The Smith Normal Form of a Matrix Associated with Young's Lattice

Tommy Wuxing Cai and Richard P. Stanley

Abstract. We prove a conjecture of Miller and Reiner on the Smith normal form of the operator DU associated with a differential poset for the special case of Young's lattice. Equivalently, this operator can be described as $\frac{\partial}{\partial p_1} p_1$ acting on homogeneous symmetric functions of degree n.

1. Introduction

Let R be a commutative ring with 1 and M an $m \times m$ matrix over R. We say that M has a *Smith normal form* (SNF) over R if there exist matrices $P, Q \in \mathrm{SL}(m, R)$ (so $\det P = \det Q = 1$) such that PMQ is a diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_m)$ with $d_i \mid d_{i+1}$ $(1 \le i \le m-1)$ in R. It is well-known that if R is an integral domain and if the SNF of M exists, then it is unique up to multiplication of each d_i by a unit u_i (with $u_1 \cdots u_m = 1$).

We see that Smith normal form is a refinement of the determinant, since $\det M = d_1 d_2 \cdots d_m$. In the case that R is a principal ideal domain (PID), it is well-known that all matrices in R have a Smith normal form. Not very much is known in general.

In this work we are interested in the ring $\mathbb{Z}[x]$ of integer polynomials in one variable. Since $\mathbb{Z}[x]$ is not a PID, a matrix over this ring need not have an SNF. For instance, it can be shown that the matrix $\begin{bmatrix} x & 0 \\ 0 & x+2 \end{bmatrix}$ does not have a Smith normal form over $\mathbb{Z}[x]$.

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We will use symmetric function notation and terminology from [**5**]. The ring Λ of symmetric functions has several standard $\mathbb{Z}$-bases: monomial symmetric functions m_λ, elementary symmetric functions e_λ, complete symmetric functions h_λ and Schur functions s_λ. The power sum symmetric functions p_λ form a $\mathbb{Q}$-basis of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The ring $\Lambda_{\mathbb{Q}}$ is a graded $\mathbb{Q}$-algebra: $\Lambda_{\mathbb{Q}} = \bigoplus_{n=0}^{\infty} \Lambda_{\mathbb{Q}}^n$, where $\Lambda_{\mathbb{Q}}^n$ is the vector space spanned over $\mathbb{Q}$ by $\{s_\lambda : \lambda \vdash n\}$. Similarly, Λ is a graded ring.

Regard elements of $\Lambda_{\mathbb{Q}}^n$ as polynomials in the p_j's. Define a linear map T_1 on $\Lambda_{\mathbb{Q}}^n$ by

$$T_1(v) = \frac{\partial}{\partial p_1}(p_1 v), \quad v \in \Lambda_{\mathbb{Q}}^n. \tag{1.1}$$

REMARK 1.1. Note that the power sum p_λ is an eigenvector of $T_1(v)$ with corresponding eigenvalue $1 + m_1(\lambda)$, where $m_1(\lambda)$ denotes the number of 1's in λ.

Denote by $M = M_1^{(n)}$ the matrix of T_1 with respect to the basis $\{s_\lambda : \lambda \vdash n\}$ (arranged in, say, lexicographic order). It is known and easy to see that M is an integer symmetric matrix of order $p(n)$, the number of partitions of n. Let $\lambda(n) = (p(n) - p(n-1), \ldots, p(2) - p(1), p(1))$, so $\lambda(n)$ is a partition of $p(n)$. Let the conjugate of $\lambda(n)$ be $\lambda(n)' = (j_{p(n)-p(n-1)}, \ldots, j_2, j_1)$ (so $j_{p(n)-p(n-1)} = n$). We prove the following result.

THEOREM 1.2. *Let $\alpha_k(x) = a_1(x)a_2(x)\cdots a_k(x)$ with $a_i(x) = i + x$ ($i = 1, 2, \ldots, n-1$) and $a_n(x) = n+1+x$. There exist $P(x), Q(x) \in \mathrm{SL}(p(n), \mathbb{Z}[x])$ such that $P(x)(M + xI_{p(n)})Q(x)$ is the following diagonal matrix*

$$\mathrm{diag}(1, 1, \ldots, 1, \alpha_{j_1}(x), \alpha_{j_2}(x), \ldots, \alpha_{j_{p(n)-p(n-1)}}(x)). \tag{1.2}$$

As an example of Theorem 1.2, let us consider the case $n = 6$. First $\lambda(6) = (4, 2, 2, 1, 1, 1)$ and $\lambda(6)' = (6, 3, 1, 1)$. Hence the diagonal entries of an SNF of $M + xI$ are seven 1's followed by

$$1 + x, 1 + x, (1 + x)(2 + x)(3 + x), (1 + x)(2 + x)(3 + x)(4 + x)(5 + x)(7 + x).$$

In general, the number of diagonal entries equal to 1 is $p(n-1)$.

