condition for when $\nu_w = 2$ and can prove the sufficiency of this condition. One can go on to consider when $\nu_w = 3$, etc., though a "nice" characterization for every m of when $\nu_w = m$ does not seem very likely.

The consideration above of small values of ν_w suggests looking at how large ν_w can be for $w \in S_n$. Write $u(n)$ for this maximum value. We have some rather crude bounds which show that as $n \to \infty$,

$$\frac{1}{4} \leq \liminf \frac{\log_2 u(n)}{n^2} \leq \limsup \frac{\log_2 u(n)}{n^2} \leq \frac{1}{2}.$$

In addition to improving these bounds, one can also ask if there is some kind of "limiting shape" for the permutation(s) $w \in S_n$ satisfying $\nu_w = u(n)$.

2. A weak order determinant

For enumeration and poset terminology see [10]. For any set X, we denote by $\mathbb{Q}X$ the $\mathbb{Q}$-vector space with basis X. Let $P = P_0 \cup P_1 \cup \cdots \cup P_m$ be a finite graded rank-symmetric poset of rank m. Thus if $p_k = \#P_k$ then $p_k = p_{m-k}$. A linear transformation $U\colon \mathbb{Q}P \to \mathbb{Q}P$ is *order-raising* if for every $t \in P$ we have $U(t) \in \mathbb{Q}C^+(t)$, where $C^+(t)$ denotes the set of

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elements of P that cover t. Thus in this case U maps $\mathbb{Q}P_k$ into $\mathbb{Q}P_{k+1}$. The poset P is *rank-unimodal* if

$$p_0 \leq p_1 \leq \cdots \leq p_j \geq p_{j+1} \geq \cdots \geq p_m$$

for some j. Note that if also P is rank-symmetric, then we can take $j = \lfloor m/2 \rfloor$. We say that P is *strongly Sperner* if for all $r \geq 1$, the largest subset S of P containing no $(r+1)$-element chain has the same number of elements as the largest union T of r levels of P (in which case we can take $S = T$). The following result is given in [7, Lemma 1.1].

Theorem 2.1. *Suppose there exists an order-raising operator $U\colon \mathbb{Q}P \to \mathbb{Q}P$ such that if $0 \leq k < \frac{m}{2}$ then the linear transformation $U^{m-2k}\colon \mathbb{Q}P_k \to \mathbb{Q}P_{m-k}$ is a bijection. Then P is strongly Sperner.*

We would like to apply Theorem 2.1 to the weak order W_n of the symmetric group S_n. A permutation w has rank k in W_n, denoted $w \in (W_n)_k$, if w has k inversions. The rank-generating function of W_n is given by

$$F(W_n, q) \coloneqq \sum_{k=0}^{\binom{n}{2}} \#(W_n)_k \cdot q^k = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Let $\ell(w)$ denote the number of inversions (or *length*) of w. Write s_i for the adjacent transposition $(i, i+1)$, so v covers u in W_n if and only if $v = us_i$ for some $1 \leq i \leq n-1$ for which $\ell(v) = 1 + \ell(u)$. We now define the order-raising operator U by

$$U(u) = \sum_{i\,:\,\ell(us_i)=1+\ell(u)} i \cdot us_i, \quad u \in W_n.$$

It follows from [4, (6.11)] or [2, Lemma 2.3] that for $u \in W_n$ and $j \geq 0$ we have

$$U^j(u) = \left(\binom{n}{2} - 2k\right)! \sum_v \nu_{vu^{-1}} v, \tag{2.1}$$

where v ranges over all elements in W_n satisying $\ell(v) = \ell(u) + j$ and $v > u$ (in weak order), and $\nu_{vu^{-1}}$ is defined by equation (1.1).

Let $D(n,k)$ denote the matrix of the linear transformation $U^{\binom{n}{2}-2k}\colon \mathbb{Q}(W_n)_k \to \mathbb{Q}(W_n)_{\binom{n}{2}-k}$ with respect to the bases $(W_n)_k$ and $(W_n)_{\binom{n}{2}-k}$ (in some order). The idea of looking at this matrix is implicit in [11]. By equation (2.1) the (u,v)-entry of this matrix (for $u \in (W_n)_k$ and $v \in (W_n)_{\binom{n}{2}-k}$) is given by

$$D(n,k)_{uv} = \begin{cases} \left(\binom{n}{2} - 2k\right)! \nu_{vu^{-1}}, & \text{if } u \leq v \text{ in } W_n \\ 0, & \text{otherwise.} \end{cases}$$

We can divide each entry of $D(n,k)$ by $\left(\binom{n}{2} - 2k\right)!$, obtaining a matrix $\tilde{D}(n,k)$ satisfying

$$\det D(n,k) = \left(\binom{n}{2} - 2k\right)!^{\#(W_n)_k} \det \tilde{D}(n,k).$$

The following conjecture has been verified for all pairs (n,k) satisfying both $n \leq 12$ and $k \leq 5$, as well as a few other cases.

