High-frequency instruments and identification-robust inference for stochastic volatility models [*]

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October 2024

[*] The authors thank the Editors Sébastien Laurent, Jean-Michel Zakoïan, Christian Francq, and Christophe Hurlin, two anonymous referees, Bertille Antoine, Christian Brownlees, Marine Carrasco, Prosper Dovonon, Firmin Doko Tchatoka, Robert Engle, Peter Fuleky, John Galbraith, Silvia Gonçalves, Joanna Jasiak, Byunguk Kang, Sayar Karmakar, Lynda Khalaf, Frank Kleibergen, Lingwei Kong, Merrick Li, Jans-Peter Kreiss, Alex Maynard, Dan Nordman, Abderrahim Taamouti, Masaya Takano, Ke-Li Xu, and Victoria Zinde-Walsh for useful comments and constructive discussions. Earlier versions of this paper were presented at CIREQ-CMP Econometrics Conference in Honor of Eric Ghysels, Concordia University, Bernoulli-IMS 10th World Congress in Probability and Statistics, 6th Canadian Conference in Applied Statistics, 2021 Australasian Meeting of the Econometric Society, 2021 China Meeting of the Econometric Society, 7th Rimini Center for Economic Analysis (RCEA) Time Series Workshop, IAAE 2021 Annual Conference, 2021 Society for Financial Econometrics (SoFiE) Annual Conference, 2021 North American Summer Meeting of the Econometric Society, 2020 Bernoulli-IMS Symposium, 2020 Joint Statistical Meetings, 2019 CESG Annual Meeting, 2019 CEA Annual Conference, CIREQ Econometrics Conference (Recent Advances on Bootstrap Methods), CIREQ Econometrics Conference (in honor of Jean-Marie Dufour), McGill University, CIREQ Ph.D. Students' Conference.

This work was supported by the William Dow Chair in Political Economy (McGill University), the Bank of Canada (Research Fellowship), the Toulouse School of Economics (Pierre-de-Fermat Chair of excellence), the Universitad Carlos III de Madrid (Banco Santander de Madrid Chair of excellence), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).

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ABSTRACT

We introduce a novel class of stochastic volatility models, which can utilize and relate many high-frequency realized volatility (RV) measures to latent volatility. Instrumental variable methods provide a unified framework for estimation and testing. We study parameter inference problems in the proposed framework with nonstationary stochastic volatility and exogenous predictors in the latent volatility process. Identification-robust methods are developed for a joint hypothesis involving the volatility persistence parameter and the autocorrelation parameter of the composite error (or the noise ratio). For inference about the volatility persistence parameter, projection techniques are applied. The proposed tests include Anderson-Rubin-type tests and their point-optimal versions. For distributional theory, we provide finite-sample tests and confidence sets for Gaussian errors, establish exact Monte Carlo test procedures for non-Gaussian errors (possibly heavy-tailed), and show asymptotic validity under weaker assumptions. Simulation results show that the proposed tests outperform the asymptotic test regarding size and exhibit excellent power in empirically realistic settings. The proposed inference methods are applied to IBM's price and option data (2009–2013). We consider 175 different instruments (IVs) spanning 22 classes and analyze their ability to describe the low-frequency volatility. IVs are compared based on the average length of the proposed identification-robust confidence intervals. The superior instrument set mostly comprises 5-minute HF realized measures, and these IVs produce confidence sets which show that the volatility process is nearly unit-root. In addition, we find RVs with higher frequency yield wider confidence intervals than RVs with slightly lower frequency, indicating that these confidence intervals adjust to absorb market microstructure noise. Furthermore, when we consider irrelevant or weak IVs (jumps and signed jumps), the proposed tests produce unbounded confidence intervals. We also find that both RV and BV measures produce almost identical confidence intervals across all 14 subclasses, confirming that our methodology is robust in the presence of jumps. Finally, although jumps contain little information regarding the low-frequency volatility, we find evidence that there may be a nonlinear relationship between jumps and low-frequency volatility.

JEL Classification: C15, C22, C53, C58.
Keywords: Realized variance, high-frequency data, identification-robust test, market microstructure noise.

1 Introduction

Alongside GARCH models [see Francq and Zakoïan (2019)], the stochastic volatility (SV) model [originally proposed by Taylor (1982)] is a fundamental framework for modelling time-varying volatility in financial and macroeconomic time series. The SV model is conceptually simple and theoretically attractive, but can pose challenges for estimation, due to the presence of latent variables.[1] In previous work [Ahsan and Dufour (2019, 2021)], we have proposed computationally simple and efficient (moment-based) methods for estimating SV models. In particular, these results show that allowing for additional explanatory variables (such as more lags in the volatility specification) can provide substantial gains in statistical fit and volatility forecasting. In this paper, we pursue this research by allowing the use of other variables (such as data on realized volatility) and developing testing procedures better adapted to a setup that includes latent variables and instruments.

Hypothesis testing on such models has remained relatively underdeveloped, relying on asymptotic standard errors – based on delta-method local approximations [for theoretical discussions of this issue, see Dufour (1997), Dufour et al. (2024)] and stationarity assumptions. High persistence in volatility data is also a common problem which complicates asymptotic distributional theory and should be accommodated. Allowing for information from additional variables may also provide power gains. In view of these issues, we first introduce a new class of discrete-time SV models, which are direct extensions of the usual state-space representation of stochastic volatility models with an instrument equation. We then achieve robustness to persistence and avoid the delta method by exploiting *instrumental variables* (IV) methods in the context of *identification-robust* (IR) procedures, which are adapted to a model with latent variables and measurement errors. Power is further improved through point-optimal tests, and Monte-Carlo test methods are used to achieve better level control (with possibly non-Gaussian error distributions). Since the choice of the instruments (IVs) plays a crucial role, we consider broad classes of IVs for the latent volatility, *e.g.*, high-frequency (HF) realized volatility (RV) measures. To the best of our knowledge, the present paper is the first one to propose such IV-based methods for testing SV models.

