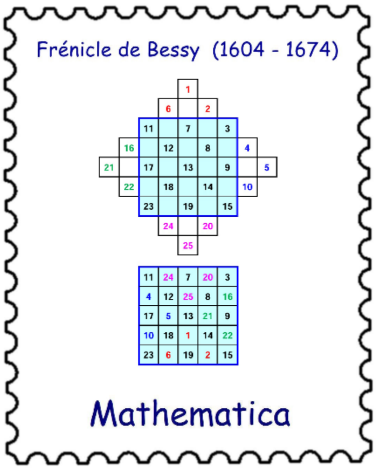
FRÉNICLE DE BESSY (1604 – 1674)

by HEINZ KLAUS STRICK , Germany

He corresponded with PIERRE DE FERMAT, RENÉ DESCARTES, CHRISTIAAN HUYGENS, MARIN MERSENNE and JOHN WALLIS; he was also one of the founding members of the *Académie Royale des Sciences* in 1666.



Despite contacts with contemporary mathematicians, very little is known about BERNARD FRÉNICLE DE BESSY. Whether he was actually born in 1604 (or not until 1605) is not known; There is also no information about the date of his death.

FRÉNICLE DE BESSY came from an old noble family in France; his father, BERNARD DE BESSY, was the *Conseiller à la Cour des Monnaies*, i.e. at the highest court for questions of coinage and state finances. He was also responsible for the administration and production of the 30 mints in France - a time-consuming office, which his son FRÉNICLE inherited after successfully completing his law studies.

FRÉNICLE devoted his free time primarily to the study of the properties of natural numbers. We know from MERSENNE's letters that he also dealt with physical problems, including GALILEO GALILEI's *Dialogue Concerning the Two Chief Systems of the World.*

He repeatedly solved - in record time - numerical problems that FERMAT had posed as challenges to his correspondents; he published one of his solutions (*Solutio*, 1657).

In 1693, many years after FRÉNICLE's death, PHILIPPE DE LA HIRE published four of his treatises (*Various works of Mathematics and Physics*) on behalf of the *Académie*, which we will discuss below.

Today, a term from the field of recreational mathematics still reminds us of the French arithmetic artist: Of the eight possible shapes of a 3x3 magic square, the arrangement on the right is known as the FRÉNICLE standard shape (the other shapes are obtained by rotating and mirroring it).

2	**7**	**6**
9	**5**	**1**
4	**3**	**8**

The problems posed by FERMAT included the task of finding cube numbers for which the sum of all divisors (including the number itself) is a square number, and the task of finding square numbers for which the sum of all divisors is a cube number.

Example for the first problem: Sum of *all* divisors of the number $7^3 = 343$ is $7^0 + 7^1 + 7^2 + 7^3 = 400 = 20^2$.

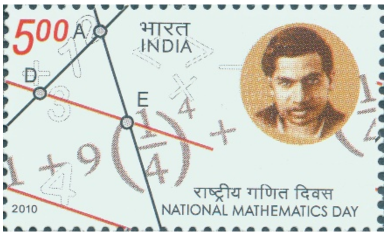
FRÉNICLE added to the List of Problems, including:

• Find a natural number *n* such that the sum of the proper divisors is 5 times the number itself and the sum of the proper divisors of 5 times this number is 25 times 5*n*.

• Find a natural number *n* such that the sum of the proper divisors is 7 times the number itself and the sum of the proper divisors of 7 times this number is 49 times 7n.

• Find two consecutive cubes whose difference is itself a cube.

He also posed a number of problems related to hexagonal numbers. And the fact that 1729 is the smallest cube number that can be represented in two ways as the sum of two cubes is something that can be found in FRÉNICLE's work - and not just in SRINIVASA RAMANUJAN's taxicab problem.



The first of FRÉNICLE's posthumously published treatises is entitled *Method for finding the Solution of Problems by Exclusion* - in this 85-page 'Method by Exclusion', he was particularly interested in explaining how - starting from simple examples - one can deduce general laws by systematically recognising similarities. Most of the examples considered deal with integer right-angled triangles.