Conjecture 2.2. *We have*

$$\det \tilde{D}(n,k) = \pm \prod_{i=0}^{k-1} \left(\frac{\binom{n}{2} - (k+i)}{k-i} \right)^{\#(W_n)_i}. \tag{2.2}$$

We can prove Conjecture 2.2 when $k = 1$. This amounts to showing that

$$\det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 2 & 0 & 1 & 1 \\ & & & \vdots & & & & \\ 1 & 1 & 2 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{bmatrix} = (-1)^{\lfloor (n-2)/2 \rfloor} \left(\binom{n}{2} - 1 \right), \tag{2.3}$$

where the matrix is $(n-1) \times (n-1)$.

Conjecture 2.2 can be formulated in terms of the *nilCoxeter algebra* $\mathcal{N}_n$ of the symmetric group S_n. By definition, $\mathcal{N}_n$ is a $\mathbb{Q}$-algebra with basis $\{\sigma_w : w \in S_n\}$ and relations

$$\sigma_u \sigma_v = \begin{cases} \sigma_{uv}, & \text{if } \ell(uv) = \ell(u) + \ell(v) \\ 0, & \text{otherwise.} \end{cases}$$

The algebra $\mathcal{N}_n$ is graded via

$$\mathcal{N}_n = (\mathcal{N}_n)_0 \oplus (\mathcal{N}_n)_1 \oplus \cdots \oplus (\mathcal{N}_n)_{\binom{n}{2}},$$

where $(\mathcal{N}_n)_i$ is spanned by elements σ_w for which $\ell(w) = i$. Thus

$$(\mathcal{N}_n)_i (\mathcal{N}_n)_j \subseteq (\mathcal{N}_n)_{i+j}.$$

Define

$$\theta = \sum_{i=1}^{n-1} i \sigma_{s_i} \in (\mathcal{N}_n)_1,$$

where s_i denotes the adjacent transposition $(i, i+1)$. Then for $0 \leq k < \frac{1}{2}\binom{n}{2}$, the matrix of the map $\mathbb{Q}(\mathcal{N}_n)_k \to \mathbb{Q}(\mathcal{N}_n)_{\binom{n}{2}-k}$, with respect to the bases $(\mathcal{N}_n)_k$ and $(\mathcal{N}_n)_{\binom{n}{2}-k}$, given by multiplication by $\theta^{\binom{n}{2}-2k}$ coincides with $D(n,k)$. Thus Conjecture 2.2 implies that θ is a kind of "hard Lefschetz element" for $\mathcal{N}_n$. Note, however, that the commutativity properties of $\mathcal{N}_n$ imply that it is not the cohomology ring of a topological space.

The right-hand side of equation (2.2) is somewhat reminiscent of formulas arising in the theory of differential posets [8, Sect. 4] [9, pp. 154–157]. This leads us to ask whether there might be some "down operator" $D \colon \mathbb{Q}W_n \to \mathbb{Q}W_n$ taking $\mathbb{Q}(W_n)_{k+1}$ to $\mathbb{Q}(W_n)_k$ with "nice" commutativity properties with respect to U. Ideally the linear transformation $DU \colon \mathbb{Q}W_n \to \mathbb{Q}W_n$ should have integer eigenvalues. We have been unable to find a suitable candidate for D. For the weak order of *affine* Coxeter groups there is a nice theory due to Lam and Shimozono [3].

3. Generalizations and refinements

We mentioned that the entries of the matrix $\tilde{D}(n,k)$ are either 0 or specializations ν_w of Schubert polynomials $\mathfrak{S}_w$. It is natural to ask what happens when we replace ν_w with $\mathfrak{S}_w$ itself. However, we have been unable to discover anything interesting in this regard.

