PARKING FUNCTIONS

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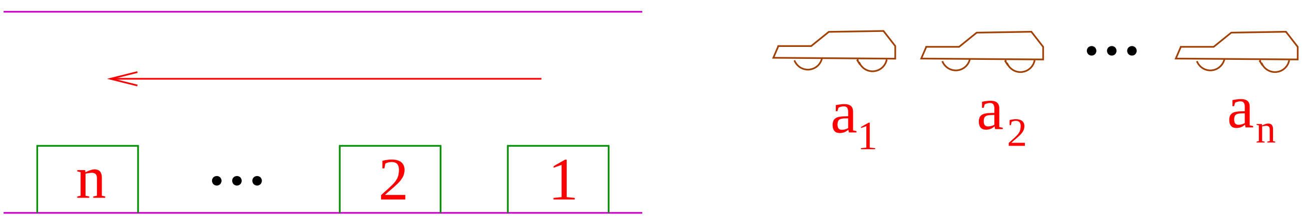
ENUMERATION OF PARKING FUNCTIONS



Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

$n = 2:$ 11 12 21

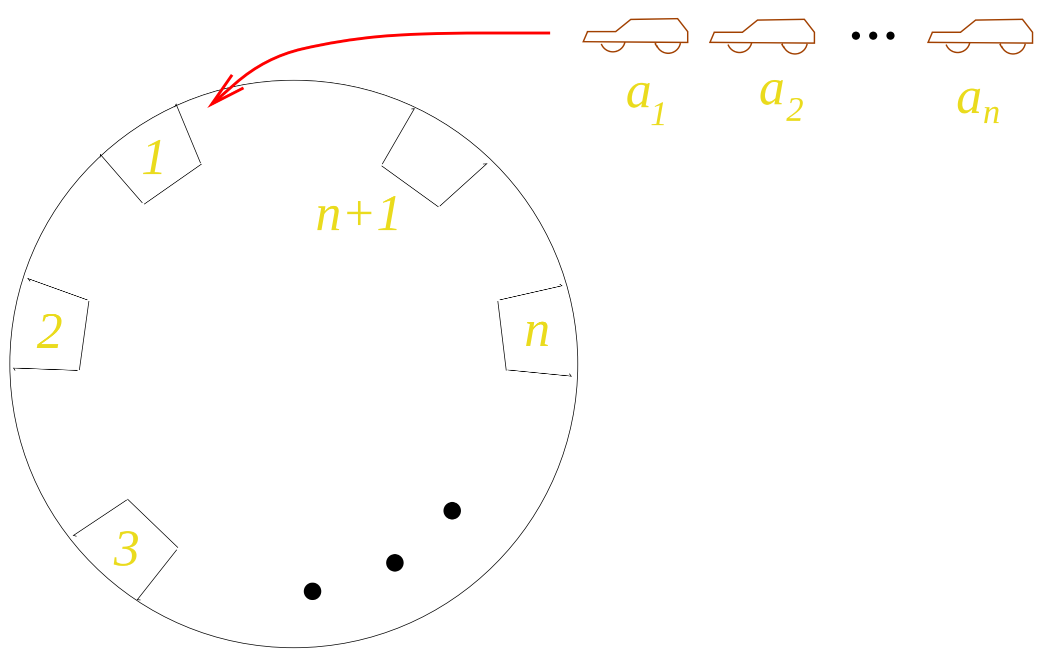
$n = 3:$ 111 112 121 211 113 131 311 122
212 221 123 132 213 231 312 321

Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of α. Then α is a parking function if and only $b_i \leq i$.

Corollary. *Every permutation of the entries of a parking function is also a parking function.*

Theorem (Pyke, 1959; Konheim and Weiss, 1966). *Let* $f(n)$ *be the number of parking functions of length* n*. Then* $f(n) = (n+1)^{n-1}$.

Proof (Pollak, c. 1974). Add an additional space $n+1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.



Now all cars can park, and there will be one empty space. α is a parking function if and only if the empty space is $n+1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

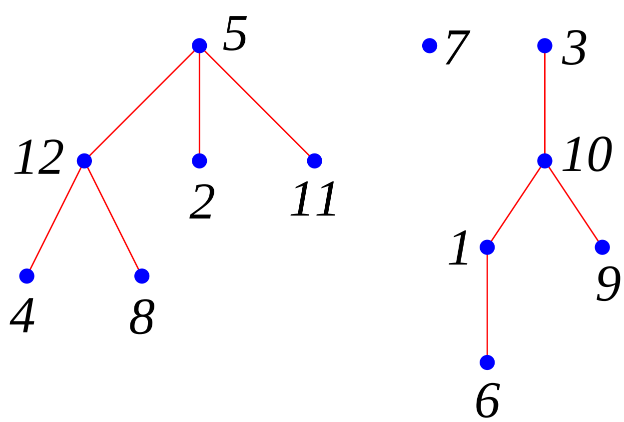
$$(a_1+i, a_2+i, \ldots a_n+i) \text{ (modulo } n+1)$$