The origin of Theorem 1.2 is as follows. Let P be a differential poset, as defined in [**3**] or [**4**, §3.21], with levels $P_0, P_1, \ldots$. Let $\mathbb{Z}[x]P_n$ denote the free $\mathbb{Z}[x]$-module with basis P_n. Let U, D be the up and down operators associated with P. Miller and Reiner [**2**] conjectured a certain Smith normal form of the operator $DU + xI \colon \mathbb{Z}[x]P_n \to \mathbb{Z}[x]P_n$. Our result is equivalent to the conjecture of Miller and Reiner for the special case of Young's lattice Y. Our result also specializes to a proof of Miller's Conjecture 14 in [**1**]. After proving the theorem, we state a conjecture which generalizes it. It seems natural to try to generalize our work to the differential poset Y^r for $r \geq 2$, but we have been unable to do this.

2. The proof of the theorem

Instead of Schur functions, we consider the matrix with respect to the complete symmetric functions $\{h_\lambda \colon \lambda \vdash n\}$. Since $\{s_\lambda \colon \lambda \vdash n\}$ and $\{h_\lambda \colon \lambda \vdash n\}$ are both $\mathbb{Z}$-bases for Λ^n, the Smith normal form does not change when we switch to the h_λ basis. We introduce a new ordering on the set $\mathcal{P}_n$ of all partitions of n. The matrix A with respect to this new ordering turns out to be much easier to manipulate than the original matrix. In fact, we show that $A + xI_{p(n)}$ can be turned into an upper triangular matrix after some simple row operations. Then we use more row/column operations to cancel the non-diagonal elements. The resulting diagonal matrix is the SNF that we are looking for.

From now on, we fix a positive integer n. The case $n = 1$ is trivial, so we assume $n \geq 2$.

2.1. A new ordering on partitions.

Definition 2.1. Let $\lambda = (\lambda_1, \lambda_2 \dots, \lambda_i, 1)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq 1$ ($i \geq 1$). We define $\lambda^+ = (\lambda_1 + 1, \lambda_2, \dots, \lambda_i)$ and write $\lambda^+ \nwarrow \lambda$. We call a partition λ *initial* if there is no μ such that $\mu^+ = \lambda$, i.e., $\lambda_1 = \lambda_2$. We call a partition μ *terminal* if μ^+ is not well-defined, i.e., $m_1(\mu) = 0$, where $m_i(\lambda)$ denotes the number of parts of λ equal to i.

For a sequence $\lambda^0, \lambda^1, \lambda^2, \dots, \lambda^t$ of partitions, we write

$$\lambda^t \nwarrow \lambda^{t-1} \nwarrow \cdots \nwarrow \lambda^0 \tag{2.1}$$

and call it a *rising string* of length t if $\lambda^{i+1} \nwarrow \lambda^i$, i.e., $\lambda^{i+1} = (\lambda^i)^+$, for $0 \leq i \leq t-1$.

In equation (2.1), if λ^0 is initial and λ^t is terminal then we call (2.1) a *full* (rising) string. Moreover, we say that λ^0 is the initial element of the string and λ^t is the terminal element of the string. One cannot add a partition to a full string to make it longer.

A partition $\mu = (\mu_1, \mu_2, \dots, \mu_i)$ with $\mu_1 = \mu_2 \geq \cdots \geq \mu_i \geq 2$ is not in any rising string of nonzero length. We also call this μ a (point) string of length 0; it is both initial and terminal.

Note that every partition μ is in exactly one full string; we denote the terminal element of this string by $T(\mu)$. Thus $T(\lambda) = T(\mu)$ if and only if λ, μ are in the same string.

Definition 2.2. We define a total ordering $\preceq$ on the set of partitions of n as follows.

(1) For two partitions in the rising string (2.1), we define $\lambda^i \preceq \lambda^j$ for $i \leq j$.
(2) For λ, μ in different full strings, we write $\lambda \preceq \mu$ if $T(\lambda) \leq_L T(\mu)$, i.e., $T(\lambda)$ is lexicographically less than $T(\mu)$.