We can also consider a q-analogue of Conjecture 2.2. Theorem 2.4 of [2] (originally conjectured by Macdonald in [4, (6.11$_q$?)]) suggests that the correct q-analogue of $\tilde{D}(n,k)$ is obtained by replacing the entry $\mathfrak{S}_w(1,1,\dots,1)$ of $\tilde{D}(n,k)$ with $\mathfrak{S}_w(1,q,q^2,\dots,q^{m-1})$ (where $\mathfrak{S}_w$ is a polynomial in $x_1,\dots,x_m$). However, the data is not too encouraging. For instance, when $(n,k)=(5,2)$ the determinant is q^{36} times an irreducible polynomial (over $\mathbb{Q}$) of degree 20. For $(n,k)=(5,3)$ the determinant has the form

$$\pm q^{26}(q^2+q+1)^3(q^4+q^3+q^2+q+1)^3P(q),$$

where $P(q)$ is an irreducible polynomial of degree 28. Perhaps $\mathfrak{S}_w(1,q,q^2,\dots,q^{m-1})$ needs to be multiplied by some power of q.

It is natural to ask what happens when we replace the weak order W_n with the (strong) Bruhat order on S_n, which we continue to denote as S_n. For the Bruhat order the strong Sperner property is quite easy to see [7, p. 182]. However, it is still interesting to ask for an analogue $E(n,k)$ of the matrix $D(n,k)$. There are reasons to define the operator $V\colon \mathbb{Q}S_n \to \mathbb{Q}S_n$ as follows. Write t_{ij} for the permutation in S_n that transposes i and j. Then for $u \in S_n$ define

$$V(u) = \sum_{\substack{1\le i<j\le n\\ \ell(ut_{ij})=1+\ell(u)}} (j-i)\cdot ut_{ij}.$$

One reason for this definition is the fact, due essentially to Chevalley and further investigated by Stembridge [13] and Postnikov and Stanley [5], that

$$V^{\binom{n}{2}}(\mathrm{id}) = \binom{n}{2}!\, w_0,$$

where id denotes the identity permutation and $w_0 = n, n-1, \dots, 1$. Moreover, for $u \in (S_n)_j$ and $v \in (S_n)_{j+1}$ we have

$$[v]V(u) = [\mathfrak{S}_v]\mathfrak{S}_u(\mathfrak{S}_{s_1}+\cdots+\mathfrak{S}_{s_{n-1}}), \tag{3.1}$$

where $[v]V(u)$ denotes the coefficient of v in $V(u)$ and $[\mathfrak{S}_v]F$ denotes the coefficient of $\mathfrak{S}_v$ when the polynomial F is written as a linear combination of Schubert polynomials. As above s_i denotes the adjacent transposition $(i,i+1)$, so $\mathfrak{S}_{s_i} = x_1+x_2+\cdots+x_i$.

Let $E(n,k)$ denote the matrix of the linear transformation $V^{n-2k}\colon \mathbb{Q}(S_n)_k \to \mathbb{Q}(S_n)_{\binom{n}{2}-k}$ with respect to the bases $(S_n)_k$ and $(S_n)_{\binom{n}{2}-k}$ (in some order). Let $e(n,k) = \det E(n,k)$. For $u \in (S_n)_k$ and $v \in (S_n)_{\binom{n}{2}-2k}$, it follows from equation (3.1) that the (u,v)-entry of $E(n,k)$ is given by

$$E(n,k)_{uv} = [\mathfrak{S}_v]\mathfrak{S}_u(\mathfrak{S}_{s_1}+\cdots+\mathfrak{S}_{s_{n-1}})^{\binom{n}{2}-2k}.$$

We have computed that

$$\begin{aligned} e(4,1) &= \pm 2^7\cdot 3\cdot 5^2\cdot 19\\ e(4,2) &= \pm 2^6\cdot 3\cdot 29\\ e(5,1) &= \pm 2^{22}\cdot 3^6\cdot 5^5\cdot 7^4\cdot 59\cdot 89. \end{aligned}$$

Obviously some further computation, or even better a theorem, is needed.

Unlike the situation for the weak order, the Bruhat analogue $\mathcal{B}_n$ of $\mathcal{N}_n$ is the cohomology ring of a (smooth, projective) variety, viz., the complete flag variety $\mathcal{F}_n = \mathrm{GL}(n,\mathbb{C})/B$, where B is a Borel subgroup. Thus the hard Lefschetz theorem for $\mathcal{F}_n$ immediately implies the strong Sperner property for S_n [7], but as stated above this is easy to prove directly. A variety related to $\mathcal{F}_n$ is the Grassmann variety $\mathrm{Gr}(n,k)$ of k-dimensional subspaces of $\mathbb{C}^n$. Here the determinants of the matrices analogous to $D(n,k)$ and $E(n,k)$ have been computed by Proctor [6]. He also computed determinants corresponding to products of complex projective spaces, where the corresponding poset is a product of chains.