In the proposed framework, the problem of testing hypotheses and building confidence sets for the *volatility persistence* parameter is explored; for discussions on the importance of this parameter, see Appendix A. We investigate restrictions on volatility persistence, including nonstationarity of the volatility process by testing for a unit root in the volatility equation, within log-squared low-frequency returns using multiple measures of volatility; for discussions on nonstationarity in conditional variance, see Appendix B. The autoregressive root of the latent volatility process is allowed to be close to or equal to one. Beyond testing, the aim is to construct a valid confidence set for the persistence parameter that can be used to

[1] On the advantages and disadvantages of the stochastic volatility specification, see the discussion in Ahsan and Dufour (2021).

determine the volatility forecast interval and/or the distribution of volatility forecasts.

Previous attempts on hypothesis testing for the volatility persistence parameter are limited. According to Harvey et al. (1994), the SV model has an ARMA representation with a large negative moving average (MA) root. Standard unit root tests are known to suffer from severe size distortions in the presence of negative MA roots [Pantula (1991), Schwert (2002)], which undermines their reliability in this context. Wright (1999) proposed to use the unit root test of Perron and Ng (1996) which is based on large-sample approximations and is not reliable in finite samples (requires extremely large samples) and different parameter settings. Furthermore, Wright (1999) considered other classical unit root tests, which were found to perform even worse under similar parameter settings. These inference procedures are based on asymptotic standard errors, which can be markedly different when a time series is nearly nonstationary and unreliable in finite samples; see Park and Phillips (2001) and Bandi and Phillips (2003) for the discussion of asymptotic theory for nonstationary nonlinear models.

Dufour and Valéry (2009) and Ahsan and Dufour (2019, 2021) developed both exact and asymptotic tests for no persistence (or no clustering) hypothesis, which are primarily based on stationarity (time invariance of unconditional variances and autocovariances) and normality assumptions. Since the latent log volatility process may be highly persistent, applying these procedures to empirical data is problematic. Simulation results (in this paper, see Panel A of Table 1) show that tests based on asymptotic standard errors fail to control the type I errors when the volatility persistence parameter approaches the unit circle. The formal hypothesis testing problem for the persistence parameter (concerning size and power) in the latent non-stationary stochastic volatility equation with additional measurements for volatility has not been studied in the literature, *i.e.*, all these previous studies did not exploit high-frequency information.

To be more specific, the other contributions of this paper can be summarized as follows.

First, we consider a variety of IVs for the latent log volatility, including realized volatility (RV) measures at a different frequency (*e.g.*, 1-second or 5-minute), sampling scheme (calendar time or tick time), and functional form (*e.g.*, jumps or kernel). We also consider subsampled versions of some of these HF IVs; these include realized semivariance, realized range RV, nearest neighbor truncated RV, and HF principal component factors.[2]

Second, we propose inference methods which are robust to *weak instruments* since potential HF IVs may

[2] We use RV measures as IVs for the daily latent volatility, in contrast with recent studies, where RV has been incorporated in traditional volatility models (GARCH or SV) by adding a measurement equation which connects the daily volatility measure and the realized volatility. It is worthwhile to note that several studies in the SV literature, such as those by Takahashi et al. (2009) and Koopman and Scharth (2012), model realized volatility and daily returns simultaneously, assuming that the realized volatility includes the market microstructure noise but still contains much information regarding the latent volatility whereas daily returns contain less noise but may not have sufficient information about the latent volatility. In the GARCH-type framework, examples of such models are the Multiplicative Error Model (MEM) model [Engle and Gallo (2006)], the HEAVY (High-frEquency-bAsed VolatilitY) model [Shephard and Sheppard (2010), Noureldin et al. (2012)] and the Realized GARCH model [Hansen et al. (2012)].

be weak due to *discretization errors* or *market microstructure noise*.[3] The discretization error is present in the estimates of the volatility since we only observe prices at intermittent and discrete points in time. The market microstructure noise is due to bid/ask bounces, the different price impact of different types of trades, limited liquidity, or other types of market frictions. These noises may lead to a divergence between the observed price process and the true or latent *frictionless equilibrium* price process.[4] Thus incorporating noisy RV estimates may lead to weak identification. As a result, standard inference procedures may produce invalid confidence tests and sets.

As pointed out by Dufour (1997), the statistical inference should be based on proper pivots, especially when a model involves locally almost unidentified parameters, *i.e.,* in the presence of weak IVs. The proposed inference methods include Anderson-Rubin-type (AR) test and point-optimal version of this test (AR^*). The AR test is considered robust to weak IVs because the test has the correct size in cases where IVs are weak and/or strong. Point-optimal tests gain power by exploiting the differences in the error covariance matrices under the null and the alternative; see King (1980), Dufour and King (1991), and Andrews et al. (2006).

Third, we consider a joint testing problem where we make an inference jointly on both the volatility persistence parameter and the autocorrelation parameter of the composite error (or the noise ratio). Hence, for inference on general (possibly nonlinear) transformations of model parameters [single parameter or a subvector], projection techniques can be applied [see Dufour (1989), Dufour (1990), Dufour and Jasiak (2001), Dufour and Taamouti (2005, 2007)].

Fourth, the proposed inference procedures are also robust to *dynamics*, *i.e.*, nonstationarity. Under the null hypothesis (even with nonstationary stochastic volatility) and appropriate assumptions on IVs, these tests can become pivotal functions with the possibility of exact inference.

Fifth, we employ three different sets of assumptions for the error distribution:

1. Assuming Gaussian errors, we provide confidence sets and tests based on standard Fisher critical values for the AR test statistic. For the point-optimal version, we propose to use the Monte Carlo

[3] In IV regressions, when IVs are not valid (the identification conditions are not satisfied), the standard asymptotic theory for estimators and test statistics typically collapses. Further, when IVs are weak, the limiting distributions of standard test statistics - like Student, Wald, likelihood ratio and Lagrange multiplier criteria - have non-standard distributions and often depend heavily on nuisance parameters; see Phillips (1989), Bekker (1994), Dufour (1997), Staiger and Stock (1997), and Wang and Zivot (1998). In particular, standard Wald-type procedures based on asymptotic standard errors are very unreliable in the presence of weak identification.