is a parking function, so

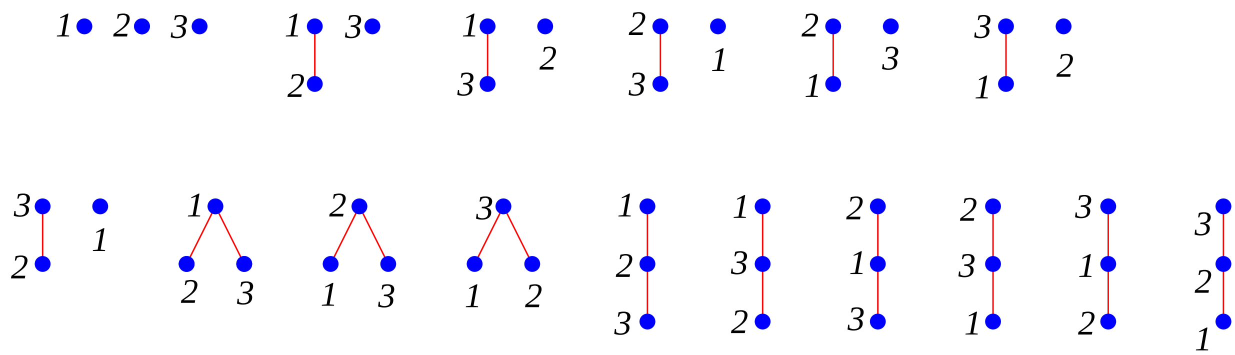
$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.$$

FOREST INVERSIONS

Let F be a rooted forest on the vertex set $\{1, \ldots, n\}$.

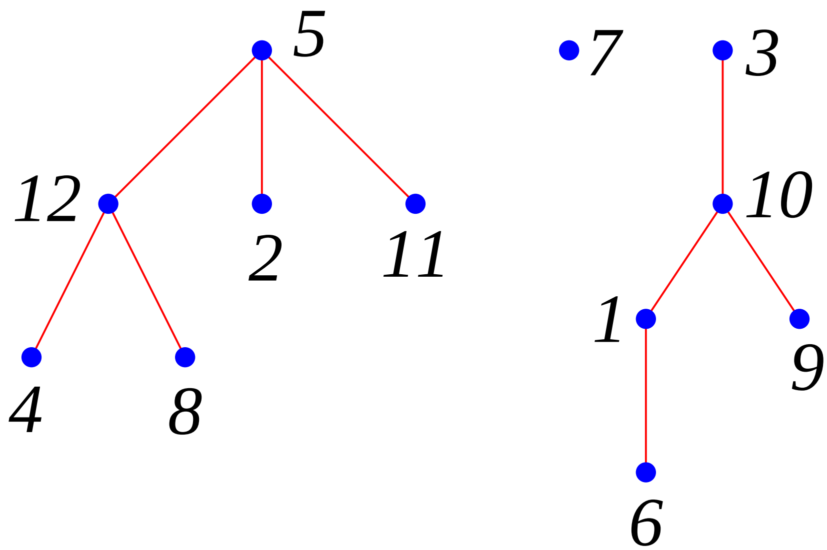


THEOREM (Sylvester-Borchardt-Cayley). *The number of such forests is* $(n+1)^{n-1}$.



An **inversion** in F is a pair (i, j) so that $i > j$ and i lies on the path from j to the root.

$$\text{inv}(F) = \#(\text{inversions of } F)$$



Inversions: $(5,4),\ (5,2),\ (12,4),\ (12,8)$

$(3,1),\ (10,1),\ (10,6),\ (10,9)$

$$\text{inv}(F) = 8$$

Let

$$I_n(q) = \sum_F q^{\mathrm{inv}(F)},$$

summed over all forests F with vertex set $\{1, \ldots, n\}$. E.g.,

$$\begin{aligned} I_1(q) &= 1 \\ I_2(q) &= 2 + q \\ I_3(q) &= 6 + 6q + 3q^2 + q^3 \end{aligned}$$

Theorem (Mallows-Riordan 1968, Gessel-Wang 1979) *We have*

$$I_n(1+q) = \sum_G q^{e(G)-n},$$

where G ranges over all connected graphs (without loops or multiple edges) on $n+1$ labelled vertices, and where $e(G)$ denotes the number of edges of G.