A canonical question suggested by Conjecture 2.2 is its possible extension to other finite Coxeter groups. We have not made any computations in this direction.

Another possible extension of Conjecture 2.2 is to refine the determinant of $\tilde{D}(n,k)$. The two most natural refinements are the characteristic polynomial (or equivalently the eigenvalues) of $\tilde{D}(n,k)$ and the Smith normal form (SNF) over $\mathbb{Z}$ of $\tilde{D}(n,k)$. The characteristic polynomial depends on the order of the rows and columns. We have done some computation using every possible order and have found nothing interesting. For instance, the eigenvalues are in general not integers. On the other hand, the SNF does not depend on the order of the rows and columns, and the data looks intriguing. For instance, the ratio between two consecutive diagonal terms is always quite small. Write $f(n,k)$ for the vector of diagonal entries of the SNF of $\tilde{D}(n,k)$. Here is some data (which we have not attempted to analyze, except for $k=1$ discussed below). We use exponents to denote repetition, e.g., $(1^5,3^2)$ stands for $(1,1,1,1,1,3,3)$.

$$
\begin{aligned}
f(4,1) &= (1^2, 5)\\
f(4,2) &= (1^2, 3^2, 6)\\
f(5,1) &= (1^3, 9)\\
f(5,2) &= (1^5, 7^3, 28)\\
f(5,3) &= (1^6, 5^6, 15^2, 105)\\
f(5,4) &= (1^8, 3^4, 6^4, 30^4)\\
f(6,1) &= (1^4, 14)\\
f(6,2) &= (1^9, 6, 12^3, 156)\\
f(6,3) &= (1^{15}, 5^4, 10^5, 110^4, 220)\\
f(6,4) &= (1^{20}, 2, 4^9, 8^5, 24^5, 72^4, 360^4, 3960)\\
f(6,5) &= (1^{31}, 3^6, 6^5, 42^{19}, 84, 168^4, 504^5)\\
f(6,6) &= (1^{28}, 2^{18}, 10^2, 20^{28}, 140^{10}, 280^3, 840)\\
f(6,7) &= (1^{52}, 2^{18}, 6^{10}, 12^6, 60^{11}, 420^3, 840)\\
f(7,1) &= (1^5, 20)\\
f(7,2) &= (1^{14}, 9, 18^4, 342)\\
f(7,3) &= (1^{29}, 8^5, 16^9, 272^5, 816)\\
f(7,4) &= (1^{49}, 7^{14}, 14^{15}, 70^6, 210^8, 420, 1680^4, 28560)\\
f(7,5) &= (1^{76}, 2^9, 6^{20}, 12^{15}, 156^{29}, 1092^{15}, 5460^4, 21840)\\
f(8,1) &= (1^6, 27)\\
f(8,2) &= (1^{20}, 25^6, 325)\\
f(8,3) &= (1^{49}, 23^{20}, 92, 276^5, 6900)\\
f(8,4) &= (1^{98}, 7^6, 21^{43}, 231^{20}, 5313^6, 10626)\\
f(9,1) &= (1^7, 35)\\
f(9,2) &= (1^{27}, 33^7, 561)\\
f(9,3) &= (1^{76}, 31^{27}, 496^7, 5456)\\
f(10,1) &= (1^8, 44)\\
f(10,2) &= (1^{35}, 21, 42^7, 1806)
\end{aligned}
$$

The data suggests the obvious conjecture $f(n,1) = (1^{n-2}, \binom{n}{2} - 1)$. We can prove this as follows. We can take $\tilde{D}(n,1)$ to be the matrix of equation (2.3). Since $\det \tilde{D}(n,1) = \binom{n}{2} - 1$ (see equation (2.3)), by basic properties of SNF (e.g., [12, Thm. 2.4]) it suffices to show that if $\tilde{D}'(n,1)$ denotes $\tilde{D}(n,1)$ with its first row and column deleted, then $\det \tilde{D}'(n,1) = 1$. The proof is by induction on n, the base case $n = 3$ being trivial to check. Subtract the next-to-last row of $\tilde{D}'(n,1)$ from the last row and expand by the last row. We obtain $\det \tilde{D}'(n,1) = \det \tilde{D}'(n-1,1)$, so the proof follows.