[4] The literature on constructing consistent volatility proxy using HF data is considerable. These include but not limited to maximum likelihood estimation [Aït-Sahalia et al. (2005)], quasi-maximum likelihood estimation [Xiu (2010)], Two Scales Realized Volatility [Zhang et al. (2005)], Multi-Scale Realized Volatility [Zhang (2006)], Realized Kernels [Hansen and Lunde (2006), Barndorff-Nielsen et al. (2008, 2011)], and Pre-Averaging volatility estimation [Jacod et al. (2009)]. Other relevant references include Bandi and Russell (2006), Fan and Wang (2007), Gatheral and Oomen (2010), Kalnina and Linton (2008), Li and Mykland (2007), and Aït-Sahalia et al. (2011).

tests (MCT) method [see Dwass (1957), Barnard (1963) and Dufour (2006)].

2. We assume that the conditional distribution of scale transformed error is completely specified up to an unknown scale factor, under which the MCT technique can apply for exact statistical inference. This assumption enables us to deal with non-standard error distributions. For example, even when errors have a heavy-tailed distribution, such as Cauchy distribution or more generally the family of stable distributions, which may not have moments and thus makes statistical inference complicated, our procedures provide exact solutions.

3. We show the asymptotic validity of these procedures under quite general distributional assumptions.

Sixth, we study the statistical properties of the proposed inference procedures by simulation experiments. We find that the usual asymptotic t-tests fail to control the level, whereas the proposed tests control the level and show excellent power. These findings hold for several empirically realistic simulation setups, where the simulated DGPs are incorrectly specified due to the violation of independence assumption and/or misspecification of error distributions together with either weak, low- or high-frequency instruments.

Finally, we apply the proposed procedures to IBM's price and option data (2009-2013). We consider 175 different instruments spanning 22 different classes and look at their ability to describe the low-frequency volatility. The average length of confidence intervals produced by the proposed tests is used to examine the strength of the IVs. The superior instrument set constitutes of 1-, 5- and 10-minute high-frequency realized measures and option implied volatilities. These IVs produce confidence sets where the persistence parameter lies roughly between 0.9 and 1.0. This result shows that the latent volatility process of IBM is highly persistent and close to unit-root.

Further, we find RVs with higher frequency produce wider confidence intervals than RVs with slightly lower frequency, pointing out that these confidence intervals adjust to incorporate the microstructure noise or discretization error. We also find jumps and signed jumps have no or little information content regarding the low-frequency volatility, whereas their log squared versions have a strong identification strength. When we consider irrelevant or weak instruments, such as jumps and signed jumps, the proposed procedures produce unbounded (valid) confidence sets with a non-zero probability.

This paper proceeds as follows. Section 2 specifies models and assumptions. Section 3 proposes finite-sample identification-robust inference procedures, whereas Section 4 extends finite-sample procedures with non-standard error distributions. Section 5 develops the asymptotic validity of the proposed tests. Section 6 presents the simulation study, and Section 7 presents the empirical applications. Section 8 offers conclusions. The mathematical proofs, other discussions, and additional results are provided in an online Supplementary Appendix.

2 Framework

This paper explores extensions of the standard log-normal SV model, which is characterized by the following equations:

$$s_t = \sigma_t z_t, \qquad \log(\sigma_t^2) = \mu + \phi \log(\sigma_{t-1}^2) + v_t, \tag{2.1}$$

where s_t is the return observed at time t, and σ_t is the corresponding volatility. The z_t's and v_t's, are i.i.d. $\mathcal{N}(0,1)$ and $\mathcal{N}(0,\sigma_v^2)$ random variables, respectively and ϕ, μ, σ_v are the fixed parameters of the model. The above SV model can be written in state-space form:

$$w_t = \mu + \phi w_{t-1} + v_t, \quad y_t = w_t + \epsilon_t, \tag{2.2}$$

where $y_t := \log(s_t^2) - \mathbb{E}\left[\log(z_t^2)\right]$, $w_t := \log(\sigma_t^2)$, and $\epsilon_t := \log(z_t^2) - \mathbb{E}\left[\log(z_t^2)\right]$. By the normality assumption on z_t, the transformed errors ϵ_t are i.i.d. according to a centered $\log(\chi^2_{(1)})$ distribution, so that $\mathbb{E}\left[\log(z_t^2)\right] \simeq -1.2704$ and $\sigma_\epsilon^2 := \mathbb{E}[\epsilon_t^2] = \mathrm{Var}\big(\log(z_t^2)\big) = \pi^2/2$.[5]

From (2.2), it is clear that using a proxy for latent volatility (*e.g.*, replacing w_t by y_t) can induce a measurement error problem. Further, the latent volatility process induces moving-average measurement errors. These problems motivate one to use IV methods. In the following assumption, we introduce a generalized stochastic volatility (GSV) model, where IVs are incorporated in $\bar{Z}_{t-2}$, which are related to w_{t-1} but uncorrelated to ϵ_{t-1}.

Assumption 2.1. GENERALIZED STOCHASTIC VOLATILITY MODEL. *The process* $\{y_t : t \in \mathbb{N}_0\}$ *satisfies the following equations*:

$$\text{State Transition Equation:} \quad w = \phi w_{-1} + X\beta + v \tag{2.3}$$

$$\text{Measurement Equation:} \quad y = w + \epsilon \tag{2.4}$$

$$\text{Instrument Equation:} \quad w_{-1} = \bar{Z}_{-2}\bar{\pi} + u_{-1} \tag{2.5}$$

where $w = (w_1, \ldots, w_T)'$, $w_{-1} = (w_0, \ldots, w_{T-1})'$, $y = (y_1, \ldots, y_T)'$ *are* $T \times 1$ *vector,* $X = [X_1', \ldots, X_T']'$ *is a* $T \times k$ *matrix of exogenous explanatory variables,* $\bar{Z}_{-2} = [\bar{Z}_{-1}', \ldots, \bar{Z}_{T-2}']'$ *is a* $T \times m$ *matrix of of variables related to* w_{-1}, *while* $\epsilon = (\epsilon_1, \ldots, \epsilon_T)'$, $v = (v_1, \ldots, v_T)'$, $u_{-1} = (u_0, \ldots, u_{T-1})'$ *are* $T \times 1$ *vector of disturbances. The*

[5] (2.2) can be expressed in an ARMA form, which can be used to derive estimators for log-normal SV model; for further details, see Francq and Zakoïan (2006) and Ahsan and Dufour (2021).