Corollary.

$$\sum_{n\geq 0} I_n(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}}$$

$$\sum_{n\geq 1} I_n(q)(q-1)^{n-1} \frac{x^n}{n!} = \log \sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}$$

Theorem (Kreweras, 1980) *We have*

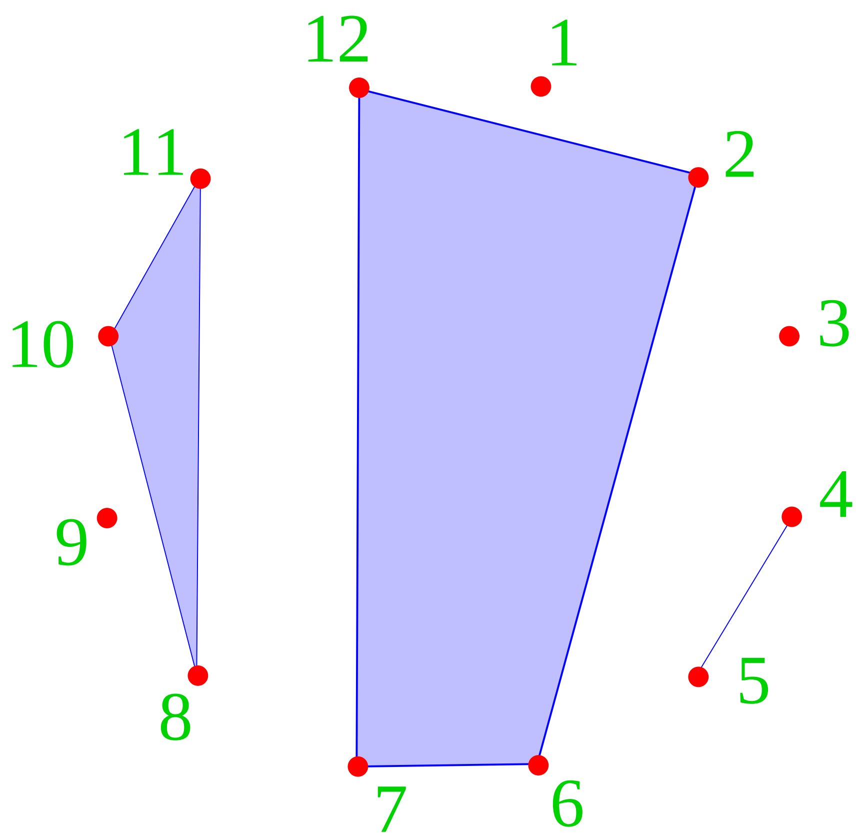
$$q^{\binom{n}{2}} I_n(1/q) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n},$$

where $(a_1, \ldots, a_n)$ *ranges over all parking functions of length* n*.*

NONCROSSING PARTITIONS

A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

$$a < b < c < d,\ a, c \in B_i,\ b, d \in B_j \Rightarrow i = j.$$



Theorem (H. W. Becker, 1948–49) *The number of noncrossing partitions of* $\{1,\ldots,n\}$ *is the* **Catalan number**

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence

$$\pi_0, \pi_1, \pi_2, \ldots, \pi_n$$

of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one. (Hence π_i has exactly $n+1-i$ blocks.)

$1-2-3-4-5 \quad 1-25-3-4 \quad 1-25-34$

$$125-34 \quad 12345$$

Define:

$$\mathbf{\min B} = \text{least element of } B$$

$$\mathbf{j < B} \;:\; j < k \;\; \forall k \in B.$$

Suppose π_i is obtained from π_{i-1} by merging together blocks B and B', with $\min B < \min B'$. Define

$$\begin{aligned} \mathbf{\Lambda_i(\mathfrak{m})} &= \max\{j \in B \;:\; j < B'\} \\ \mathbf{\Lambda(\mathfrak{m})} &= (\Lambda_1(\mathfrak{m}), \ldots, \Lambda_n(\mathfrak{m})). \end{aligned}$$

For above example:

$$1-2-3-4-5 \;\; 1-25-3-4 \;\; 1-25-34$$

$$125-34 \;\; 12345$$

we have

$$\Lambda(\mathfrak{m}) = (2, 3, 1, 2).$$

Theorem. Λ *is a bijection between the maximal chains of noncrossing partitions of* $\{1, \ldots, n+1\}$ *and parking functions of length* n.

Corollary (Kreweras, 1972) *The number of maximal chains of noncrossing partitions of* $\{1, \ldots, n+1\}$ *is*

$$(n+1)^{n-1}.$$

Is there a connection with Voiculescu's theory of free probability?

THE SHI ARRANGEMENT

Braid arrangement $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \quad 1 \leq i < j \leq n,$$

