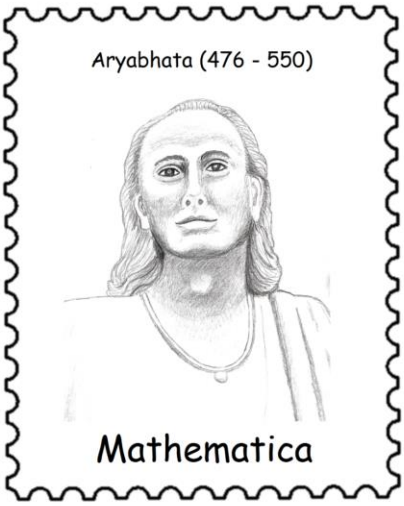
ĀRYABHATA THE ELDER (476 – 550)

by HEINZ KLAUS STRICK, Germany



(drawing © Andreas Strick)

ĀRYABHATA was the first important Indian mathematician and astronomer whose name has been handed down to posterity. To distinguish him from another astronomer of the same name who lived in the 10th or 11th century, he is often referred to as ĀRYABHATA I or ĀRYABHATA THE ELDER.

There is evidence that ĀRYABHATA was born in Kusumapura, near present-day Patna (Bihar state), the capital of the once powerful Gupta Empire, which stretched from Punjab (now Pakistan) to the Bay of Bengal, and that he was head of the university and a teacher there. Other sources give Ashmaka (Assaka) in southern India as the region of his birth.

The importance of ĀRYABHATA in the history of science in India is evident from the fact that the first Indian earth satellite, which was launched into space in 1975 with the help of a Soviet launch vehicle, bore the name of the famous scientist.

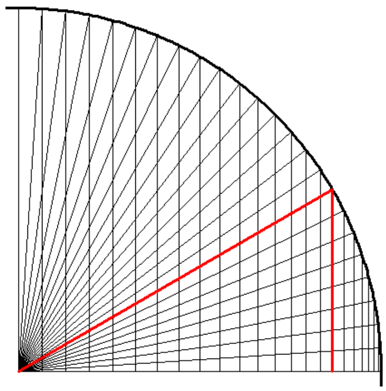
ĀRYABHATA wrote at least two books, the existence of one of which is only certain through quotations from later living authors. The other work, called *Āryabhatīya* by posterity, was written in 499, as can be inferred from calendar calculations contained in the work. It was among the writings translated into Arabic in the *House of Wisdom* in Baghdad around 820. MOHAMMED AL KHWARIZMI referred to this book in his *Algebra*.

Āryabhatīya was written in Sanskrit, the ancient Indian language of the scholars and ritual language of the scriptures of Hinduism, Buddhism and Jainism (comparable to the earlier role of Latin in Europe), for which PĀNINI produced a grammar in the 4th century BC, the first grammar in human history.

Āryabhatīya consists of 118 verses dealing with topics from mathematics, astronomy and chronology. The scripture begins with a praise of *Brahma*, the creator of the earth and the universe. Then follows a description of the astronomical system. *Āryabhata* assumes that the earth revolves around itself daily and thus explains the movement of the starry sky. Otherwise, he advocates a geocentric view of the world: the sun, moon and planets move around the earth and he explains deviations from the uniform movement by epicycles of different sizes. He determines the orbital periods of the sun, moon and planets and calculates from this that the common conjunction of these celestial bodies repeat itself every 4.32 million years. One day for *Brahma* lasts 4.32 billion years for humans. His explanation of lunar and solar eclipses as natural processes replaced the traditional ideas that these eclipses are caused by demons.

The last verse of the first part contains a list of 24 numbers. It says:

> The 24 values of the sine are: 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7.



Later Āryabhata explains:

If one divides a quarter circle of radius 3438 into 24 equal sectors, then the heights opposite the angles of 3° 45', 7° 30', 11° 15', 15°, ..., 90° have the lengths 225, 225 + 224 = 449, 449 + 222 = 671, 671 + 219 = 890, ..., 3438.

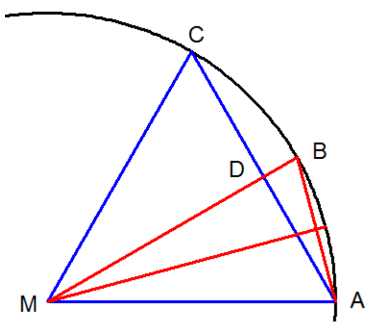
The value for the radius, which seems unusual, is explained thus:

A full angle comprises $360° = 360 \cdot 60' = 21600'$;
the circumference of a circle with a radius of 3438 units of length is almost exactly 21600 units of length, so that each minute of arc can be assigned an arc of length 1 unit of length.