We arrange the partitions in $\mathcal{P}_n$ from largest to smallest according to the following order:

$$\lambda^{11}, \lambda^{12}, \ldots, \lambda^{1i_1}; \lambda^{21}, \lambda^{22}, \ldots, \lambda^{2i_2}; \ldots; \lambda^{t1}, \lambda^{t2}, \ldots, \lambda^{ti_t}, \tag{2.2}$$

where $\lambda^{j1} \nwarrow \lambda^{j2} \nwarrow \cdots \nwarrow \lambda^{ji_j}$ is the jth full string of partitions of n, and we use semicolons to separate neighboring strings. It's easy to see that $\lambda^{11} = (n)$, $i_1 = n$, $\lambda^{1i_1} = (1^n)$, and t is the number of terminal elements of n, which is equal to the number of partitions of n with no part equal to 1, viz., $p(n) - p(n-1)$. In fact, these cardinalities $i_1, i_2, \ldots, i_t$ of strings can be expressed explicitly. We will see this point after the following example.

Example 2.3. The following is the list of partitions of 6:

$$6,\ 51,\ 41^2,\ 31^3,\ 21^4,\ 1^6;\ 42,\ 321,\ 2^21^2;\ 3^2;\ 2^3.$$

The eigenvalues of M arranged in accordance with this order are by Remark 1.1 as follows:

$$1, 2, 3, 4, 5, 7; 1, 2, 3; 1; 1.$$

On the other hand, we can arrange the eigenvalues of M in the following form:

$$\begin{aligned} p(6) - p(5) &= 4 : 1\ \ 1\ \ 1\ \ 1 \\ p(5) - p(4) &= 2 : 2\ \ 2 \\ p(4) - p(3) &= 2 : 3\ \ 3 \\ p(3) - p(2) &= 1 : 4 \\ p(2) - p(1) &= 1 : 5 \\ p(1) &= 1 : 7. \end{aligned}$$

We see that the eigenvalues (to the right side of the colons) form a tableau (with constant rows and increasing columns) of shape $\lambda(6) = (4, 2, 2, 1, 1, 1)$.

Notice that the eigenvalues associated with the first string are 1,2,3,4,5, and 7 and they form the first column of the above tableau. In fact, the eigenvalues corresponding to every string form a column of the tableau. This is not a coincidence, but rather because the eigenvalues corresponding to a string form a sequence of consecutive integers starting from 1: $1, 2, 3 \ldots, i$, except there is a gap for those corresponding to the first string. Therefore the cardinality of a string is the length of a column of the tableau.

We can easily formalize this argument to prove the following.

Lemma 2.4. *Rearrange the cardinalities of the strings $i_1, i_2, \ldots, i_t$ in (2.2) in weakly decreasing order: $J = (j_t, \ldots, j_2, j_1)$. Then J is exactly the conjugate of the partition $\lambda(n) = (p(n) - p(n-1), \ldots, p(2) - p(1), p(1))$ as defined in the introduction.*

Note that $i_1, i_2, \ldots, i_t$ are not necessarily in weakly decreasing order.

2.2. The transition matrix with respect to the new ordering. Let $A = (a_{\mu\lambda})$ be the $p(n) \times p(n)$ matrix of the action of $\frac{\partial}{\partial p_1} p_1$ on the basis h_λ, i.e., $\frac{\partial}{\partial p_1} p_1 \cdot h_\lambda = \sum_\mu a_{\mu\lambda} h_\mu$. Here we use the total ordering $\preceq$, with the greatest partition (n) corresponding to the first row and column. Recall that the notation $\frac{\partial}{\partial p_1} v$ means that we write v as a polynomial in the power sums $p_1, p_2, \ldots$ and regard each p_i, $i \geq 2$, as a constant when we differentiate.

LEMMA 2.5. *The matrix $A = (a_{\mu\lambda})$ has the following properties.*

(1) $a_{\mu\lambda} \neq 0$ *only if* $\lambda \nwarrow \mu$, $\mu = \lambda$ *or* $T(\mu) > T(\lambda)$ *(greater but not equal in dominance order).*
(2) $a_{\mu\lambda} = 1$ *if* $\lambda \nwarrow \mu$;
$a_{\lambda\lambda} = m_1(\lambda) + 1$;
$m_1(\mu)$ *equals* $m_1(\lambda) + 1$ *or* $m_1(\lambda) + 2$ *if* $a_{\mu\lambda} \neq 0$ *and* $T(\mu) > T(\lambda)$.