matrices of unknown coefficients ϕ, β, *and* $\bar{\pi}$ *have dimensions* 1×1, $k \times 1$ *and* $m \times 1$, *respectively.*

We do not impose any stationary restriction on the latent volatility process, *i.e.*, $|\phi| = 1$ belongs to the parameter space. This is allowed by the fact that we focus on hypothesis testing (as opposed to point estimation), as is typically done by identification-robust and finite-sample methods. Indeed, difficulties with non-stationarity or identification failure can often be bypassed by using such an approach; see, for example, Dufour (2003), Dufour and King (1991), and Dufour and Kiviet (1998). The assumption that the latent autoregressive volatility process is first-order is not essential to the analysis. Indeed, higher-level dynamics could be allowed, but in this paper we focus on the first-order case which illustrates the main points of the paper without needless generality. The matrix X_t is a set of exogenous variables, which may predict the latent volatility, capture the leverage effect, or incorporate heterogeneous autoregressive (HAR) structure, and jump components. For instance, consider a basic formulation of the leverage function as: $X_t = \tau_1 \tilde{z}_t + \tau_2(\tilde{z}_t^2 - 1)$, where $\tilde{z}_t = s_t / RV_t$ and RV_t is the realized volatility at time t. This specification can generate an asymmetric response in volatility to return shocks; for a discussion regarding this type of leverage function, see Hansen et al. (2012).

To derive finite distributional theory for test statistics (proposed in Section 3), we employ the following assumptions.

Assumption 2.2. INDEPENDENCE. *The* $T \times k$ *matrix* X *and the* $T \times m$ *matrix* $\bar{Z}_{-2}$ *are independent of the* $T \times 1$ *vectors* v *and* ϵ.

Assumption 2.3. FULL RANK. $\text{rank}(X) = k$, $1 \le \text{rank}(\bar{Z}_{-2}) = m < T$, $1 \le \text{rank}[Z_{-2}, X_1, X_2] = l + k < T$, *where* Z_{-2}, X_1, *and* X_2 *are* $T \times l$, $T \times k_1$, *and* $T \times k_2$ *matrices respectively,* $k = k_1 + k_2$ *and* $m = l + k_2$.

Assumption 2.4. GAUSSIAN NOISE. *The* ϵ_t*'s and* v_t*'s are i.i.d.* $\mathcal{N}(0, \sigma_\epsilon^2)$ *and* $\mathcal{N}(0, \sigma_v^2)$ *random variables, respectively.*

In order to handle common variables (*e.g.*, the constant term) in equations (2.3) and (2.5), Assumption 2.3 allows for the presence of common columns in the matrices $\bar{Z}_{-2}$ and X. If $\bar{Z}_{-2}$ and X have k_2 columns in common ($0 \le k_2 < m$) then the other k_1 columns of X are linearly independent of $\bar{Z}_{-2}$. The full-rank Assumption 2.3 guarantees unique least-squares estimates in AR-type regressions. It would be easy to allow for rank-deficiency, but degree-of-freedom corrections would then be required; for ways this can be done in a similar IV context, see Dufour and Taamouti (2007). For exposition simplicity, we focus here on the full-rank case. Due to the robustness of AR-type procedures to missing instruments, any subset of the instrument matrix that satisfies the rank condition yields a valid test; we call this important feature robustness to missing instruments (or instrument exclusion) [Dufour and Taamouti (2007). Note also that using too many instruments (which may be quasi-collinear or redundant) can reduce the power the proposed procedures. No restriction is imposed on the distribution of u and it may follow any distribution (heteroskedastic or autocorrelated) since no statistical property of u has effects on the validity of the tests proposed in this paper.

Note that we change the distributional assumption of ϵ_t by an i.i.d. $\mathcal{N}(0, \sigma_\epsilon^2)$ distribution. This is consistent with several previous studies where the distribution of ϵ_t is approximated by a normal distribution characterized by a mean of zero and a variance of $\pi^2/2$; see Harvey et al. (1994), Wright (1999). We relax the above assumptions in Sections 4 and 5.

The IV regression requires valid IVs for the observable volatility proxy y_t, which is typically the low-frequency (LF) daily squared return. As a result, IVs are also connected to the logarithm of latent daily volatility [see equation (2.5)]. To find valid IVs, we first look at the properties of the observed volatility proxy y_t. If y_t is autocorrelated with a sufficiently long lag and the ϵ_t's are uncorrelated, then the lag values of observed proxy $(y_{t-2}, y_{t-3}, y_{t-4}, \ldots)$ are potential IVs for y_{t-1}. For the sake of efficiency, it is typically preferable to avoid using too many lagged values as IVs, because this requires truncating the sample. We can also use realized volatility as IV ($\bar{Z}_{t-2}$ contains past realized volatilities) since HF price data contain valuable information regarding the latent volatility. In Section 7, we consider daily and HF IVs, as well option implied volatility.

3 Finite-sample test procedures

In this section, we consider the problem of testing the volatility persistence parameter in a GSV model (as specified by Assumption 2.1), *i.e.*, testing a restriction on volatility clustering. We propose two finite-sample procedures, which are valid under Assumptions 2.2 - 2.4. We first focus on the null hypothesis:

$$H_0 : \phi = \phi_0 \,. \tag{3.1}$$

To do this, we consider an *instrument substitution method*, based on replacing unobserved variables with a set of IVs. We first substitute (2.4) into (2.3):

$$y = \phi y_{-1} + X\beta + v + \epsilon - \phi\epsilon_{-1} \,. \tag{3.2}$$

Subtracting $\phi_0 y_{-1}$ on both sides of (3.2), we get:

$$y - \phi_0 y_{-1} = (\phi - \phi_0) y_{-1} + X\beta + v + \epsilon - \phi\epsilon_{-1} \,. \tag{3.3}$$