It has been known since EUCLID how primitive Pythagorean number triples can be generated: If *a, b* are any two numbers that are mutually independent, then the triple ($a^2 - b^2$; $2ab$; $a^2 + b^2$) fulfils the desired condition. FRÉNICLE applies this …

• We are looking for two square numbers whose difference results in a certain square number. Example (even square number): $144 = 12^2$, divide the number by 4 and break the result down into factors: $36 = 1 \cdot 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9$. Then calculate the difference of the squares for the sum and the difference of these factors: The (generating) factors 1, 36 give the numbers 35, 37, and further: $37^2 - 35^2 = 144$. The numbers 2, 18 give the numbers 16, 20, and further: $20^2 - 16^2 = 144$. The numbers 3, 12 give the numbers 9, 15, and further: $15^2 - 9^2 = 144$. The numbers 4, 9 give the numbers 5, 13, and further: $13^2 - 5^2 = 144$.

Example (odd square number): $81 = 9^2 = 1 \cdot 81 = 3 \cdot 27$. The factors 1, 81 give the numbers 80, 82. If you halve these, you get: $41^2 - 40^2 = 81$; similarly, the factors 3, 27 give the numbers 24, 30, halving these gives: $15^2 - 12^2 = 81$.

• We are looking for whole-number right-angled triangles whose leg lengths differ by a certain amount.

Example: The Pythagorean number triples (5; 12; 13) and (15; 8; 17) belong to the whole-number right-angled triangles whose leg lengths differ by 7. Further examples can be found by finding suitable number sequences for the generating factors *a, b*:

a	b	b^2-a^2	$2ab$	a^2+b^2
2	3	5	12	13
3	8	55	48	73
8	19	297	304	425
19	46	…	…	…

a	b	b^2-a^2	$2ab$	a^2+b^2
1	4	15	8	17
4	9	65	72	97
9	22	403	396	565
22	53	…	…	…

Some of his questions lead to astonishingly large numbers ...

• We are looking for a right-angled triangle in which the length of the hypotenuse and the sum of the lengths of the legs are a square number. (Solution: Leg lengths: 1,061,652,293,520, 4,565,486,027,761, hypotenuse length: 4,687,298,610,289)

• We are looking for a right-angled triangle in which the length of the hypotenuse is a square number and the difference in length of the shorter leg to the other two sides is also a square number. (Solution: Legs: 473,304, 2,276,953, hypotenuse: 2,325,625)

In another posthumous work, the 79-page *Traité des Triangles rectangles en Nombres* (Treatise on integer right-angled triangles), FRÉNICLE first systematically discusses the properties of square numbers:

- Square numbers cannot end in 2, 3, 7 or 8. If the last digit is 1, 4 or 9, then the second to last digit must be even. If the last digit is 6, then the second to last digit must be odd.
- Square numbers that are not divisible by 3 leave a remainder of 1 when divided by 3.
- Odd square numbers leave a remainder of 1 when divided by 8; odd square numbers that are not divisible by 3 leave a remainder of 1 when divided by 24.
- Square numbers that are not divisible by 5 leave a remainder of 1 or -1 when divided by 5.

FRÉNICLE also discovered a remarkable feature:

- If *x, y* are the legs and *z* is the hypotenuse of a right-angled triangle, then the square of the difference between the two leg lengths, the square of the hypotenuse length and the square of the sum of the two leg lengths form an arithmetic sequence: $(y - x)^2$, $x^2 + y^2$, $(x + y)^2$.
- Example: $x = 5$, $y = 12$: $(y - x)^2 = 49$; $x^2 + y^2 = 169$; $(x + y)^2 = 289$

In the Pythagorean triples, prime numbers that leave a remainder of 1 after division by 4 (5, 13, 17, 29, 37, ...) play a special role. FRÉNICLE states:

- These prime numbers can be uniquely represented as the sum of two square numbers ($5 = 1^2+2^2$, $13 = 2^2+3^2$, $17 = 1^2+4^2$, $29 = 2^2+5^2$, $37 = 1^2+6^2$, …); corresponding primitive triples: $(2^2-1^2; 2 \cdot 1 \cdot 2; 1^2+2^2) = (3; 4; 5)$, $(3^2-2^2; 2 \cdot 2 \cdot 3; 2^2+3^2) = (5; 12; 13)$, $(4^2-1^2; 2 \cdot 1 \cdot 4; 1^2+4^2) = (15; 8; 17)$, …
- Products of two different prime numbers of this type can be represented in two ways
- ($65 = 5 \cdot 13 = 1^2+8^2 = 4^2+7^2$, $85 = 5 \cdot 17 = 2^2+9^2 = 6^2+7^2$, $13 \cdot 17 = 221 = 5^2+14^2 = 10^2+11^2$, …); corresponding primitive triples: (63; 16; 65), (33; 56; 65) or (77; 36; 85), (13; 84; 85) or
- (171; 140; 221), (21; 220; 221), …
- Products of three different prime numbers can be represented in four ways,
- e.g. $1105 = 5 \cdot 13 \cdot 17 = 4^2+33^2 = 9^2+32^2 = 12^2+31^2 = 23^2+24^2$), corresponding triples:
- (264; 1073; 1105), (576; 943; 1105), (744; 817; 1105), (47; 1104; 1105), …
- Products of four different prime numbers of the type can be represented in eight ways, products of five different prime numbers of the type can be represented in sixteen ways, etc.
- In addition to these primitive Pythagorean number triples, in the case of the product of two different prime numbers, two further multiple triples can be found, e.g. $5 \cdot (5; 12; 13) = (25; 60; 65)$ and $13 \cdot (3; 4; 5) = (39; 52; 65)$, in the case of the product of three different prime numbers there are nine multiple triples, of four prime numbers 32 multiple triples, etc.
- In total, for products of 2 different prime numbers of this type there are 2+2 = 4 triples, of 3 different prime numbers 4+9 = 13 triples, of 4 different prime numbers 8+32 = 40 triples, of 5 prime numbers 16+105 = 121 triples, of 6 different prime numbers 32+332 = 364 triples, …

The mathematical genius FRÉNICLE was not afraid to determine all forty triples in the case of 4 prime numbers for example.

He then investigated the number of primitive Pythagorean triples for powers and for products of powers of prime numbers $a, b \in \{5, 13, 17, 29, \dots\}$:

Power	a	a^2	a^3	a^4	a^5	a^6
Number	1	1	2	2	3	3

Power	$a \cdot b$	$a \cdot b^2$	$a \cdot b^3$	$a \cdot b^4$	$a \cdot b^5$	$a \cdot b^6$
Number	2	3	4	5	6	7

Power	$a^2 \cdot b$	$a^2 \cdot b^2$	$a^2 \cdot b^3$	$a^2 \cdot b^4$	$a^2 \cdot b^5$	$a^2 \cdot b^6$
Number	3	4	6	7	9	10

Power	$a^3 \cdot b$	$a^3 \cdot b^2$	$a^3 \cdot b^3$	$a^3 \cdot b^4$	$a^3 \cdot b^5$	$a^3 \cdot b^6$
Number	4	6	8	10	12	14

Please note that the following instructions are given below:

- If the generating numbers of a primitive Pythagorean triple are multiplied by a natural number, the numbers of the triple are multiplied by the square of that number and vice versa. If the numbers of a primitive Pythagorean triple are multiplied by twice a square number, a pair of generating numbers also exists; however, for other multiples of a primitive Pythagorean triple, such a pair does not exist.

a	b		b^2-a^2	$2ab$	a^2+b^2	
1	2	↔	3	4	5	
2	4	↔	12	16	20	↵ · 4
3	6	↔	27	36	45	↵ · 9
4	8	↔	48	64	80	↵ · 16

1	2	↔	3	4	5	
1	3	←	6	8	10	↵ · 2
2	6	←	32	24	40	↵ · 8
3	9	←	72	54	90	↵ · 18
4	12	←	128	96	160	↵ · 32

a	b	↔	b^2-a^2	$2ab$	a^2+b^2	
1	2	↔	3	4	5	
–	–		9	12	15	↵ · 3
–	–		15	20	25	↵ · 5
–	–		21	28	35	↵ · 7

FRÉNICLE also demonstrated the following properties, among others:

- The difference between the length of the hypotenuse and the odd-length side in a primitive whole-number right-angled triangle is twice a square number, the sum and difference of the hypotenuse and the even-length side is a square number.
- The length of the hypotenuse of an integer primitive right-angled triangle is not divisible by 3. The side length of one of the sides of a whole-number right-angled triangle is divisible by 3. The side length of one of the sides of a right-angled triangle is divisible by 4. Therefore, there can be no integer right-angled triangle in which the lengths of the two sides are prime numbers.
- One of the side lengths of a whole-number right-angled triangle is divisible by 5.
- In an integer primitive right-angled triangle, both the sum and the difference of the side lengths leave the remainder -1 or +1 when divided by 8.
- The area of an integer right-angled triangle is always divisible by 6.
- There is no integer right-angled triangle with a square or doubly square area.

Finally, FRÉNICLE gives a general method for the problem of finding a right-angled triangle with the same area as an integer right-angled triangle. He proceeds as shown in the table on the right.