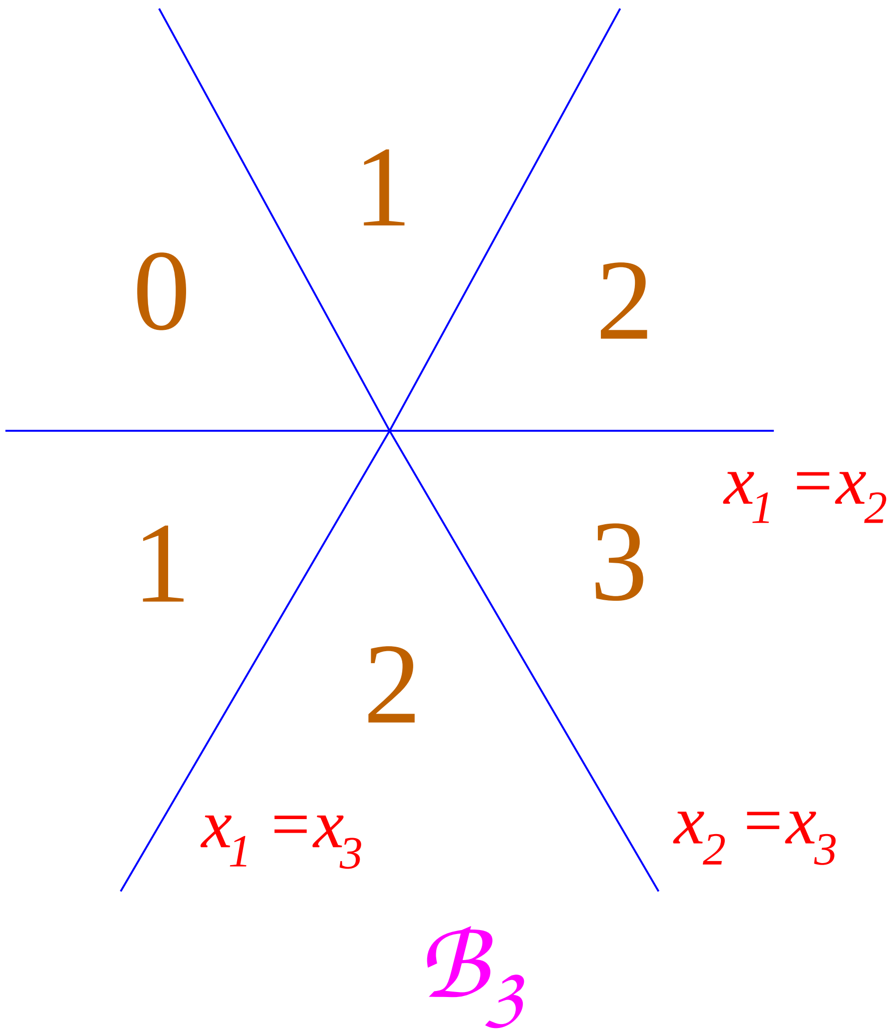
in $\mathbb{R}^n$.

$$\mathcal{R} = \text{set of regions of } \mathcal{B}_n$$
$$\#\mathcal{R} = n!$$

Let R_0 be the "base region"

$$R_0 : \; x_1 > x_2 > \cdots > x_n.$$

Let $\mathbf{d(R)}$ be the number of hyperplanes in $\mathcal{B}_n$ separating R_0 from R.


1
0
2
1
3
2
$x_1 = x_2$
$x_1 = x_3$
$x_2 = x_3$
$\mathcal{B}_3$

Proposition.

$$\#\{R : d(R) = j\} =$$

$$\#\{w \in \mathfrak{S}_n : \ell(w) = \binom{n}{2} - j\},$$

where

$$\ell(w) = \#\{(r,s) : r < s,\ w(r) > w(s)\},$$

the **number of inversions** of w.