PROOF. Let $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k} \cdots h_{\lambda_i} h_1^j$, with $\lambda_1 \geq \cdots \geq \lambda_i \geq 2$. From e.g. the basic identity

$$\sum_{m \geq 0} h_m t^m = \exp \sum_{i \geq 1} p_i \frac{t^i}{i}$$

it follows that $\frac{\partial h_m}{\partial p_1} = h_{m-1}$. Then (since $h_1 = p_1$)

$$\begin{aligned} \frac{\partial}{\partial p_1} p_1 \cdot h_\lambda &= \frac{\partial}{\partial p_1} h_{\lambda_1} \cdots h_{\lambda_k} \cdots h_{\lambda_i} h_1^{j+1} \\ &= (j+1) h_\lambda + \sum_{k=1}^{i} h_{\lambda_1} \cdots h_{\lambda_k - 1} \cdots h_{\lambda_i} h_1^{j+1}. \end{aligned}$$

We see that $a_{\lambda\lambda} = m_1(\lambda) + 1$. Let μ be the partition such that

$$h_{\lambda_1} \cdots h_{\lambda_k - 1} \cdots h_{\lambda_i} h_1^{j+1} = h_\mu, \quad 1 \leq k \leq i.$$

In the case that $k = 1$ and λ is not an initial element, i.e., $\lambda_1 > \lambda_2$, then $\mu = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_i, 1^{j+1})$ and thus $\lambda \nwarrow \mu$.

In the other cases, μ is of the form $\mu = (\lambda_1, \lambda_2, \ldots, \lambda_r - 1, \ldots, \lambda_i, 1^{j+1})$ for some $1 < r \leq i$. Hence $T(\mu) = (\lambda_1 + j + 1, \lambda_2, \ldots, \lambda_r - 1, \ldots, \lambda_i)$ or $T(\mu) = (\lambda_1 + j + 2, \lambda_2, \ldots, \lambda_{i-1})$ (when $i = r$ and $\lambda_r = 2$). Note that $T(\lambda) = (\lambda_1 + j, \lambda_2, \ldots, \lambda_i)$. We see that $T(\mu) > T(\lambda)$ (in dominance order). This proves (1) and (2). $\square$

In the following we set $a'_i = i$ for $i = 1, 2, \ldots, n-1$ and $a'_n = n + 1$. We separate rows and columns corresponding to different strings and write A in the block matrix form $A = (A_{kl})_{t \times t}$. We have the following properties of these A_{kl}'s by Lemma 2.5.

(1) For $k > l$, $A_{kl} = 0$.
(2) A_{kk} is an $i_k \times i_k$ lower triangular matrix. Its diagonal entries are $a'_1, a'_2, \ldots, a'_{i_k}$; the entries on the line right below and parallel to the diagonal are all 1 and all the other entries are 0.

It looks as follows:

$$A = \begin{bmatrix}
a'_1 & & & & b^{12}_{11} & b^{12}_{12} & \cdots & b^{12}_{1i_2} & \cdots & b^{1t}_{11} & b^{1t}_{12} & \cdots & b^{1t}_{1i_t} \\
1 & a'_2 & & & b^{12}_{21} & b^{12}_{22} & \cdots & b^{12}_{2i_2} & \cdots & b^{1t}_{21} & b^{1t}_{22} & \cdots & b^{1t}_{2i_t} \\
 & \ddots & \ddots & & \cdots & & & & & \cdots & & & \\
 & & 1 & a'_{i_1} & b^{12}_{i_1 1} & b^{12}_{i_1 2} & \cdots & b^{12}_{i_1 i_2} & \cdots & b^{1t}_{i_1 1} & b^{1t}_{i_1 2} & \cdots & b^{1t}_{i_1 i_t} \\
 & & & & a'_1 & & & & \cdots & b^{2t}_{11} & b^{2t}_{12} & \cdots & b^{2t}_{1i_t} \\
 & & & & 1 & a'_2 & & & \cdots & b^{2t}_{21} & b^{2t}_{22} & \cdots & b^{2t}_{2i_t} \\
 & & & & & \ddots & \ddots & & & \cdots & & & \\
 & & & & & & 1 & a'_{i_2} & \cdots & b^{2t}_{i_2 1} & b^{2t}_{i_2 2} & \cdots & b^{2t}_{i_2 i_t} \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & a'_1 & & & \\
 & & & & & & & & & 1 & a'_2 & & \\
 & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & 1 & a'_{i_t}
\end{bmatrix}.$$