4. Two-term Schubert polynomials

The appearance of the numbers $\nu_w = \mathfrak{S}_w(1,1,\ldots,1)$ as entries of the matrix $\tilde{D}(n,k)$ suggest the question of how "complicated" these numbers can be. The simplest situation is when $\mathfrak{S}_w$ is a single monomial, so $\nu_w = 1$. It is well-known [4] that $\nu_w = 1$ if and only if w is 132-avoiding (or *dominant*).

Conjecture 4.1. *We have $\nu_w = 2$ if and only if $w = a_1 \cdots a_n$ has exactly one subsequence a_i, a_j, a_k in the pattern 132, i.e., $a_i < a_k < a_j$.*

If $f(n)$ denotes the number of permutations with exactly one subsequence with pattern 132, then it is known that $f(n) = \binom{2n-3}{n-3}$, $n \geq 3$ [1, eqn. (2)].

We can prove the "if" direction of Conjecture 4.1. Namely, if $w = a_1 \cdots a_n$ has exactly one subsequence with pattern 132, then it is easily seen that the unique such subsequence has the form a_i, a_{i+1}, a_j. If we transpose a_i and a_{i+1}, then we get a permutation w' that is 132-avoiding. For any permutation $v = v_1 \cdots v_n$ let

$$\lambda_i(v) = \#\{j > i \,:\, v_j < v_i\}.$$

By the theory of Schubert polynomials [4], we have (since w' is 132-avoiding)

$$\mathfrak{S}_{w'} = x_1^{\lambda_1(w')} x_2^{\lambda_2(w')} \cdots x_{n-1}^{\lambda_{n-1}(w')} \tag{4.1}$$

and (since w covers w' in W_n)

$$\mathfrak{S}_w = \frac{\mathfrak{S}_{w'}(x_1,\ldots,x_i,x_{i+1},\ldots,x_{n-1}) - \mathfrak{S}_{w'}(x_1,\ldots,x_{i+1},x_i,\ldots,x_{n-1})}{x_i - x_{i+1}}. \tag{4.2}$$

Since a_i, a_{i+1}, a_j is the unique subsequence of w with the pattern 132, we have $\lambda_i(w') = \lambda_{i+1}(w') + 2$. It follows from equations (4.1) and (4.2) that $\mathfrak{S}_w$ has two terms.

We suspect that the converse, i.e., if $\nu_w = 2$ then w has exactly one subsequence with pattern 132, will not be difficult to prove.

One can go on to consider the number $f_j(n)$ of permutations $w \in \mathfrak{S}_n$ for which $\nu_w = j \geq 3$, and the characterization of such permutations. It seems unlikely that there will be a nice result in general.

5. The maximum value of ν_w for $w \in S_n$

The previous section dealt with small values of ν_w, so we can also ask for large values. As in Section 1, define

$$u(n) = \max\{\nu_w \,:\, w \in S_n\}.$$

Setting each $x_i = y_i = 1$ in the "Cauchy identity"

$$\prod_{i+j\leq n} (x_i + y_j) = \sum_{\substack{w=v^{-1}u \\ \ell(w)=\ell(u)+\ell(v)}} \mathfrak{S}_u(x)\mathfrak{S}_v(y)$$

immediately yields the estimates

$$\frac{1}{4} \leq \liminf \frac{\log_2 u(n)}{n^2} \leq \limsup \frac{\log_2 u(n)}{n^2} \leq \frac{1}{2}.$$

Presumably the limit $L = \lim_{n\to\infty} n^{-2} \log_2 u(n)$ exists (in which case $\frac{1}{4} \leq L \leq \frac{1}{2}$), and it would be interesting to compute this limit. It would also be interesting to determine whether the permutations $w \in S_n$ achieving $\nu_e = u(n)$ have some "limiting" description. Below is the value of $u(n)$ for $3 \leq n \leq 9$, together with the set of all permutations $w \in S_n$ achieving $\nu_w = u(n)$.

n	$u(n)$	w
3	2	132
4	5	1432
5	14	12543, 15432, 21543
6	84	126543, 216543
7	660	1327654
8	9438	13287654
9	163592	132987654
10	4424420	1,4,3,2,10,9,8,7,6,5

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