Some Linear Recurrences Motivated by Stern's Diatomic Array

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Abstract

We define a triangular array closely related to Stern's diatomic array and show that for a fixed integer $r \geq 1$, the sum $u_r(n)$ of the rth powers of the entries in row n satisfy a linear recurrence with constant coefficients. The proof technique yields a vast generalization. In certain cases we can be more explicit about the resulting linear recurrence.

1 Introduction.

We first define an array of numbers analogous to Pascal's triangle (or the arithmetic triangle). The rows will be numbered $0, 1, \ldots$. The first row consists of a single 1, and every subsequent row begins and ends with a 1, just like Pascal's triangle. We add two consecutive numbers in row n and place the sum in row $n+1$ to the right of the first number and left of the second number, again just like Pascal's triangle. However, we also bring down (copy) into row $n+1$ each entry in row n, placing it directly below. The first four rows look like

$$
\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&1&&&&1&&&&1&&& \\
&1&&1&&2&&1&&2&&1&&1& \\
1&1&2&1&3&2&3&1&3&2&3&1&2&1&1 \\
&&&&&&&\vdots&&&&&&&
\end{array}.
$$

We will call this array *Stern's triangle*, though "triangle" is somewhat of a misnomer since the number of entries in each row grows exponentially, not linearly.

Stern's triangle is closely related to a well-known array called *Stern's diatomic array* [7]. It has the same recursive rules as Stern's triangle, but the first row consists of two 1's which are brought down to form the first and last element of each row. It looks like

$$
\begin{array}{ccccccccccccccccc}
1&&&&&&&&&&&&&&&&1 \\
1&&&&&&&&2&&&&&&&&1 \\
1&&&&3&&&&2&&&&3&&&&1 \\
1&&4&&3&&5&&2&&5&&3&&4&&1 \\
1&5&4&7&3&8&5&7&2&7&5&8&3&7&4&5&1 \\
&&&&&&&&\vdots&&&&&&&&
\end{array}.
$$

Remark 1. Let R_i denote the ith row of Stern's diatomic array, beginning with row 0. Form the concatenation

$$R_0 R_1 \cdots R_{n-2} R_{n-1} R_{n-1} R_{n-2} \cdots R_1 R_0$$

and then merge together the last 1 in each row with the first 1 in the next row. We then obtain row n of Stern's triangle. From this observation almost any property of Stern's triangle can be carried over straightforwardly to Stern's diatomic array and *vice versa*, including the properties that we discuss below. See in particular Remark 2.

We prefer Stern's triangle to Stern's diatomic array because its properties are more elegant and simple. In particular, let $\left\langle {n \atop k} \right\rangle$ denote the kth entry (beginning with $k = 0$) in row n of Stern's triangle, in analogy with Pascal's triangle. Naturally we define $\left\langle {n \atop k} \right\rangle = 0$ if k is "out of range," i.e., if $k < 0$ or $k > 2^{n+1} - 2$. We then have the following "Stern analogue" of the binomial theorem.

Theorem 1. *Let $n \geq 1$. Then*

$$F_n(x) \coloneqq \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right). \tag{1}$$

Proof. This formula is immediate from the recursive definition of Stern's triangle. The result is clearly true for $n = 1$, so we need to show that $F_{n+1}(x) = (1 + x + x^2)F_n(x^2)$, $n \geq 1$. The product $xF_n(x^2)$ accounts for the entries in row $n + 1$ that are brought down from row n, while $(1 + x^2)F_n(x^2)$ accounts for the entries in row $n + 1$ that are the sum of two consecutive entries in row n. $\square$

Note that it follows immediately from Theorem 1 that $\left\langle {n \atop k} \right\rangle$ has the following combinatorial interpretation: it is the number of partitions of k (as defined, e.g., in [1] or [5, §1.8]) into the parts $1, 2, 4, \ldots, 2^{n-1}$, where each part may be used at most twice.

Theorem 1 implies that row n of Stern's triangle approaches a limiting sequence $b_0, b_1, \ldots$ as $n \to \infty$, meaning that for all $k \geq 0$ we have $\left\langle {n \atop k} \right\rangle = b_k$ for n sufficiently large (depending on k). Moreover, letting $n \to \infty$ in equation (1) shows that

$$\sum_{k \geq 0} b_k x^k = \prod_{i=0}^{\infty} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right).$$

The sequence $b_0, b_1, \ldots$ is the famous *Stern's diatomic sequence*, with many remarkable properties. (Sometimes $0, b_0, b_1, \ldots$ is called Stern's diatomic sequence.) Perhaps the most amazing property, though irrelevant here, is that every positive rational number appears exactly once as a ratio b_k/b_{k+1}, and that this fraction is always in lowest terms. Northshield [2] has a nice survey.