$$\sum_{R \in \mathcal{R}} q^{d(R)} =$$

$$(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

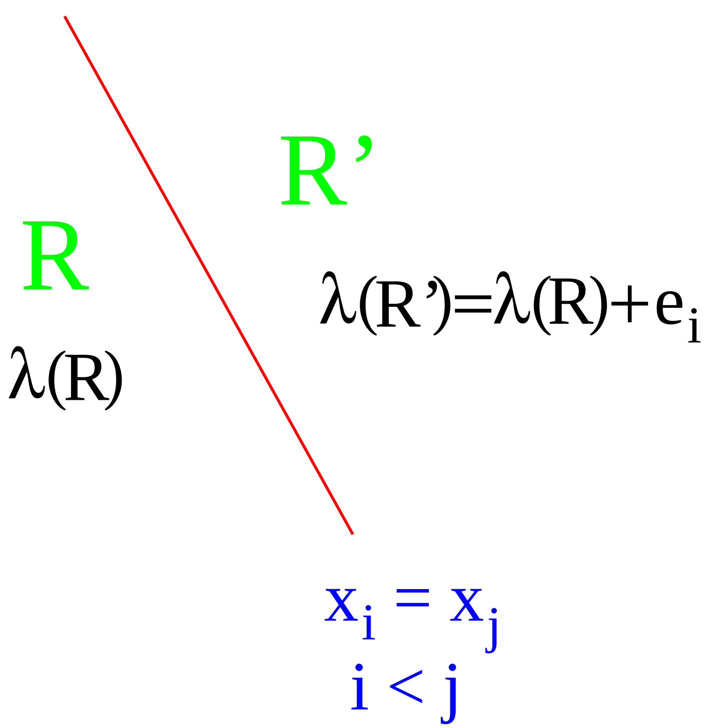
Label R_0 with

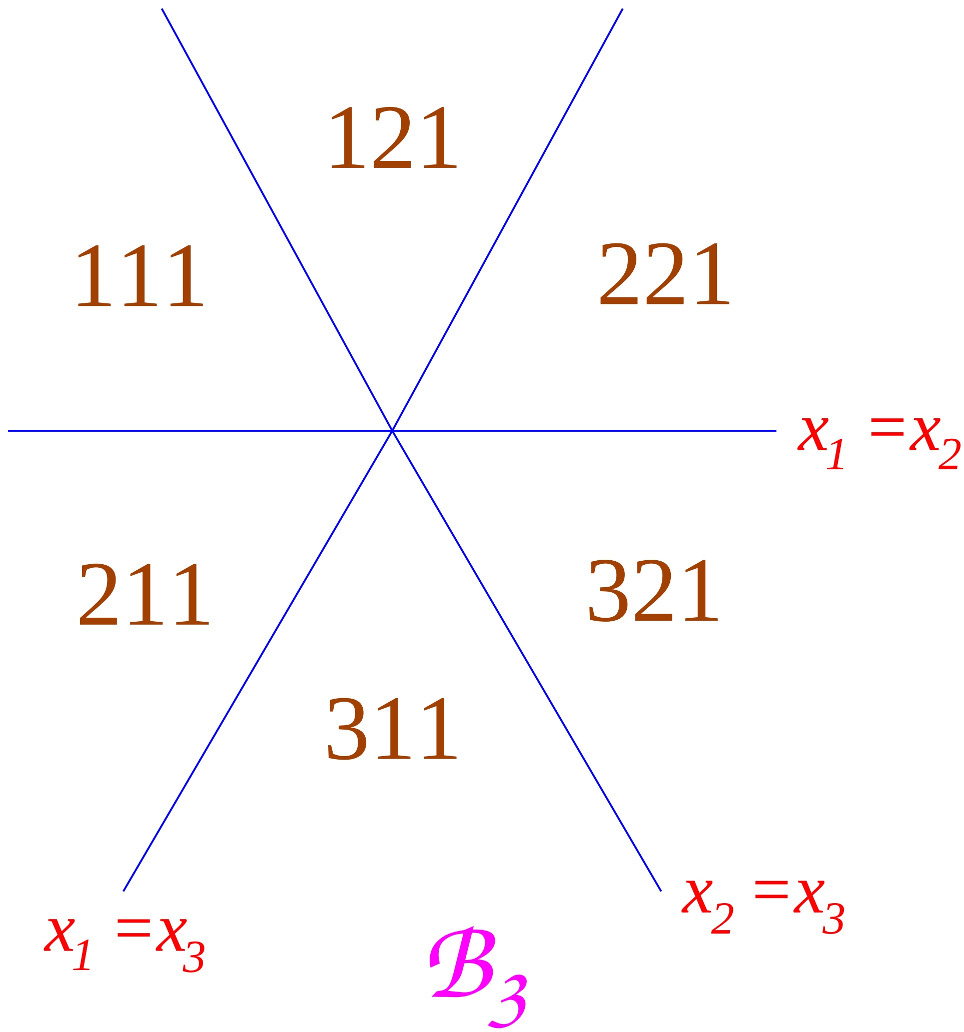
$$\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n.$$

If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i,$$

where e_i = ith unit coordinate vector.