Since $\mathbb{E}[y_{t-1}\epsilon_{t-1}] \neq 0$, we need to find IVs for w_{-1} to tackle this endogeneity problem. Substituting (2.4) into (2.5), we have $y_{-1} = \bar{Z}_{-2}\bar{\pi} + \eta_{-1}$, where $\eta_{-1} := \epsilon_{-1} + u_{-1}$.[6] By Assumption 2.2, $\bar{Z}_{-2}$ is independent of ϵ_{-1}. On introducing the expression fro y_{-1} into (3.3), we get:

$$y - \phi_0 y_{-1} = \bar{Z}_{-2}\bar{\pi}(\phi - \phi_0) + X\beta + \xi, \quad \xi := (\phi - \phi_0)u_{-1} + v + \epsilon - \phi_0\epsilon_{-1}. \tag{3.4}$$

Using Assumption 2.3, we can write (3.4) as

$$y - \phi_0 y_{-1} = Z_{-2}\delta + X\beta_* + \xi \tag{3.5}$$

where $\delta := \bar{\pi}_1(\phi - \phi_0)$, $\beta_* := (\beta_1', \beta_{2*}')'$, $\beta_{2*} := \beta_2 + \bar{\pi}_2(\phi - \phi_0)$, $\bar{\pi} := (\bar{\pi}_1', \bar{\pi}_2')'$ and $\bar{\pi}_i$ is a $k_i \times 1$ vector.

3.1 Anderson-Rubin-type procedure

Since $\epsilon_t - \phi_0\epsilon_{t-1}$ is an MA(1) process, the components of ξ are serially correlated. However, when $\phi = \phi_0 = 0$, ξ is distributed according to $\mathcal{N}(0, \sigma_\xi^2 I_T)$ distribution, with $\sigma_\xi^2 = \sigma_v^2 + \sigma_\epsilon^2$. Consequently the model (3.5) satisfies all the assumptions of the classical linear model when $\phi_0 = 0$. Furthermore, since $\delta = 0$ when $\phi = \phi_0$, we can test H_0 by a standard F-test of the null hypothesis: $H_0^* : \delta = 0$. This F-statistic has the form

$$AR(\phi_0) = \frac{(y - \phi_0 y_{-1})'(M[X] - M[X, Z_{-2}])(y - \phi_0 y_{-1})/l}{(y - \phi_0 y_{-1})' M[X, Z_{-2}](y - \phi_0 y_{-1})/(T - l - k)} \tag{3.6}$$

where $M(A) = I - A(A'A)^{-1}A'$. $AR(\phi_0)$ can be interpreted as an Anderson-Rubin-type statistic. When normality holds $\left[\xi \sim \mathcal{N}(0, \sigma_\xi^2 I_T)\right]$ and X and Z_{-2} are exogenous, we have $AR(\phi_0) \sim F(l, T - l - k)$, and $H_0(\phi_0)$ can be tested by using a critical region of the form $\{AR(\phi_0) > f(\alpha)\}$ where $f(\alpha) = F_\alpha(l, T - l - k)$ is the $(1 - \alpha)$-quantile of the $F(l, T - l - k)$ distribution.[7]

Unfortunately, this property does not extend to a more general $AR(\phi_0)$ statistic where $\phi_0 \neq 0$, because in this case the errors ξ_t are not i.i.d. under H_0. When $\phi_0 \neq 0$, it is easy to see that the model (3.5) under H_0 does not satisfy all the assumptions of the classical linear model. In this case, under the null hypothesis, $\xi = v + \epsilon - \phi_0\epsilon_{-1}$ is an MA(1) process which makes the standard t-tests and F-tests are invalid because the standard errors are wrong. We could correct the standard errors by a Generalized Least Squares (GLS) type transformation. The model defined by equation (3.5) can be transformed under the H_0 to a model such that an AR-type test is valid, and the distribution of the test statistic follows an F-distribution.

Under the null hypothesis, $\xi = v + \epsilon - \phi_0\epsilon_{-1}$ is an MA(1) process with $\xi \sim \mathcal{N}\left[0, \sigma_\xi^2 \Sigma(\rho)\right]$ [by Assumption

[6] The MA(1) assumption follows naturally from the basic SV(1) specification. It would be of interest to consider more general error structures, such as MA(q), and indeed our approach can be extended to deal with more complex models. As this raises a wide range of associated problems, it is left for further work.

[7] When the disturbances are i.i.d with finite fourth-order moments, the AR-statistic converges under H_0 to a χ^2 distributed random variable when the sample size gets large. This large sample distribution of the AR-statistic does not depend on the value of $\bar{\pi}$ which makes it a more reliable statistic for practical purposes than the Wald statistic.

2.4] and

$$\Sigma(\rho) := \begin{pmatrix} 1 & -\rho & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\rho & 1 & -\rho & 0 & & & & \vdots \\ 0 & -\rho & 1 & -\rho & \ddots & & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & -\rho & 1 & -\rho & 0 \\ \vdots & & & & 0 & -\rho & 1 & -\rho \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\rho & 1 \end{pmatrix}, \tag{3.7}$$

$$\sigma_\xi^2 := (1+\phi_0^2)\sigma_\epsilon^2 + \sigma_v^2, \quad \rho := \frac{-\mathrm{Cov}(\xi_t \xi_{t-1})}{\mathrm{Var}(\xi_t)} = \frac{\phi_0 \sigma_\epsilon^2}{(1+\phi_0^2)\sigma_\epsilon^2 + \sigma_v^2}. \tag{3.8}$$

Clearly, ρ is a function of ϕ_0, σ_v^2, and σ_ϵ^2. $\Sigma(\rho)$ is a *Toeplitz* matrix (or diagonal-constant matrix) with dimension $T \times T$. Because $\Sigma(\rho)$ is a symmetric positive-definite matrix, there exists a $T \times T$ matrix C such that $C\Sigma(\rho)C' = I_T$. If $\Sigma(\rho)$ is known, we can propose the following transformation. Multiply equation (3.5) by C to make the error covariance matrix to an identity matrix. However, ρ is not known. On setting the noise ratio $\lambda := \sigma_\epsilon^2/\sigma_v^2 \in [0, \infty)$, we can write ρ as $\rho(\phi_0, \lambda) = \phi_0\lambda/[(1+\phi_0^2)\lambda + 1]$. Hence, we can do a joint test such that under the null ρ is known.