Furthermore, we have

$$b^{kl}_{ij} = 0 \text{ if } i \leq j, \tag{2.3}$$

i.e., A_{kl} is a strict lower triangular matrix if $k < l$. The reason is that $b^{kl}_{ij} = a_{\lambda^{ki}\lambda^{lj}}$, and if it is nonzero, then we have the following by Lemma 2.5:

$$i - 1 = m_1(\lambda^{ki}) \geq m_1(\lambda^{lj}) + 1 = j - 1 + 1 = j.$$

(Here we use that $m_1(\lambda^{pr}) = r - 1$ for $\lambda^{pr} \neq (1^n)$. It is possible that $\lambda^{ki} = (1^n)$, but then again $i = n \geq i_l + 1 \geq j + 1$.)

Now we replace the a'_i on the diagonal of A with an arbitrary $f_i = f_i(x) \in \mathbb{Z}[x]$ and change A into a matrix $A(x)$ with entries in $\mathbb{Z}[x]$. We will apply some row/column operations to $A(x)$ and transform it into an SNF in $\mathbb{Z}[x]$. (Some of these operations depend on the f_i's.) This is the same as to say that there are $P_1(x), Q_1(x) \in \mathrm{SL}(p(n), \mathbb{Z}[x])$ such that $P_1(x)A(x)Q_1(x)$ is an SNF in $\mathbb{Z}[x]$.

Notice that the original matrix M is equal to PAP^{-1} for some $P \in \mathrm{SL}(p(n), \mathbb{Z})$. If we take $f_i = a'_i + x$ (which is $a_i(x)$ in our theorem) in the beginning, then $A(x) = A + xI_{p(n)}$, and thus the SNF $P_1(x)A(x)Q_1(x)$ is equal to $P_1(x)(P^{-1}MP + xI_{p(n)})Q_1(x) = P_1(x)P^{-1}(M + xI_{p(n)})PQ_1(x)$, as desired.

2.3. Transformation into an upper triangular matrix. If we use horizontal lines to separate rows of $A(x)$ corresponding to different full strings, then $A(x)$ is partitioned into t submatrices. We see that we should consider a matrix of the following form:

$$B = \begin{bmatrix} f_1 & & & & b_{11} & b_{12} & \cdots & b_{1m} \\ 1 & f_2 & & & b_{21} & b_{22} & \cdots & b_{2m} \\ & \ddots & \ddots & & \cdots & & & \\ & & 1 & f_s & b_{s1} & b_{s2} & \cdots & b_{sm} \end{bmatrix}.$$

We can apply row operations to B and transform it first into

$$B_1 = \begin{bmatrix} 1 & f_2 & & & b_{21} & b_{22} & \cdots & b_{2m} \\ & \ddots & \ddots & & \cdots & & & \\ & & 1 & f_s & b_{s1} & b_{s2} & \cdots & b_{sm} \\ f_1 & & & & b_{11} & b_{12} & \cdots & b_{1m} \end{bmatrix},$$

and then into

$$B_2 = \begin{bmatrix} 1 & f_2 & & & b_{21} & b_{22} & \cdots & b_{2m} \\ & \ddots & \ddots & & \cdots & & & \\ & & 1 & f_s & b_{s1} & b_{s2} & \cdots & b_{sm} \\ 0 & \cdots & 0 & \alpha & \beta_1 & \beta_2 & \cdots & \beta_m \end{bmatrix},$$

with $\alpha = f_1 \cdots f_s$ and

$$(-1)^{s-1}\beta_j = b_{1j} - f_1 b_{2j} + f_1 f_2 b_{3j} + \cdots + (-1)^{s-1} f_1 \cdots f_{s-1} b_{sj}.$$