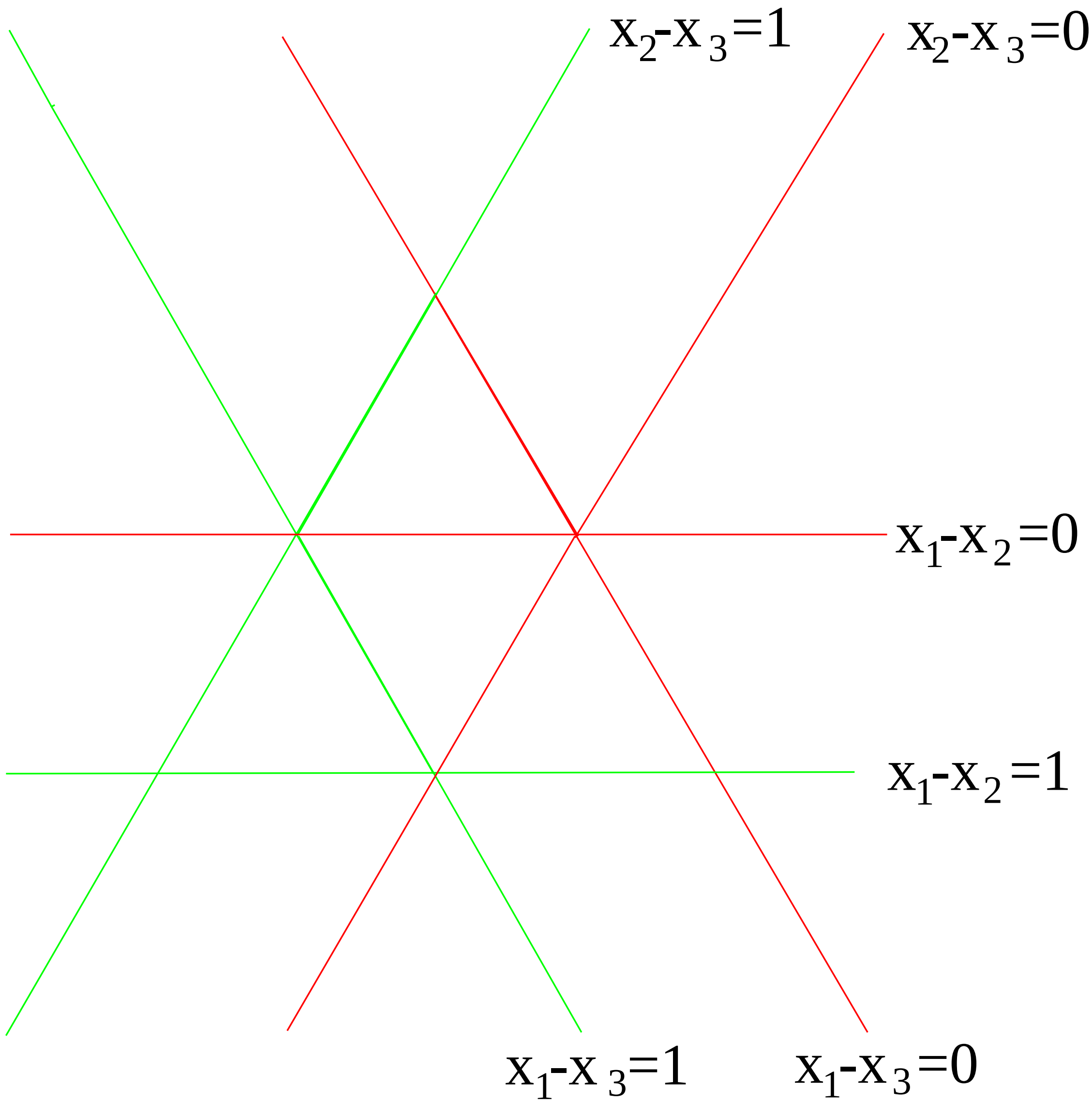




Theorem (easy). *The labels of* $\mathcal{B}_n$ *are the sequences* $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ *such that* $1 \leq a_i \leq n - i + 1$.

Shi arrangement $\mathcal{S}_n$: the set of hyperplanes

$$x_i - x_j = 0, 1,\ 1 \leq i < j \leq n,\ \text{in } \mathbb{R}^n.$$



base region

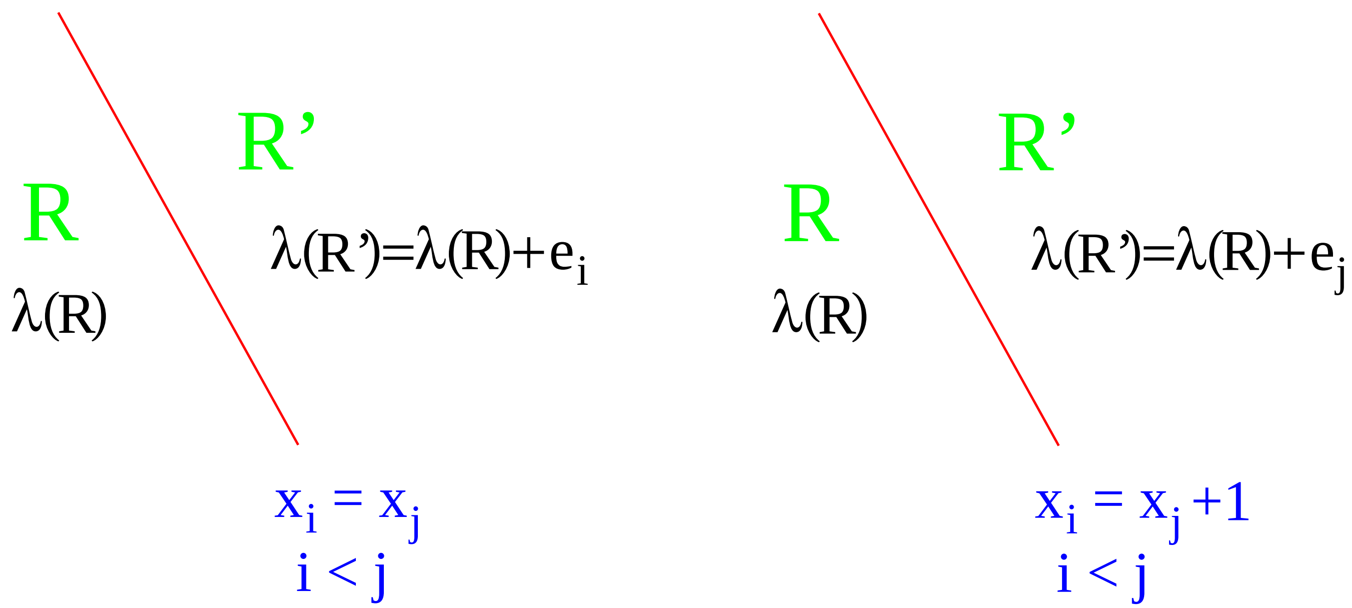
$$R_0: \quad x_n + 1 > x_1 > \cdots > x_n$$