2 Sums of products of powers.

In this section we will consider sums $\sum_{k \geq 0} \left\langle {n \atop k} \right\rangle^r$, where $r \in \mathbb{N} = \{0, 1, \ldots\}$, and more generally

$$u_\alpha(n) \coloneqq \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle^{\alpha_0} \left\langle {n \atop k+1} \right\rangle^{\alpha_1} \cdots \left\langle {n \atop k+m-1} \right\rangle^{\alpha_{m-1}},$$

where $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^m$. For notational simplicity we write $u_{\alpha_0, \ldots, \alpha_{m-1}}(n)$ as short for $u_{(\alpha_0, \ldots, \alpha_{m-1})}(n)$.

We will use the following terminology concerning linear recurrences with constant coefficients. Suppose that $f \colon \mathbb{N} \to \mathbb{C}$ satisfies such a recurrence for $n \geq n_0$. Thus there are complex numbers $c_1, \ldots, c_\ell$, with $c_\ell \neq 0$, such that

$$f(n + \ell) + c_1 f(n + \ell - 1) + c_2 f(n + \ell - 2) + \cdots + c_\ell f(n) = 0, \quad n \geq n_0.$$

Equivalently, $f(n)$ has a generating function of the form

$$\sum_{n \geq 0} f(n) x^n = \frac{P(x)}{1 + c_1 x + \cdots + c_\ell x^\ell}, \tag{2}$$

where $P(x) \in \mathbb{C}[x]$ and $\deg P(x) < \ell + n_0$. We say that $f(n)$ has a *rational generating function* with *characteristic polynomial*

$$R(x) = x^\ell + c_1 x^{\ell-1} + \cdots + c_\ell.$$

Thus if $Q(x)$ denotes the denominator of the right-hand side of equation (2), then $R(x) = x^\ell Q(1/x)$. For further information on linear recurrences with constant coefficients and rational generating functions, see [5, Chapter 4].

Clearly the number of entries in row n of Stern's triangle is $2^{n+1}-1$, so $u_0(n) = 2^{n+1}-1$. It is also clear that $u_1(n) = 3^n$, e.g., by putting $x = 1$ in equation (1) or directly from the recursive structure of Stern's triangle, since each entry in row n contributes to three entries of row $n+1$.

Thus let us turn to $u_2(n)$. The first few values (beginning at $n = 0$) are 1, 3, 13, 59, 269, 1227, 5597, 25531,.... By various methods, such as trial-and-error, using the *Online Encyclopedia of Integer Sequences* (OEIS) , or using the Maple package gfun, we are led to conjecture that

$$u_2(n+2) - 5u_2(n+1) + 2u_2(n) = 0, \quad n \geq 0. \tag{3}$$

Equivalently (using the initial values $u_2(0) = 1$ and $u_2(1) = 3$),

$$\sum_{n\geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}.$$

Note the difference from Pascal's triangle, where $\sum_{k\geq 0}\binom{n}{k}^2 = \binom{2n}{n}$ and $\sum_{n\geq 0}\binom{2n}{n}x^n = 1/\sqrt{1-4x}$. For Stern's triangle the generating function is rational, while for Pascal's triangle it is only algebraic.

How do we prove the recurrence (3)? From the definition of Stern's triangle we have

$$\begin{aligned}
u_2(n+1) &= \sum_k \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \sum_k \left\langle {n \atop k} \right\rangle^2 \\
&= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle \\
&= 3u_2(n) + 2u_{1,1}(n).
\end{aligned}$$

We therefore need to play a similar game with $u_{1,1}$:

$$\begin{aligned}
u_{1,1}(n+1) &= \sum_k \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle \right) \left\langle {n \atop k} \right\rangle \\
&\quad + \sum_k \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \\
&= 2u_2(n) + 2u_{1,1}(n).
\end{aligned}$$

Hence we get the matrix recurrence

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}, \tag{4}$$

where $A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. This is a standard type of simultaneous linear recurrence. To solve it, we have

$$A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}.$$

The minimum polynomial of A, i.e., the (nonzero) monic polynomial $M(x)$ of least degree satisfying $M(A) = 0$, is easily computed to be $x^2 - 5x + 2$. Hence

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= A^n(A^2 - 5A + 2) \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} \\
&= (A^{n+2} - 5A^{n+1} + 2A^n) \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} \\
&= \begin{bmatrix} u_2(n+2) \\ u_{1,1}(n+2) \end{bmatrix} - 5 \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix} + 2 \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix},
\end{aligned}$$

so we get $u_2(n+2) - 5u_2(n+1) + u_2(n) = 0$, as well as the same recurrence for $u_{1,1}(n)$.