	Leg 1	Leg 2	Hypotenuse	Area	
Given triangle	A	B	C	½ AB	
Auxiliary 1	$D := B^2 - A^2$	2AB	$C^2 = A^2 + B^2$	ABD	↵ ·2D
Auxiliary 2	D^2	$4ABC^2$	$E := A^2B^2 + C^4$	$2ABC^2D^2$	↵ $\cdot 2C^2D$
Solution	D/2C	2ABC/D	E/2CD	½ AB	

The numerical example in the table below leads from a right-angled triangle with the leg lengths 3 and 4 to a triangle with the leg lengths and – both with an area of 6.

	Leg 1	Leg 2	Hypotenuse	Area	
Given triangle	3	4	5	6	
Auxiliary 1	7	24	25	84	↵ ·2·7
Auxiliary 2	49	1200	1201	4900	↵ $\cdot 2 \cdot 5^2 \cdot 7$
Solution	7/10	120/7	1201/70	6	

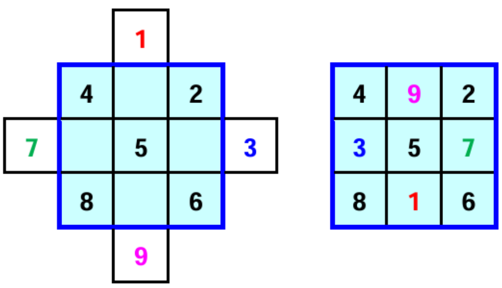
Using this general approach, he discovered, for example: (20; 21; 29) and (12; 35; 37) both have an integer area of 210; the triangles (48; 55; 73) and (22; 120; 122) have an area of 1320; the triangles (27; 364; 365) and (39; 252; 255) have an area of 4914.

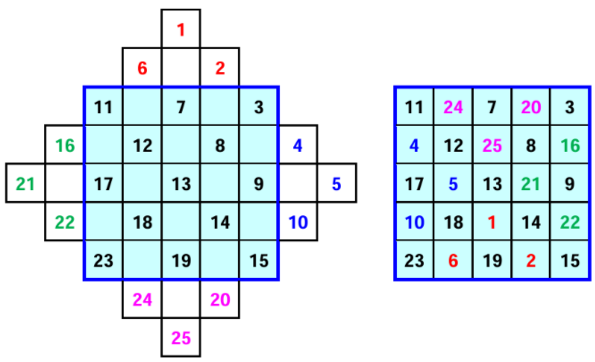
FRÉNICLE even found examples where three integer triangles have the same area, including (56; 390; 394), (105; 208; 233) and (120; 182; 218) with an area of 10920. In an example with six integer right-angled triangles of the same size, the area is a number with 32 digits.

In another text, the 39-page treatise *Abregé des Combinaisons*, FRÉNICLE explains the most important rules of combinatorics using numerous examples; he does not go beyond the issues known up to that point.

The last of the four texts is entitled *Des Quarréz ou Tables Magique* and comprises 146 pages. Without any reference to any sources that may have been used, FRÉNICLE describes various methods that can be used to create magic squares.

After explaining the basic rules, he first presents a method that can be used for magic squares with an odd order (side length): the numbers outside the framed square are moved to the cells that are furthest away horizontally or vertically (see the following examples). By symmetrically swapping rows and columns, further variants can be developed from this.





For 4x4 squares, FRÉNICLE explains a method in which the diagonal elements of the starting square (black) remain, the remaining elements are mirrored – the magic sum of the square is 34.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

A 4x4 square can then be expanded to a 6x6 square (magic number: 111) using the so-called frame method, for example by first considering the numbers from 1 to 8 and from 29 to 36 (magic number: 74) and then adding the remaining numbers in the frame in such a way that opposite numbers each add up to the missing sum of 37.

	1	35	34	4	
	32	6	7	29	
	8	30	31	5	
	33	3	2	36	

9	25	26	23	18	10
16	1	35	34	4	21
20	32	6	7	29	17
24	8	30	31	5	13
15	33	3	2	36	22
27	12	11	14	19	28

On the following pages, FRÉNICLE then explains the next construction steps - up to a 14x14 magic square.

The frame method can also be applied to magic squares with odd order. Since there are different ways of selecting the numbers for the inner square, different extended squares can be created accordingly, as FRÉNICLE explains in detail.

His article concludes with a list of all 880 4th order magic squares. Since each of these squares can be represented in eight different ways by rotations and reflections, there are a total of 8 · 880 = 7040 magic squares of 4th order.

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