In contrast to the ancient Greek mathematicians, Āryabhata did not tabulate the length of the chords that are opposite an angle, but was the first to tabulate the lengths of the half chords. He called them *ardha-jya*, or *jya* for short, which in the Arabic translation became *jiba*, a word without meaning.

In translating the Toledan tables of Al-Zarqali into Latin, Gerhard of Cremona confused *jiba* with the actually existing Arabic word *jaib*, which translates as *sinus*.

The calculation of the individual table values is based on $AD = \sin(30°) = \frac{1}{2}$.



Then using the Pythagorean theorem, $MD = \cos(30°)$ can be calculated as well as the versine of the angle:

$BD = \text{versin}(30°) = 1 - \cos(30°) = ... = 2 \cdot \sin^2(15°)$

and from this then the value of $\sin(15°)$ and so on.

The second part of the *Āryabhatīya* contains treatises (*siddhānta*) on mathematics (*ganita*, from *gana* = to count). For the representation of numbers, ĀRYABHATA uses artificial words, which he obtains through an encoding he invented:

- For the numbers 1, 2, 3, ..., 25 and 30, 40, 50, ..., 100, he uses the 25 + 8 = 33 consonants of the Sanskrit alphabet, supplemented by the 9 vowels, by which it results with which powers of ten the numbers are multiplied.

The sequence of syllables of these words of art plays an important role in the *sūtras* (mnemonic verses).

Mathematical methods and theorems were traditionally taught in India in this form. The *sūtras* served as a thought-support for the procedure to be applied and were to be learnt by heart by the students. For inexperienced readers, the algorithm described in the 4th and 5th verses for taking the square or cube root of a number in the system of 10 may seem incomprehensible at first.

Only through an example does it become comprehensible:

- *Always divide the non-square digit by twice the square root. When the square is then subtracted from the square digit, enter the quotient in the next digit.*

An example of this is:

	3	2	9	4	7	6	$= (574)^2$
	2	5	↓				subtract 5^2
		7	9				How many times can $(2 \cdot 5)$ fit in? Answer: 7
−		7	0	↓			$= 2 \cdot 5 \cdot 7$
			9	4			
−			4	9	↓		subtract 7^2
			4	5	7		How many $(2 \cdot 57)$ fit in? Answer: 4
−			4	5	6	↓	$= 2 \cdot 57 \cdot 4$
					1	6	
−					1	6	subtract 4^2
						0	

Obviously Āryabhata has mastered the underlying formula:

$$(100a + 10b + 1c)^2$$
$$= (100a)^2 + 2 \cdot (100a) \cdot (10b) + (10b)^2 + 2 \cdot (100a + 10b) \cdot (1c) + (1c)^2$$

The procedure for extracting the cube root results from:

$$(10a + 1b)^3 = (10a)^3 + 3 \cdot (10a)^2 \cdot (1b) + 3 \cdot (10a) \cdot (1b)^2 + (1b)^3$$

- *Divide the second non-cubic digit by three times the square of the cube root. The square multiplied by three and the previously obtained must be subtracted from the first non-cubic digit and the third power from the cubic digit.*

4	3	8	9	7	6	$= (76)^3$
3	4	4	↓			subtract 7^3
	9	5	9			How many times can $(3 \cdot 7^2)$ fit in? Answer: 6
	8	8	2	↓		$= 3 \cdot 7^2 \cdot 6$
		7	7	7		
		7	5	6	↓	$= 3 \cdot 7 \cdot 6^2$
			2	1	6	
			2	1	6	subtract 6^3
					0	

Note: Until the 7th century, the zeros occurring in decimal numbers were recognisable by the gaps in the sequence of digits (*sūnya* (Sanskrit) = emptiness, Arabic: *sifr*); only after that were the gaps replaced by a dot or a small circle, the precursor of the digit "0".

In verse 6, the area of a triangle is given as the product of half the base times the height, and the volume of a tetrahedron ("hexagon") is given with an analogously formed formula, but incorrectly as half the base times the height.

Verse 7 contains the correct formula for the area of a circle (half the circumference times the radius) and a rather inaccurate approximation formula for the volume of a sphere (area of a circle times the square root of the area, i.e. $V \approx 1.77 \cdot \pi \cdot r^3$).