Let us apply this procedure to $u_3(n) = \sum_k \left\langle {n \atop k} \right\rangle^3$. We get

$$\begin{aligned} u_3(n+1) &= \sum_k \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^3 + \sum_k \left\langle {n \atop k} \right\rangle^3 \\ &= 3u_3(n) + 3u_{2,1}(n) + 3u_{1,2}(n). \end{aligned}$$

Because of the symmetry of Stern's triangle about a vertical axis, we have

$$u_{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}}(n) = u_{\alpha_{m-1},\ldots,\alpha_1,\alpha_0}(n),$$

so in particular $u_{2,1}(n) = u_{1,2}(n)$. Thus

$$u_3(n+1) = 3u_3(n) + 6u_{2,1}(n).$$

Similarly,

$$\begin{aligned} u_{2,1}(n+1) &= \sum_k \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle \right)^2 \left\langle {n \atop k} \right\rangle \\ &\quad + \sum_k \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 \\ &= 2u_3(n) + 4u_{2,1}(n). \end{aligned}$$

The matrix $\left[\begin{array}{cc} 3 & 6 \\ 2 & 4 \end{array} \right]$ has minimum polynomial $x(x-7)$, so we get the recurrence

$$u_3(n+1) = 7u_3(n), \quad n \geq 1,$$

and similarly for $u_{2,1}(n)$. (The recurrence is not valid at $n = 0$ since the minimum polynomial is $x(x-7)$, not $x - 7$.) In fact, we have the surprisingly simple formulas

$$u_3(n) = 3 \cdot 7^{n-1}, \quad u_{2,1}(n) = 2 \cdot 7^{n-1}, \quad n \geq 1.$$

Here we see an even larger divergence from the behavior of Pascal's triangle—the generating function for $f(n) \coloneqq \sum_{k \geq 0} \binom{n}{k}^3$ is not even algebraic. The best we can say is that it is D-finite [6, §6.4], meaning that $f(n)$ satisfies a linear recurrence with polynomial coefficients, namely,

$$(n+2)^2 f(n+2) - (7n^2 + 21n + 16) f(n+1) - 8(n+1)^2 f(n) = 0, \ n \geq 0.$$

For more on the sums $\sum_k \binom{n}{k}^r$, see [6, Exercise 6.54] and the references given there.

We have shown that $u_2(n), u_{1,1}(n), u_3(n)$, and $u_{2,1}(n)$ have rational generating functions. The same technique yields the following general result.

Theorem 2. *For any $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^m$, the function $u_\alpha(n)$ has a rational generating function.*

Proof. We have

$$\begin{aligned} u_\alpha(n+1) &= \sum_k \left\langle {n \atop k} \right\rangle^{\alpha_0} \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^{\alpha_1} \left\langle {n \atop k+1} \right\rangle^{\alpha_2} \qquad (5) \\ &\quad \cdot \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle \right)^{\alpha_3} \cdots \\ &\quad + \sum_k \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^{\alpha_0} \left\langle {n \atop k+1} \right\rangle^{\alpha_1} \\ &\quad \cdot \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle \right)^{\alpha_2} \left\langle {n \atop k+2} \right\rangle^{\alpha_3} \cdots . \qquad (6) \end{aligned}$$

When the summands are expanded, we obtain an expression for $u_\alpha(n+1)$ as a linear combination of u_β's. Define the *spread* of u_α, denoted $\mathrm{spread}(u_\alpha)$, to be largest length (number of terms) of any β for which $u_\beta(n)$ appears in this expression for $u_\alpha(n+1)$. For instance, from $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$ we see that $\mathrm{spread}(u_2) = 2$, coming from $\beta = (1,1)$ of length two.

From equation (6) we see that that when $u_\alpha(n+1)$ is written as a linear combination of u_β's, the indices β that occur satisfy (a) $|\alpha| := \sum \alpha_i = \sum \beta_i = |\beta|$, and (b) $\mathrm{spread}(u_\alpha) = 2 + \lfloor \frac{1}{2}(\ell - 1) \rfloor$, where α has length ℓ. Since $2 + \lfloor \frac{1}{2}(\ell - 1) \rfloor \leq \ell$ for $\ell \geq 2$, it follows that for $\alpha = (r)$ we will obtain a (finite) matrix recurrence like equation (4), where one of the functions is $u_\alpha(n)$. The size (number of rows or columns) of the matrix is $1 + \lfloor r/2 \rfloor$, the number of weakly decreasing sequences of positive integers with sum r and length 1 or 2. Similarly, when α has length $\ell \geq 2$, then the size of the matrix will not exceed the number of equivalence classes of sequences of nonnegative integers of length at most ℓ summing to $|\alpha|$, where a sequence α is equivalent to its reverse. The point is that there are only finitely many such equivalence classes, so we obtain a finite matrix equation. By the same argument used to show $u_2(n+2) - 5u_2(n+1) + u_2(n) = 0$, we get that $u_\alpha(n)$ has a rational generating function. $\square$