Apply this process to the t submatrices of $A(x)$. We turn $A(x)$ into the matrix

$$A_1(x) = \begin{bmatrix} 1 & f_2 & & & b^{12}_{21} & b^{12}_{22} & \cdots & b^{12}_{2i_2} & \cdots & b^{1t}_{21} & b^{1t}_{22} & \cdots & b^{1t}_{2i_t} \\ & \ddots & \ddots & & \cdots & & & & & \cdots & & & \\ & & 1 & f_{i_1} & b^{12}_{i_1 1} & b^{12}_{i_1 2} & \cdots & b^{12}_{i_1 i_2} & \cdots & b^{1t}_{i_1 1} & b^{1t}_{i_1 2} & \cdots & b^{1t}_{i_1 i_t} \\ & & & \alpha_{i_1} & \beta^{12}_1 & \beta^{12}_2 & \cdots & \beta^{12}_{i_2} & \cdots & \beta^{1t}_1 & \beta^{1t}_2 & \cdots & \beta^{1t}_{i_t} \\ & & & & 1 & f_2 & & & \cdots & b^{2t}_{21} & b^{2t}_{22} & \cdots & b^{2t}_{2i_t} \\ & & & & & \ddots & \ddots & & & \cdots & & & \\ & & & & & & 1 & f_{i_2} & \cdots & b^{2t}_{i_2 1} & b^{2t}_{i_2 2} & \cdots & b^{2t}_{i_2 i_t} \\ & & & & & & & \alpha_{i_2} & \cdots & \beta^{2t}_1 & \beta^{2t}_2 & \cdots & \beta^{2t}_{i_t} \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & 1 & f_2 & & \\ & & & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & & & 1 & f_{i_t} \\ & & & & & & & & & & & & \alpha_{i_t} \end{bmatrix},$$

where $\alpha_k = f_1 f_2 \cdots f_k$ and

$$(-1)^{i_k-1}\beta^{kl}_j = b^{kl}_{1j} - f_1 b^{kl}_{2j} + f_1 f_2 b^{kl}_{3j} + \cdots + (-1)^{i_k-1} f_1 f_2 \cdots f_{i_k-1} b^{kl}_{i_k j}.$$

Recalling that $b_{ij}^{kl} = 0$ for $i \leq j$ (see (2.3)), we find that

$$f_1 f_2 \cdots f_j \mid \beta_j^{kl}, \text{ and as a special case, } \alpha_{i_l} \mid \beta_{i_l}^{kl}. \tag{2.4}$$

This property is crucial for later cancellation.

Remark 2.6. Next we will show that we can cancel the nondiagonal entries without altering the diagonal. Then by the definition of α_k, we see that the matrix $A(x)$ has the Smith normal form

$$\operatorname{diag}(1, 1, \ldots, 1, \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_t}),$$

where $(j_1, \ldots, j_t)$ is the rearrangement of $i_1, \ldots, i_t$ in weakly increasing order. Combining Lemma 2.4, everything in our theorem is clear now.

2.4. The cancellation of the nondiagonal entries. Now we want to cancel the nondiagonal elements, completing the proof. For those nonzero elements above the diagonal, we can do the following.

(C1) First apply column operations to cancel the entries on the rows with diagonal elements equal to 1 (starting from the first row).

(C2) Then apply row operations to cancel the entries on the columns with diagonal elements equal to 1.

The matrix turns into the following:

$$A_2(x) = \begin{bmatrix}
1 & & & & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
& \ddots & \ddots & & \cdots & & & & & \cdots & & & \\
& & 1 & & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
& & & \alpha_{i_1} & 0 & 0 & \cdots & \beta^{12} & \cdots & 0 & 0 & \cdots & \beta^{1t} \\
& & & & 1 & & & & & 0 & 0 & \cdots & 0 \\
& & & & & \ddots & \ddots & & & \cdots & & & \\
& & & & & & 1 & & & 0 & 0 & \cdots & 0 \\
& & & & & & & \alpha_{i_2} & & 0 & 0 & \cdots & \beta^{2t} \\
& & & & & & & & \ddots & & & & \\
& & & & & & & & & 1 & & & \\
& & & & & & & & & & 1 & & \\
& & & & & & & & & & & \ddots & \\
& & & & & & & & & & & & \alpha_{i_t}
\end{bmatrix}.$$

The only entries we cannot cancel in (C1) and (C2) are those on the intersection of rows and columns with α_{i_l}'s, i.e., the β^{kl}'s in A_2(x). If we can prove that each β^{kl} is a multiple of α_{i_l}, then we can apply row operations to cancel all those β^{kl}'s, and we are done.