To deal with the presence of two nuisance parameters in the serial covariance structure, we shall perform joint inference, as typically done for identification-robust and finite-sample inference.[8] This is motivated by the fact that nuisance parameters may be difficult to eliminate in nonlinear models or when identification is weak [see Dufour (2003), Dufour and Taamouti (2005, 2007)]. As far as we know, this is one of the original contributions of this paper, in the context of inference on stochastic volatility. Consider the following null hypothesis:

$$H_0(\phi_0, \lambda_0) : \phi = \phi_0, \quad \lambda = \lambda_0. \tag{3.9}$$

Under $H_0(\phi_0, \lambda_0)$, we can write $\rho_0 := \phi_0\lambda_0/[(1+\phi_0^2)\lambda_0 + 1] \in [-1/2, 1/2]$, and the joint null hypothesis [in (3.9)] becomes

$$\bar{H}_0(\phi_0, \rho_0) : \phi = \phi_0, \quad \rho = \rho_0. \tag{3.10}$$

Under $\bar{H}_0(\phi_0, \rho_0)$, we have $\lambda_0 = \rho_0/[\phi_0 - \rho_0(1+\phi_0)^2] \in [0, \infty)$; see Table A1 of Appendix F for testable null values (ϕ_0, ρ_0) with corresponding values of λ_0. Since ρ_0 is known under $H_0(\phi_0, \lambda_0)$ or $\bar{H}_0(\phi_0, \rho_0)$, we can

[8] This is motivated by the fact that nuisance parameters may be difficult to eliminate in nonlinear models or when identification is weak [see Dufour (2003), Dufour and Taamouti (2005, 2007)]. As far as we know, this approach has not been applied in earlier work on stochastic volatility models. No other paper has focused on developing tests for a stochastic volatility model.

consider the following transformed model:

$$C_0(y - \phi_0 y_{-1}) = C_0 Z_{-2}\delta + C_0 X\beta_* + C_0\xi \tag{3.11}$$

where $C_0 = C(\rho_0)$ is a $T \times T$ matrix such that $C_0\Sigma(\rho_0)C_0' = I_T$. The variance-covariance matrix of $\xi^* := C_0\xi$ is now an i.i.d. $\mathcal{N}(0, \sigma_\xi^2 I_T)$ distribution. The F-statistic for testing $\delta = 0$ (or $\phi = \phi_0$) in (3.11) is:

$$AR(\phi_0, \rho_0) = \frac{y(\phi_0, \rho_0)'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])y(\phi_0, \rho_0)/l}{y(\phi_0, \rho_0)'M_{C_0}[X, Z_{-2}]y(\phi_0, \rho_0)/(T - l - k)} \tag{3.12}$$

where $y(\phi_0, \rho_0) = C_0(y - \phi_0 y_{-1})$, $M_{C_0}[A] = I - A[A'\Sigma(\rho_0)^{-1}A]^{-1}A'\Sigma(\rho_0)^{-1}$. A central feature of most situations where IV methods are required come from the fact that IVs may be used to solve an endogeneity or an errors-in-variables problem. It is very rare that one can or should use all the possible valid IVs. A drawback of the AR method is that it loses power when too many IVs are used. However, the AR procedure is *robust to missing IVs* (or *instrument exclusion*) [see Dufour and Taamouti (2007)]. Alternative methods of inference aimed at being robust to weak identification [Wang and Zivot (1998), Kleibergen (2002), Moreira (2003)] do not enjoy this type of robustness. In the case of feasible GLS-type transformations, where ρ is replaced by an estimate $\hat{\rho}$, the test statistic is no longer F-distributed, but it converges under $\bar{H}_0(\phi_0, \rho_0)$ to a χ^2 distribution in large samples. The tests and confidence sets obtained by the instrument substitution method can be interpreted as likelihood ratio (LR) procedures (based on appropriately chosen reduced form alternatives), or equivalently as profile likelihood techniques [for further discussion of such techniques, see Bates and Watts (1988), Meeker and Escobar (1995) and Chen and Jennrich (1996)].

3.2 Anderson-Rubin-type point-optimal procedure (AR^*)

In this section, we propose a point-optimal (PO) version of AR-type tests. PO tests provide simple and effective methods for building tests with excellent power properties in a wide variety of problems in linear regression. The empirical evidence in the literature indicates that in general, PO tests often outperform other testing methods in terms of power. Besides, exact small-sample critical values for PO tests can be computed in most cases. Thus, one does not have to rely on the asymptotic properties of the test statistic to make inferences. For a general review of PO tests, the reader may consult King (1980), King (1987) and Dufour and King (1991).

Following Dufour and King (1991), the PO test for $\rho = \rho_0$ against $\rho = \rho_1$ under Gaussian assumptions is given by

$$S(\rho_0, \rho_1) = \frac{\hat{\xi}'\Sigma(\rho_0)^{-1}\hat{\xi}}{\tilde{\xi}'\Sigma(\rho_1)^{-1}\tilde{\xi}} \tag{3.13}$$

where $|\rho_0| \leq 1/2$, $|\rho_1| \leq 1/2$, and $\hat{\xi}$ and $\tilde{\xi}$ are the GLS residual vectors corresponding to covariance matrices

$\Sigma(\rho_0)$ and $\Sigma(\rho_1)$, respectively. The test rejects the null for large values of $S(\rho_0, \rho_1)$. However, the choice of ρ_1 is important. To obtain a test of $\rho = \rho_0$ against $\rho > \rho_0$, we select a value of ρ_1, such that $\rho_0 < \rho_1 \leq 1/2$ and apply the test based on $S(\rho_0, \rho_1)$. Similarly, testing $\rho = \rho_0$ against $\rho < \rho_0$, we select ρ_1, such that $-1/2 \leq \rho_1 < \rho_0$. For example, we may choose ρ_1 such that $\rho_1 = \rho_0 - \bar{\Delta}$ where $0 < \bar{\Delta} < 1$. The test based on (3.13) is point-optimal, and it gains power by exploiting the differences in the error covariance matrices under the null and the alternative.