Here are the characteristic polynomials of the recurrences satisfied by $u_r(n)$ for n sufficiently large (denoted $n \gg 0$), for $1 \leq r \leq 10$:

$$
\begin{gathered}
x - 3, \\
x^2 - 5x + 2, \\
x - 7, \\
(x+1)(x^2 - 11x + 2), \\
x^2 - 14x - 47, \\
x^4 - 20x^3 - 161x^2 - 40x + 4, \\
x^3 - 29x^2 - 485x - 327, \\
(x+1)(x^4 - 44x^3 - 1313x^2 - 88x + 4), \\
x^3 - 65x^2 - 3653x - 3843, \\
(x+1)(x^4 - 100x^3 - 9601x^2 - 200x + 4).
\end{gathered}
$$

We can say quite a bit more about the recurrence satisfied by $u_\alpha(n)$. We can assume that $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^m$ with $\alpha_0 > 0$ and $\alpha_{m-1} > 0$. We then write $m = \ell(\alpha)$, the length of α.

Write $\mathrm{mmp}(\alpha)$ (for "matrix minimum polynomial") for the minimum polynomial $M(A_\alpha)$ of the matrix A_α used to compute $u_\alpha(n)$ by the method above. Write $\mathrm{rmp}(\alpha)$ (for "recurrence minimum polynomial") for the characteristic polynomial of the linear recurrence with constant coefficients *of least degree* satisfied by $u_\alpha(n)$ for $n \gg 0$. Note that the proof of Theorem 2 shows that $\mathrm{mmp}(\alpha)$ is divisible by $\mathrm{rmp}(\alpha)$.

Linearly order all sequences $\alpha = (\alpha_0, \ldots, \alpha_{m-1})$ with $\alpha_0 > 0$ and $\alpha_{m-1} > 0$, and where $|\alpha| = r$ is fixed, in such a way that the following conditions are satisfied: (a) if $\ell(\alpha) < \ell(\beta)$ then $\alpha < \beta$; and (b) if $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ (with $|\alpha| = |\beta|$), then define $\alpha \leq \beta$ if $\alpha_1 \geq \beta_1$, $\alpha_2 \leq \beta_2$, and $\alpha_3 \geq \beta_3$. (We do not specify the ordering when $\ell(\alpha) = \ell(\beta) \neq 3$ except for condition (a).)

Order the rows and columns of A_α using the order defined in the previous paragraph. It is easy to check that A_α is block lower-triangular. The first block is A_r, where $r = |\alpha|$. This corresponds to rows and columns indexed by β with $\ell(\beta) \leq 2$. For $\ell(\beta) = 3$, the blocks are 1×1 with entry 1. All the other blocks are 1×1 with entry 0.

Example. For $\alpha = (1,1,1,1)$ we use the ordering (writing e.g. 121 for $(1,2,1)$)

$$4 < 31 < 22 < 121 < 211 < 1111.$$

(No other β's occur in computing the recurrence satisfied by $u_{1,1,1,1}$.) We get the matrix

$$A_{(1,1,1,1)} = \begin{bmatrix} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

In particular, the minimum polynomial of $A_{(1,1,1,1)}$ is $x(x+1)(2x^2-11x+1)(x-1)^2$. The factor $(x+1)(2x^2-11x+1)$ is the minimum polynomial of the block A_4. The characteristic polynomial $\mathrm{rmp}(1,1,1,1)$ of the least order recurrence satisfied by $u_{1,1,1,1}(n)$ for $n \gg 0$ turns out to be $(x-1)^2(x+1)(2x^2-11x+1)$.

The above argument yields the following theorem.

Theorem 3. *Let $\alpha \in \mathbb{N}^m$ and $|\alpha| = r$. Then the polynomial* $\mathrm{mmp}(\alpha)$ *has the form $x^{w_\alpha}(x-1)^{z_\alpha}\mathrm{mmp}(r)$ for some $w_\alpha, z_\alpha \in \mathbb{N}$.*

We have not considered whether there is a "nice" description of the integers w_α and z_α, nor the largest power of $x-1$ dividing $\mathrm{rmp}(\alpha)$.