To see this, let us first go back to $A_1(x)$. We know that at the beginning $\beta_{i_l}^{kl}$ was a multiple of α_{i_l} by (2.4). Then this entry was changed to β^{kl} after we applied (C1) and (C2). More precisely, (C1) changed it but (C2) did not. If we look more closely, we find that terms were added to $\beta_{i_l}^{kl}$ only when we were doing the column operations to cancel the nonzero entries below this entry $\beta_{i_l}^{kl}$. We claim that each term which was added to this entry was actually a multiple of α_{i_l}. Thus the new term β^{kl} in $A_2(x)$ is still a multiple of α_{i_l}.

Now let us prove our claim. For simplicity we consider only the entry $\beta_{i_t}^{1t}$. The general case can be treated similarly. The entries which were below this $\beta_{i_t}^{1t}$ and which were canceled in (C1) were $b_{ci_t}^{kt}$ $(2 \le k \le t-1,\ 2 \le c \le i_k)$ together with f_{i_t} which was right above α_{i_t}.

(a) If $b_{ci_t}^{kt}$ was nonzero, then $c \ge i_t + 1$ by (2.3). To cancel this $b_{ci_t}^{kt}$, we added $-b_{ci_t}^{kt}$ times the $\lambda^{k,(c-1)}$ column (i.e., the column indexed by $\lambda^{k,(c-1)}$) to the λ^{ti_t} column. Thus $-b_{ci_t}^{kt}\beta_{c-1}^{1k}$ was added to $\beta_{i_t}^{1t}$. By the fact (2.4), $f_1 \cdots f_{c-1} \mid \beta_{c-1}^{1k}$. But $c - 1 \ge i_t$, so this term added to $\beta_{i_t}^{1t}$ did have $f_1 \cdots f_{i_t} = \alpha_{i_t}$ as a factor.

(b) To cancel f_{i_t}, we added $-f_{i_t}$ times the λ^{t,i_t-1} column to the λ^{ti_t} column. This added $-f_{i_t}\beta_{i_t-1}^{1t}$ to $\beta_{i_t}^{1t}$. But again $f_1 \cdots f_{i_t-1} \mid \beta_{i_t-1}^{1t}$ by (2.4); we see that this term added is a multiple of α_{i_t}. $\square$

3. A conjecture.

We conjecture that our theorem can be generalized to the action $k\frac{\partial}{\partial p_k}p_k$ for $k \ge 1$.

Conjecture 3.1. Let $M_k^{(n)}$ be the matrix of the map $k\frac{\partial}{\partial p_k}p_k$ with respect to an integral basis for homogeneous symmetric functions of degree n. Then there exists $P(x), Q(x) \in \mathrm{SL}(p(n), \mathbb{Z}[x])$ such that $P(x)(M_k^{(n)} + xI_{p(n)})Q(x)$ is the diagonal matrix $\mathrm{diag}(f_1(x), \ldots, f_{p(n)}(x))$, where $f_i(x)$ may be described as follows. Let $\mathcal{M}$ be the multiset of all numbers $m_k(\lambda)$ for $\lambda \vdash n$. First, $f_{p(n)}(x)$ is a product of factors $x + k(a_i + 1)$ where the a_i's are the *distinct* elements of $\mathcal{M}$. Then $f_{p(n)-1}(x)$ is a product of factors $x + k(b_i + 1)$ where the b_i's are the remaining *distinct* elements of $\mathcal{M}$, etc. (After a while we will have exhausted all the elements of $\mathcal{M}$. The remaining diagonal elements are the empty product 1.)

We can prove the following special case of the above conjecture. The proof is based on the result that for a partition $\lambda \vdash n$ there is at most one k-border strip if and only if $k > n/2$, though we omit the details here.

Proposition 3.2. *If $k > n/2$, then an SNF of $M_k^{(n)} + xI_{p(n)}$ over $\mathbb{Z}[x]$ is given by*

$$\mathrm{diag}(1, \ldots, 1, x + k, \ldots, x + k, (x + k)(x + 2k), \ldots, (x + k)(x + 2k)),$$

where there are $p(n - k)$ 1's and $p(n - k)$ $(x + k)(x + 2k)$'s.

Thus it is known that Conjecture 3.1 is true for $k = 1$ or $k > n/2$.

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School of Sciences, SCUT, Guangzhou 510640, China
E-mail address: caiwx@scut.edu.cn

Department of mathematics, M.I.T., Cambridge, MA, USA
E-mail address: rstan@math.mit.edu