As pointed out by King (1987), a PO test can be viewed as a partition of the sample space into two regions, a rejection region and a non-rejection region. If the observed sample falls in the rejection region, the null is rejected. Otherwise, the null is not rejected. Consider an AR-type PO test statistic $\overline{AR}(\phi_0, \rho_0, \rho_1)$ similar to (3.13) for $\rho = \rho_0$ against $\rho = \rho_1$ (under $\phi = \phi_0$):

$$\overline{AR}(\phi_0, \rho_0, \rho_1) = \frac{y(\phi_0, \rho_0)' M_{C_0}[X] y(\phi_0, \rho_0)}{y(\phi_0, \rho_1)' M_{C_1}[X, Z_{-2}] y(\phi_0, \rho_1)} \tag{3.14}$$

where $y(\phi_0, \rho_0) = C_0(y - \phi_0 y_{-1})$, $y(\phi_0, \rho_1) = C_1(y - \phi_0 y_{-1})$ and $M_{C_i}[A] = I - A\left[A'\Sigma(\rho_i)^{-1} A\right]^{-1} A'\Sigma(\rho_i)^{-1}$ for $i = 0, 1$. Note that it is difficult to derive the analytical null distribution of (3.14) even under the Gaussian assumption, while the MCT method described in Section 4 can be implemented and confidence set for ϕ and ρ with level $(1-\alpha)$ is obtained by inverting the tests.

It is worth noting that $\overline{AR}(\phi_0, \rho_0, \rho_1)$ can become degenerate in the limit. Thus we consider a monotonic transformation of $\overline{AR}(\phi_0, \rho_0, \rho_1)$, which is given as:

$$AR^*(\phi_0, \rho_0, \rho_1) = T\left[\overline{AR}(\phi_0, \rho_0, \rho_1) - 1\right]. \tag{3.15}$$

For finite-sample inference, both $\overline{AR}(\phi_0, \rho_0, \rho_1)$ and $AR^*(\phi_0, \rho_0, \rho_1)$ lead to identical results since a monotonic transformation does not change the rank of the statistic in the MCT method. On the other hand, $AR^*(\phi_0, \rho_0, \rho_1)$ is more appropriate for proving the asymptotic validity.

3.3 Inference on general transformations

In Sections 3.1-3.2, we make joint inference on $(\phi, \rho)'$. These tests are based on extensions of Anderson-Rubin statistics and designed to test hypotheses fixing the entire vector of the endogenous (or unobserved) regressor coefficients. When one is interested in its subsets, or more generally in any functions of the parameters, projection technique can be applied; see Dufour (1989), Dufour and Jasiak (2001), Dufour and Taamouti (2005, 2007).

Let $\theta := (\phi, \rho)'$ for notational convenience. A confidence set associated with one of the tests for $H_0(\theta_0)$: $\theta = \theta_0$ in the previous subsections can be written as

$$C_\alpha(\theta) = \left\{\theta_0 \mid H_0(\theta_0) \text{ is not rejected}\right\}. \tag{3.16}$$

If the test has level α, the confidence set $C_\alpha(\theta)$ has level $1-\alpha$. Note that all the four tests are based on pivotal functions and have size α. Thus, the confidence sets in (3.16) from these tests have size $1-\alpha$.

Now consider an arbitrary (possibly nonlinear) transformation $\delta = g(\theta)$ of θ, then a confidence set of δ, with the level at least $1-\alpha$, can be constructed as

$$C_\alpha(\delta) = \{\delta_0 \mid \delta_0 = g(\theta) \text{ for some } \theta \in C_\alpha(\theta)\}. \tag{3.17}$$

Since $\theta \in C_\alpha(\theta)$ implies $\delta = g(\theta) \in C_\alpha(\delta)$, and further, $Pr[\delta \in C_\alpha(\delta)] \geq Pr[\theta \in C_\alpha(\theta)] \geq 1-\alpha$, so that $C_\alpha(\delta)$ has level $1-\alpha$. We reject $H_0(\delta_0) : \delta = \delta_0$ when $\delta_0 \notin C_\alpha(\delta)$ and get a test of level α.

One can use numerical optimization technique or grid search over economically or statistically plausible parameter space to implement the projection method. However, if the parameter transformation of interest is a linear scalar function, an analytical expression for $C_\alpha(\delta)$ is available in Dufour and Taamouti (2005).

If $\delta = \phi$ where $\theta = (\phi, \rho)'$, the projection method can be implemented more efficiently. Let $F(\theta_0)$ and c_α denote a test statistic used in confidence set in (3.16) and a corresponding critical value, respectively. Then, the confidence set in (3.17) is rewritten as

$$C_\alpha(\phi) = \left\{\phi_0 \mid \inf_{\rho \in \bar{\rho}} F(\phi_0, \rho) \leq c_\alpha\right\} \tag{3.18}$$

where $\bar{\rho}$ is the parameter space for ρ. An alternative projection technique improves efficiency by restricting ρ. The procedure can be described in two steps: (1) construct $C_{\alpha_1}(\rho \mid \phi_0)$, a confidence set for ρ under $H_0 : \phi = \phi_0$ with level $(1-\alpha_1)$; (2) reject $H_0 : \phi = \phi_0$ if $C_{\alpha_1}(\rho \mid \phi_0) = \emptyset$, or $\inf_{\rho \in C_{\alpha_1}(\rho|\phi_0)} F(\phi_0, \rho) > c_{\alpha_2}$, where $\alpha = \alpha_1 + \alpha_2$ and c_{α_2} is a critical value chosen in the same manner as c_α but with α_2 instead of α. By Bonferroni inequality, the test has level α, and it can be inverted to get confidence set for ϕ with level $1-\alpha$. Since the infimum is computed over $C_{\alpha_1}(\rho \mid \phi_0)$, this procedure is expected to be more efficient. Furthermore, it is worthwhile noting that, even though the simultaneous confidence set $C_\alpha(\theta)$ for θ may be interpreted as a confidence set based on inverting LR-type tests for $\theta = \theta_0$ [see Meeker and Escobar (1995) or Chen and Jennrich (1996)], projection-based confidence sets, such as $C_\alpha(\delta)$, are not (strictly speaking) LR confidence sets. For more details and further discussion about the projection technique; see Dufour (1989, 1990) and Chaudhuri and Zivot (2011).