3 A conjecture on the order of the recurrence.

Can we say more about the actual recurrence satisfied by $u_\alpha(n)$? We have not investigated this question systematically, but we do have a conjecture about the order of the recurrence and some special properties of the characteristic polynomial. For instance, is it just an "accident" that the matrix $A_3 = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ has a zero eigenvalue, thereby reducing the order of the recurrence from two to one? Or that the polynomials $\mathrm{rmp}(4)$, $\mathrm{rmp}(8)$, and $\mathrm{rmp}(10)$, are divisible by $x+1$?

We noted before that the matrix A_r has size $\lceil (r+1)/2 \rceil$ (the number of weakly decreasing sequences of positive integers with sum r and length 1 or 2) so $u_r(n)$ satisfies a linear recurrence with constant coefficients of this order. However, on the basis of empirical evidence ($r \le 125$), we conjecture that the least order of such a recurrence is actually $\frac{1}{3}r + O(1)$. In fact, we have the following more precise conjecture. Write $[a_0, \ldots, a_{q-1}]_q$ for the periodic function $f\colon \mathbb{N} \to \mathbb{R}$ satisfying $f(n) = a_i$ if $n \equiv i \pmod{q}$. Let $e_r(\theta)$ denote the number of eigenvalues of A_r equal to θ. Recall that an eigenvalue θ of a matrix A is *semisimple* if the minimum polynomial of A is not divisible by $(x-\theta)^2$. Equivalently, all the Jordan blocks with eigenvalue θ of the Jordan canonical form of A have size one.

Conjecture. *(a) We have*

$$e_{2s-1}(0) = \frac{1}{3}s + \left[0, -\frac{1}{3}, \frac{1}{3}\right]_3,$$

and the eigenvalue 0 is semisimple. There are no other multiple eigenvalues, and 1 is not an eigenvalue.

(b) We have

$$\begin{aligned} e_{2s}(1) &= \frac{1}{6}s + \left[-1, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}, \frac{1}{6}\right]_6 \\ e_{2s}(-1) &= e_{2s+6}(1). \end{aligned}$$

The eigenvalues 1 and -1 are semisimple, and there are no other multiple eigenvalues.

Let $\mathrm{mo}(r)$ be the minimum order of a linear recurrence with constant coefficients satisfied by $u_r(n)$ for $n \gg 0$. Conjecture 3 reduces the "naive" bound $\mathrm{mo}(r) \le \lceil r/2 \rceil$ to $\mathrm{mo}(r) \le \lceil r/2 \rceil - e_r(0)$ when r is odd, and

$$\mathrm{mo}(r) \le \left\lceil \frac{r}{2} \right\rceil - \max\{0, e_r(1) - 1\} - \max\{0, e_r(-1) - 1\}$$

when r is even. However, it appears that the eigenvalue 1 of A_{2s} is always superfluous, that is, $x-1$ is not a factor of the characteristic polynomial $\mathrm{rmp}_{2s}(x)$. This will lower the upper bound for $\mathrm{mo}(r)$ by 1 when $e_r(1) > 0$. The evidence suggests that we then get a best possible result. The resulting conjecture is the following.

Conjecture. *We have* $\mathrm{mo}(2) = 2$, $\mathrm{mo}(6) = 4$, *and otherwise*

$$\begin{aligned}
\mathrm{mo}(2s) &= 2\left\lfloor \frac{s}{3} \right\rfloor + 3, \quad s \neq 1, 3 \\
\mathrm{mo}(6s+1) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+3) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+5) &= 2s+2, \quad s \geq 0.
\end{aligned}$$

After the above conjectures were communicated in a lecture, David Speyer [4] made some important progress. He showed that the conjectured values of $e_{2s-1}(0)$ and $e_{2s}(\pm 1)$ are lower bounds for their actual values. Moreover, A_r can be conjugated by a diagonal matrix to give a symmetric matrix, thereby showing that the eigenvalues of A_r are semisimple (and real). As a consequence, the conjectured value of $\mathrm{mo}(r)$ is an upper bound on its actual value. The key to Speyer's argument is that if B_r is defined like the matrix A_r except that we don't take into account the symmetry $u_{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}}(n) = u_{\alpha_{m-1},\ldots,\alpha_1,\alpha_0}(n)$, then B_r is the matrix of the linear transformation $\phi\colon V \to V$, with respect to the basis of monomials, defined by

$$\phi(f)(x,y) = f(x+y,y) + f(x,x+y),$$

where V is the vector space of complex homogeneous polynomials of degree r in the two variables x and y. We will not give further details here.