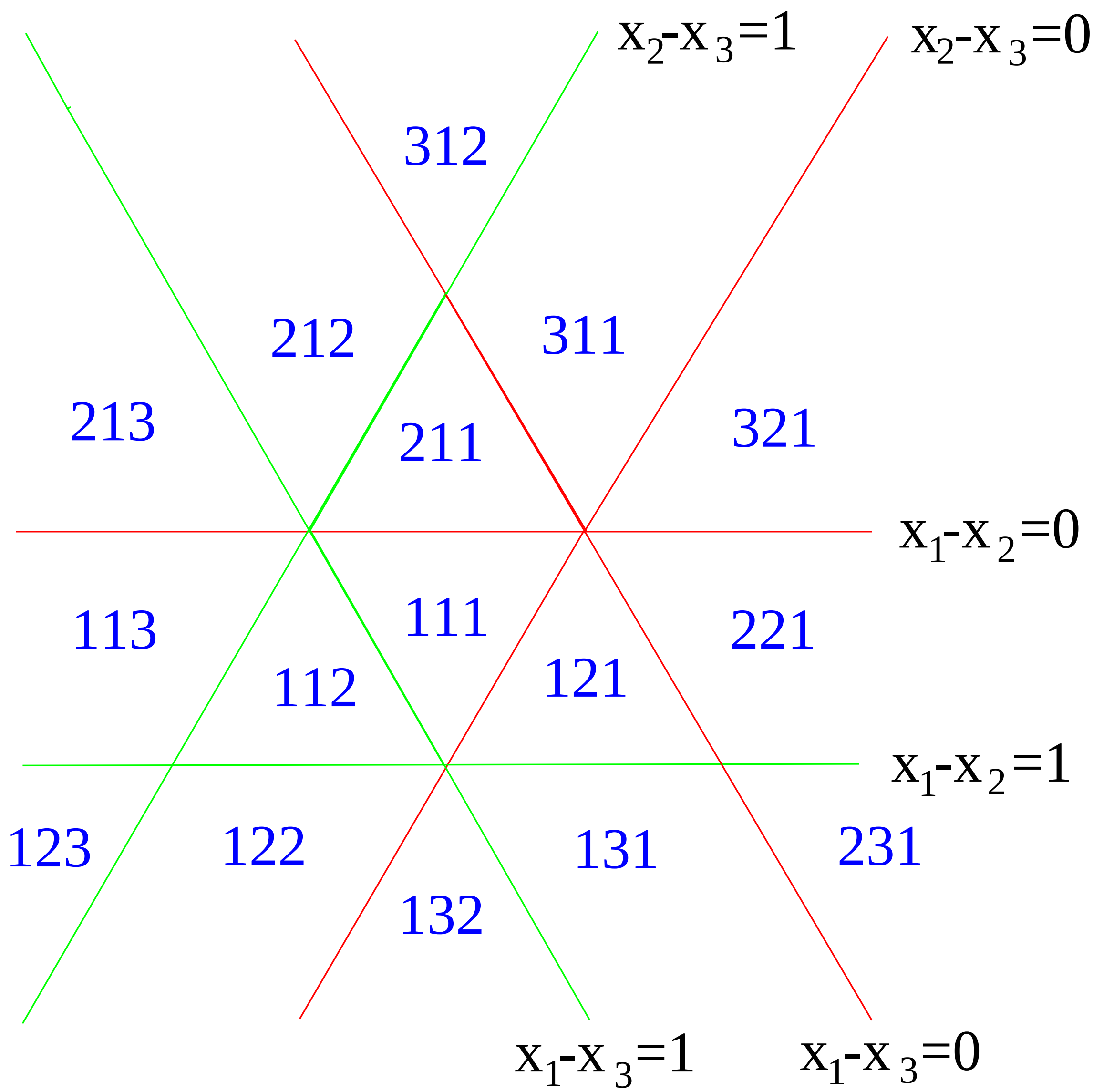
- $\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n$
- If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i.$$

- If R is labelled, R' is separated from R only by $x_i - x_j = 1$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_j.$$


R'
R
λ(R')=λ(R)+e_i
λ(R)
x_i = x_j
i < j
R'
R
λ(R')=λ(R)+e_j
λ(R)
x_i = x_j +1
i < j



Theorem (Pak, S.). *The labels of $\mathcal{S}_n$ are the parking functions of length n (each occurring once).*

Corollary (Shi, 1986)

$$r(\mathcal{S}_n) = (n+1)^{n-1}$$

As for $\mathcal{B}_n$, let **d(R)** be the number of hyperplanes in $\mathcal{S}_n$ separating R_0 and R.