In verse 9, ĀRYABHATA gives, without justification, a method by which the number *π* could be calculated:

- *Add 4 to 100, multiply by 8, then add 62 000, and the result is approximately the circumference of a circle 20 000 in diameter.*

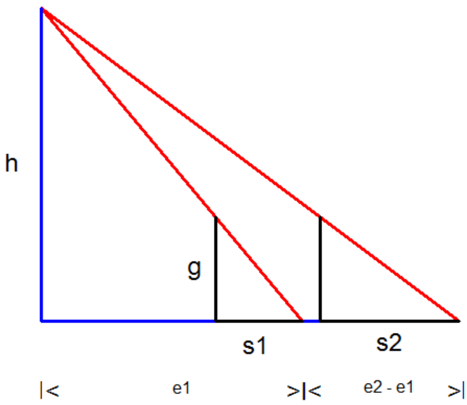
In fact, $\frac{62832}{20000} \approx 3.1416$ is a better approximation for *π* than the value of $\sqrt{10} \approx 3.1623$ commonly used before ĀRYABHATA.

This approximation was also used in China until the 5th century (for example, by ZHANG HENG), until ZU CHONGZHI determined *π* to seven decimal places from calculations of a regular 24 576-sided polygon – at the same time as ĀRYABHATA, who "only" considered a regular 384-gon.

Verses 14 to 16 deal with the shadow lengths of *gnomons* (rods set vertically into the ground) and the possibility of using two rods of equal length *g*, one behind the other, at distances e_1 and e_2 to find the height *h* of a light source. From the lengths of the shadows: s_1 and s_2 and the distance $a = e_2 - e_1$ between the ends of the shadows we get: $\frac{s_1}{e_1} = \frac{g}{h} = \frac{s_2}{e_2}$ and

$s_1 = e_1 \cdot \frac{g}{h}$, $s_2 = e_2 \cdot \frac{g}{h}$, $s_2 - s_1 = a \cdot \frac{g}{h}$ and finally

$e_1 = a \cdot \frac{s_1}{s_2 - s_1}$, $e_2 = a \cdot \frac{s_2}{s_2 - s_1}$, $h = e_1 \cdot \frac{g}{s_1} = e_2 \cdot \frac{g}{s_2}$.



Verses 19 to 22 contain various rules on arithmetical sequences, as well as formulas for the sum of the first *n* natural numbers, the first *n* square or *n* cubic numbers as well as for the sum of the first *n* triangular numbers:

$$(1)+(1+2)+(1+2+3)+...+(1+2+3+...+n) = \tfrac{1}{6} \cdot n \cdot (n+1) \cdot (n+2) \; = \tfrac{1}{6} \cdot [(n+1)^3 - (n+1)]$$

Verses 23 to 24 give rules for sums, differences and products of numbers resulting from binomial formulae:

- *The product of two numbers is equal to half the difference of the square of the sum and the sum of the squares of the two numbers*: $a \cdot b = \tfrac{1}{2} \cdot [(a+b)^2 - (a^2 + b^2)]$

If we know the product $a \cdot b$ and the difference $a - b$ of two numbers, then the numbers *a, b* can be determined as follows:

$a = \frac{1}{2} \cdot \left[\sqrt{4ab + (a-b)^2} + (a-b) \right]$ and $b = \frac{1}{2} \cdot \left[\sqrt{4ab + (a-b)^2} - (a-b) \right]$

The next verses explain how to calculate with fractions and how to solve ratio equations.

From a task on the *calculation of interest*, it becomes clear that solving quadratic equations was also known, even if this topic was not explicitly addressed.

In verse 31, he examines when two objects whose location and speed are known will meet, or when they would have met in the past if they were currently moving away from each other. This method was important in astronomy when determining conjunctions of celestial bodies.

ĀRYABHATA devotes the last two verses of the section on mathematical methods to solving congruence equations, which he calls *kuttaka* (literally: a grinding machine with which something is crushed). As in the EUCLIDean algorithm, the coefficients that occur gradually reduce.

Example: We are looking for the smallest natural number *n* which, when divided by 13 leaves the remainder 4 and by division by 19 the remainder 7.

The following therefore holds: $n = 13a + 4 = 19b + 7$.

After $a = \frac{19b+3}{13} = 1b + \frac{6b+3}{13} = 1b + c$ it follows that $b = \frac{13c-3}{6} = 2c + \frac{1c-3}{6} = 2c + d$ and $c = \frac{6d+3}{1} = 6d + 3$.

If one sets for *d* the smallest possible natural number, i.e. *d* = 1, then one obtains, going backwards, one after another: *c* = 9, *b* = 2 · 9 + 1 = 19 and *a* = 1 · 19 + 9 = 28 and thus

n = 13 · 28 + 4 = 368 = 19 · 19 + 7

ĀRYABHATA's work had considerable influence on the development of mathematics, not only in India. Around 630, BHASKARA I wrote an extensive commentary on it, and BRAHMAGUPTA, who lived at the same time, continued ĀRYABHATA's work.

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https://www.spektrum.de/wissen/aryabhata-indischer-mathematiker-lehrmeister-der-arabischen-mathematik/1300797

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