Remark 2. Let $v_\alpha(n)$ be the analogue for Stern's diatomic array of $u_\alpha(n)$. That is, if $\left[{n \atop k}\right]$ denotes the kth entry (beginning with $k=0$) in row n (beginning with $n=0$) in Stern's diatomic array, then

$$v_\alpha(n) := \sum_{k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}^{\alpha_0} \begin{bmatrix} n \\ k+1 \end{bmatrix}^{\alpha_1} \cdots \begin{bmatrix} n \\ k+m-1 \end{bmatrix}^{\alpha_{m-1}},$$

where $\alpha = (\alpha_0,\ldots,\alpha_{m-1}) \in \mathbb{N}^m$.

Write $U_\alpha(x) = \sum_{n\geq 0} u_\alpha(n)x^n$ and $V_\alpha(x) = \sum_{n\geq 0} v_\alpha(n)x^n$. It follows from Remark 1 that

$$\frac{2V_r(x)}{1-x} = \frac{U_r(x)-1}{x} + \frac{1+x}{(1-x)^2}. \tag{7}$$

Write $R_\alpha(x)$ for the characteristic polynomial of the linear recurrence with constant coefficients *of least degree* satisfied by $v_\alpha(n)$ for $n \gg 0$. If the empirical observation above, that $\mathrm{rmp}_r(x)$ is not divisible by $x-1$, holds, then it follows from equation (7) that $R_r(x) = (x-1)\mathrm{rmp}_r(x)$.

4 A generalization.

A much more general result can be proved by exactly the same method. Let $p(x)$ and $q(x)$ be any complex polynomials, and let $b \geq 2$ be an integer. Define

$$F_{p,q,b,n}(x) = q(x)\prod_{i=0}^{n-1} p(x^{b^i}).$$

Let $\alpha = (\alpha_0,\ldots,\alpha_{m-1}) \in \mathbb{N}^m$. If

$$F_{p,q,b,n}(x) = \sum_{i\geq 0} c_i(n)x^i,$$

then set

$$u_{p,q,b,\alpha}(n) = \sum_k c_k(n)^{\alpha_0} c_{k+1}(n)^{\alpha_1} \cdots c_{k+m-1}(n)^{\alpha_{m-1}}.$$

Theorem 4. *For fixed p, q, b and α, the function $u_{p,q,b,\alpha}(n)$ has a rational generating function.*

Proof. Just as in the previous section we can express $u_{p,q,b,\alpha}(n+1)$ as a linear combination of $u_{p,q,b,\beta}(n)$'s. If $u_{p,q,b,\beta}(n)$ actually appears (i.e., has a nonzero coefficient), then call β a *child* of α. Successive children of α are called *descendants* of α. The only issue is whether α has only finitely many descendants. It is clear that all the descendants γ satisfy $|\alpha| = |\gamma|$.

In the previous section we observed that if α has length ℓ, then $\operatorname{spread}(u_\alpha) = 1 + \lceil \ell/2 \rceil$, so that we can take $s = 2$. In the present situation, we can assume that $p(0) \neq 0$ and $q(0) \neq 0$. Let $h = \deg p + \deg q$. If α has length ℓ, then

$$\operatorname{spread}(u_{p,q,b,\beta}) = 1 + \left\lfloor \frac{h}{b} \right\rfloor + \left\lfloor \frac{\ell - 1}{b} \right\rfloor.$$

(The precise formula is irrelevant. One just needs to see that if ℓ increases by b then the spread increases by 1.) Since $b \geq 2$, for sufficiently large ℓ we will have

$$\operatorname{spread}(u_{p,q,b,\beta}) \leq \ell, \tag{8}$$

showing that α has finitely many descendants. $\square$

Note that the above proof breaks down for Pascal's triangle, as it should. For then $b = 1$, so the inequality (8) does not hold for sufficiently large ℓ.

Remark 3. There is a straightforward multivariate generalization of Theorem 4. The polynomials $p(x)$ and $q(x)$ are replaced by complex multivariate polynomials $p(x_1, \ldots, x_d)$ and $q(x_1, \ldots, x_d)$, and b is replaced by d integers $b_1, \ldots, b_d \geq 2$. We define

$$F_{p,q,b,n}(x) = q(x_1, \ldots, x_d) \prod_{i=0}^{n-1} p(x_1^{b_1^i}, \ldots, x_d^{b_d^i}),$$

and the development proceeds as before. Details are omitted. As some random examples, extend the definition of $\operatorname{rmp}(\alpha)$ to $\operatorname{rmp}(p, q, \alpha, b)$, where $b = (b_1, \ldots, b_d)$. Then

$$\begin{aligned}
\operatorname{rmp}((1 + x_1 + x_2)^2, 1, (2), (2)) &= x^2 - 27x + 132, \\
\operatorname{rmp}((1 + x_1 + x_2)^2, 1, (3), (2)) &= x^3 - 67x + 1020x^2 - 4704, \\
\operatorname{rmp}((1 + x_1 + x_2)^2, 1, (2), (2, 3)) &= x^2 - 23x + 104, \\
\operatorname{rmp}((1 + x_1 + x_2)^2, 1, (3), (2, 3)) &= x^2 - 45x + 402, \\
\operatorname{rmp}((1 + x_1 + x_2)^2, 1, (4), (2, 3)) &= x^3 - 107x^2 + 3176x - 28320.
\end{aligned}$$