Note: If $\lambda(R) = (a_1, \ldots, a_n)$, then

$$d(R) = a_1 + \cdots + a_n - n.$$

Let $\mathcal{R}$ = set of regions of $\mathcal{S}_n$.

Corollary.

$$\sum_{R \in \mathcal{R}} q^{d(R)} = q^{\binom{n}{2}} I_n(1/q).$$

THE PARKING FUNCTION $\mathfrak{S}_n$-MODULE

The symmetric group acts on the set $\mathcal{P}_{\mathbf{n}}$ of all parking functions of length n by permuting coordinates.

Sample properties:

- Multiplicity of trivial representation (number of orbits) $= C_n = \frac{1}{n+1}\binom{2n}{n}$

  $n = 3: \quad 111 \quad 211 \quad 221 \quad 311 \quad 321$

- Number of elements of $\mathcal{P}_n$ fixed by $w \in \mathfrak{S}_n$ (character value at w):

  $$\#\mathrm{Fix}(w) = (n+1)^{(\#\,\text{cycles of } w)-1}$$

For symmetric function aficionados: Let $\mathrm{PF}_n = \mathrm{ch}(\mathcal{P}_n)$.

$$\begin{aligned}
\mathrm{PF}_n &= \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} z_\lambda^{-1} p_\lambda \\
&= \sum_{\lambda \vdash n} \frac{1}{n+1} s_\lambda(1^{n+1}) s_\lambda \\
&= \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{\lambda_i + n}{n} \right] m_\lambda \\
&= \sum_{\lambda \vdash n} \frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{m_1(\lambda)! \cdots m_n(\lambda)!} h_\lambda.
\end{aligned}$$

$$\omega \mathrm{PF}_n = \sum_{\lambda \vdash n} \frac{1}{n+1} \left[\prod_i \binom{n+1}{\lambda_i} \right] m_\lambda.$$

Moreover

$$\sum_{n\geq 0} \mathrm{PF}_n\, t^{n+1} = (tE(-t))^{\langle -1\rangle}\,,$$

where $E(t) = \sum_{n\geq 0} e_n t^n$, and ${}^{\langle -1\rangle}$ denotes compositional inverse.

THE HAIMAN MODULE

The group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables, i.e., $w \cdot x_i = x_{w(i)}$. Let

$\mathbf{R}^{\mathfrak{S}_\mathbf{n}} = \{f \in R \,:\, w{\cdot}f = f \text{ for all } w \in \mathfrak{S}_n\}$.

Well-known:

$$R^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n],$$

where

$$e_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Let

$$D = R/\left(R_+^{\mathfrak{S}_n}\right) = R/(e_1, \ldots, e_n).$$

Then $\dim D = n!$, and $\mathfrak{S}_n$ acts on D according to the **regular representation**.

Now let $\mathfrak{S}_n$ act **diagonally** on

$$R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

i.e,

$$w \cdot x_i = x_{w(i)}, \quad w \cdot y_i = y_{w(i)}.$$

As before, let

$$\mathbf{R}^{\mathfrak{S}_n} = \{f \in R : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\}$$

$$D = R/\left(R_+^{\mathfrak{S}_n}\right).$$

Conjecture (Haiman, 1994). $\dim D = (n+1)^{n-1}$, and the action of $\mathfrak{S}_n$ on D is isomorphic to the action on $\mathcal{P}_n$, tensored with the sign representation. (Connections with Macdonald polynomials, Hilbert scheme of points in the plane, etc.)

A GENERALIZATION

Let

$$\lambda = (\lambda_1, \ldots, \lambda_n), \;\; \lambda_1 \geq \cdots \geq \lambda_n > 0.$$

A **λ-parking function** is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq \lambda_{n-i+1}$.

Ordinary parking functions:

$$\lambda = (n, n-1, \ldots, 1)$$

Number (Steck 1968, Gessel 1996):

$$\mathbf{N}(\lambda) = n! \det \left[\frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!} \right]_{i,j=1}^{n}$$

The Parking Function Polytope

(with J. Pitman)

Given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, define $\mathcal{P} = \mathcal{P}(\mathbf{x_1}, \ldots, \mathbf{x_n}) \subset \mathbb{R}^n$ by: $(y_1, \ldots, y_n) \in \mathcal{P}_n$ if

$$0 \leq y_i, \quad y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$$

for $1 \leq i \leq n$.

Theorem. (a) *Let* $x_1, \ldots, x_n \in \mathbb{N}$. *Then*

$$n!\, V(\mathcal{P}_n) = N(\lambda),$$

where $\lambda_{n-i+1} = x_1 + \cdots + x_i$.

(b) $$n!\, V(\mathcal{P}_n) = \sum_{\substack{\text{parking functions} \\ (i_1,\ldots,i_n)}} x_{i_1} \cdots x_{i_n}.$$

Note. If each $x_i > 0$, then $\mathcal{P}_n$ has the combinatorial type of an n-cube.

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