Compare with the univariate analogue $p(x) = (1 + x)^d$, where for instance

$$\begin{aligned}
\operatorname{rmp}((1 + x)^2, 1, (2), 2) &= (x - 2)(x - 8), \\
\operatorname{rmp}((1 + x)^2, 1, (3), 2) &= (x - 4)(x - 16), \\
\operatorname{rmp}((1 + x)^2, 1, (4), 2) &= (x - 2)(x - 8)(x - 32), \\
\operatorname{rmp}((1 + x)^3, 1, (2), 2) &= (x - 2)(x - 8)(x - 32), \\
\operatorname{rmp}((1 + x)^3, 1, (3), 2) &= (x - 2)(x - 8)(x - 32)(x - 128), \\
\operatorname{rmp}((1 + x)^3, 1, (4), 2) &= (x - 2)(x - 8)(x - 32)(x - 128)(x - 512).
\end{aligned}$$

The reason for this nice factorization is discussed in the next section.

5 A special case.

A natural problem is to say more about the recurrence (or equivalently its characteristic polynomial) satisfied by $u_{p,q,b,\alpha}$ in general, or at least in special situations. In this section we give one such result.

For simplicity we first consider the case $p(x) = (1+x)^3$, $q(x) = 1$, $b = 2$, and $\alpha = (r)$. We then state a more general result that is proved by exactly the same technique.

Theorem 5. *For $r \geq 1$ we have*

$$u_{(1+x)^3,1,2,r}(n) = \sum_{i=0}^{r} c_i 2^{(2i+1)n}$$

for certain rational constants c_i (depending on r).

Proof. The key to the proof is the simple and well-known identity

$$(1+x)(1+x^2)(1+x^4)\cdots\left(1+x^{2^{t-1}}\right) = \frac{1-x^{2^t}}{1-x}.$$

Thus

$$\begin{aligned}\prod_{j=0}^{t-1}\left(1+x^{2^j}\right)^3 &= \frac{\left(1-x^{2^t}\right)^3}{(1-x)^3} \\ &= \frac{1-3x^{2^t}+3x^{2\cdot 2^t}-x^{3\cdot 2^t}}{(1-x)^3}.\end{aligned}$$

Let us consider more generally the generating function

$$\begin{aligned}H_m(x) &= \frac{(1-x^m)^3}{(1-x)^3} \\ &= \frac{1-3x^m+3x^{2m}-x^{3m}}{(1-x)^3}.\end{aligned}$$

Now

$$\frac{x^p}{(1-x)^3} = \sum_{k\geq 0}\binom{k+2}{2}x^{p+k}.$$

Since $H_m(x)$ is a polynomial in x of degree $3(m-1)$, we get

$$\begin{aligned}H_m(x) &= \sum_{k=0}^{m}\binom{k+2}{2}x^k + \sum_{k=m}^{2m-1}\left[\binom{k+2}{2}-3\binom{k-m+2}{2}\right]x^k \\ &\quad + \sum_{k=2m}^{3m-1}\left[\binom{k+2}{2}-3\binom{k-m+2}{2}+3\binom{k-2m+2}{2}\right]x^k.\end{aligned}$$

Let

$$\begin{aligned}P(m) &= \sum_{k=0}^{m}\binom{k+2}{2}^r + \sum_{k=m}^{2m-1}\left[\binom{k+2}{2}-3\binom{k-m+2}{2}\right]^r \\ &\quad + \sum_{k=2m}^{3m-1}\left[\binom{k+2}{2}-3\binom{k-m+2}{2}+3\binom{k-2m+2}{2}\right]^r. \qquad (9)\end{aligned}$$

The binomial coefficient $\binom{k-jm+2}{2}$ is a polynomial in m of degree two. For any polynomial $Q(m)$ of degree d, the sum $\sum_{k=0}^{m} Q(k)$ is a polynomial of degree $d+1$, so the same is true of $\sum_{k=m}^{2m-1} Q(k)$, etc. Hence $P(m)$ is a polynomial of degree at most $2r+1$, say $P(m)=\sum_{i=0}^{2r+1} a_i m^i$. Then,

$$\begin{aligned} u_{(1+x)^3,1,2,r}(n) &= P(2^n) \\ &= \sum_{i=0}^{2r+1} a_i 2^{ni}. \end{aligned}$$

It remains to prove that $a_i=0$ if i is even. The coefficient of x^k in $H_m(x)$ is 0 for $k>3(m-1)$, so

$$\binom{k+2}{2} - 3\binom{k-m+2}{2} + 3\binom{k-2m+2}{2} - \binom{k-3m+2}{2} = 0 \tag{10}$$

for $k>3(m-1)$. The left-hand side is a polynomial in k and m. Since it is 0 for $k>3(m-1)$, it must be 0 as a polynomial in k and m.

In general, for integers $a<b<c$ and any function $f(i)$, we have

$$\sum_{i=a}^{b} f(i) + \sum_{i=b+1}^{c} f(i) = \sum_{i=a}^{c} f(i). \tag{11}$$

If we want to define $\sum_{i=a}^{b} f(i)$ for integers $a>b$ so that equation (11) is valid for *all* integers a,b,c, then we will have the identity

$$\sum_{i=a}^{b} f(i) = -\sum_{b+1}^{a-1} f(i).$$

It is easy to check (using the fact that if two univariate complex polynomials agree for infinitely many values, then they are the same polynomial) that if

$$G(m) = \sum_{i=A(m)}^{B(m)} Q(m)$$

for polynomials Q,A,B (so G is also a polynomial), then indeed we have

$$G(-m) = \sum_{i=A(-m)}^{B(-m)} Q(-m).$$

Write the polynomial $P(m)$ of equation (9) as $P_1(m)+P_2(m)+P_3(m)$, corresponding to the three sums. If we substitute $-m$ for m in

$$P_3(m) = \sum_{k=2m}^{3m-1} \left[\binom{k+2}{2} - 3\binom{k-m+2}{2} + 3\binom{k-2m+2}{2}\right]^r,$$

then we get

$$\begin{aligned} P_3(-m) &= \sum_{k=-2m}^{-3m+1} \left[\binom{k+2}{2} - 3\binom{k+m+2}{2} + 3\binom{k+2m+2}{2}\right]^r \\ &= -\sum_{k=-3m}^{-2m-1} \left[\binom{k+2}{2} - 3\binom{k+m+2}{2} + 3\binom{k+2m+2}{2}\right]^r \\ &= -\sum_{k=0}^{m-1} \left[\binom{k-3m+2}{2} - 3\binom{k-2m+2}{2} + 3\binom{k-m+2}{2}\right]^r. \end{aligned}$$

From equation (10) there follows

$$\begin{aligned} P_3(-m) &= -\sum_{k=0}^{m-1} \binom{k+2}{2}^r \\ &= -P_1(m). \end{aligned}$$

Hence $P_1(m) + P_3(m)$ is an odd polynomial, i.e.,

$$P_1(-m) + P_3(-m) = -(P_1(m) + P_3(m)),$$

so all powers of m appearing in this polynomial have odd exponents.

In exactly the same way, this time using equation (10) in the form

$$\binom{k+2}{2} - 3\binom{k-m+2}{2} = -3\binom{k-2m+2}{2} + \binom{k-3m+2}{2},$$

we obtain that $P_2(-m) = -P_2(m)$. Thus $P(-m) = -P(m)$, completing the proof. □

The reader who has followed the previous proof should have no trouble extending it to the following more general result. We only point out one possible subtlety: when d is even in the theorem below, the polynomial $(1-x^b)^d$ has an odd number of terms. Hence the analogue of the equation $P_i(-m) = -P_i(m)$ becomes $P_i(-m) = -(-1)^r P_i(m)$. Thus $P_i(m)$ is an even polynomial when d is even and r is odd.

Theorem 6. *(a) Let $b \geq 2$, $d \geq 1$, and $p(x) = (1 + x + \cdots + x^{b-1})^d$. For any $\alpha \in \mathbb{N}^m$ and $q(x) \in \mathbb{C}[x]$ we have*

$$u_{p,q,b,\alpha}(n) = \sum_{i=0}^{1+(d-1)|\alpha|} c_i b^{in},$$

where $c_i \in \mathbb{C}$.

(b) If $q(x) = 1$, $\alpha = (r)$ and either r is even or d is odd, then $c_i = 0$ when i is even.

(c) If $q(x) = 1$, $\alpha = (r)$, r is odd, and d is even, then $c_i = 0$ when i is odd.

We leave as an open problem to see to what extent Theorem 6 can be generalized.

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