4 Finite-sample procedures with possibly non-Gaussian errors

In this section, we extend the exact tests proposed in the previous section, by allowing non-Gaussian distributions. The use of Gaussian assumptions, when the volatility distributions are not normal, can be hazardous; such a practice could lead us to invalid inferences. Under the non-Gaussian assumptions, we can build an exact test based on the MCT technique. We can take the observed test statistic (derived under Gaussian assumptions) and perform simulations to obtain an exact test. In order to do that, we need the null distribution of the test statistic under non-Gaussian errors. Under the Assumption 2.4, the GLS transformed composite error $\xi^* \sim \mathcal{N}(0, \sigma_\xi^2 I_T)$, where $\sigma_\xi^2 = (1+\phi_0^2)\sigma_\epsilon^2 + \sigma_v^2$. We need the following additional assumption about the transformed composite error to get the finite-sample inference under non-Gaussian errors.

Assumption 4.1. CONDITIONAL SCALE MODEL OF TRANSFORMED COMPOSITE ERROR. *$\xi^* = \sigma_\xi \vartheta$, where σ_ξ is a (possibly random) scalar such that $P[\sigma_\xi \neq 0] = 1$, and the conditional distribution of ϑ is completely or incompletely specified such that*

$$\vartheta \mid \overline{X} := (\vartheta_1, \ldots, \vartheta_T) \sim \mathcal{F}(v) \tag{4.1}$$

where $\mathcal{F}(\cdot)$ represents a known distribution function and $\overline{X} = [X, Z_{-2}]$.

We consider both the case where the error distribution does not involve nuisance parameters,

$$\vartheta \mid \overline{X} \sim \mathcal{F}(v_0), \quad \text{where } v_0 \text{ is specified} \tag{4.2}$$

and the one where it does

$$\vartheta \mid \overline{X} \sim \mathcal{F}(v), \quad \text{where } v \text{ is unknown.} \tag{4.3}$$

The above assumption includes the Gaussian distribution, all elliptically symmetric distributions, such as the multivariate t, and cases where $\vartheta_1, \ldots, \vartheta_T$ are i.i.d. according to any given distribution.

In the following proposition, we characterize the null distribution of $AR(\phi_0, \rho_0)$ given in (3.12) under the above assumption.

Proposition 4.1. NULL DISTRIBUTION OF AR-TEST STATISTIC UNDER NON-GAUSSIAN ERRORS. *Suppose that Assumptions 2.1-2.3 and 4.1 hold. If $\phi = \phi_0$ and $\rho = \rho_0$, we have*

$$AR(\phi_0, \rho_0) = \kappa \frac{\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta}{\vartheta' M_{C_0}[X, Z_{-2}]\vartheta} \tag{4.4}$$

where $\kappa = (T - l - k)/l$, and the conditional distribution of $AR(\phi_0, \rho_0)$ given $\overline{X}$ only depends on $\overline{X}$ and the distribution of ϑ, which is given in Assumption 4.1.

Proposition 4.1 covers the null distribution of $AR(\phi_0, \rho_0)$. It is easy to see that the null distribution of the other proposed test statistic under non-Gaussian errors can be derived in the same way upon employing Assumption 4.1. Proposition 4.1 means that the conditional null distribution of $AR(\phi_0, \rho_0)$ given $\overline{X}$, only depends on the distribution of ϑ. If the distribution of $\vartheta \mid \overline{X}$ can be simulated, one can get exact tests based on $AR(\phi_0, \rho_0, \vartheta \mid \overline{X})$ through the MCT method [see Dufour (2006)], even if this distribution is non-Gaussian. Furthermore, the exact test obtained in this way is robust to weak IVs as well as if the distribution does not have moments (*e.g.*, the Cauchy distribution).

The MCT technique was originally proposed by Dwass (1957) for implementing permutation tests and did not involve nuisance parameters. This technique was also independently proposed by Barnard (1963); for a general discussion and proofs, see Dufour (2006). It has the great attraction of providing exact (randomized) tests based on any statistic whose finite-sample distribution may be intractable but can be simulated. Here we have briefly summarized the procedure.

Let $S(Y,\overline{X})$ be a test statistic which can be rewritten in the form $S(Y,\overline{X}) = \bar{S}(\vartheta,\overline{X})$ under the null hypothesis, where ϑ is defined by (4.1) and the distribution of ϑ is known. For example, $S(Y,\overline{X})$ could be the AR-type statistic considered in Proposition 4.1. Then the conditional distribution of $S(Y,\overline{X})$, given $\overline{X}$, is completely determined by the matrix $\overline{X}$ and the conditional distribution of ϑ given $\overline{X}$, *i.e.*, $S(Y,\overline{X})$ is pivotal. We can then proceed as follows to obtain an exact critical region.

1. Compute the statistic $S^{(0)}$ (based on data), where $S^{(0)} = AR^{(0)}(\phi_0, \rho_0)$.
2. By Monte Carlo methods, draw N i.i.d. replications of $\vartheta : \vartheta_{(j)} = [\vartheta_1^{(j)}, \ldots, \vartheta_T^{(j)}]$, $j = 1,\ldots, N$.
3. From each simulated error matrix $\vartheta_{(j)}$, compute the statistics, $S^{(j)} = \bar{S}(\vartheta_{(j)}, X)$, $j = 1,\ldots, N$, according to the fully specified distribution of $\vartheta \mid \overline{X}$. For instance, in the case of the AR statistic underlying Proposition 4.1, calculate
$$AR^{(j)} := AR(\vartheta_{(j)}) = \frac{\vartheta'_{(j)}\big(M_{C_0}[X] - M_{C_0}[X, Z_{-2}]\big)\vartheta_{(j)}}{\vartheta'_{(j)}M_{C_0}[X, Z_{-2}]\vartheta_{(j)}}, \quad 1,\ldots, N. \tag{4.5}$$
4. Compute the MC p-value $\hat{p}_N[S] := p_N(S^{(0)}; S)$, where
$$p_N(x,S) := \frac{NG_N(x;S)+1}{N+1},, \quad G_N(x;S) := \frac{1}{N}\sum_{j=1}^{N} I_{[0,\infty)}(S^{(j)} - x), \quad I_{[0,\infty)}(x) = \begin{cases} 1 & \text{if } x \in [0,\infty) \\ 0 & \text{if } x \notin [0,\infty) \end{cases}. \tag{4.6}$$
In other words, $p_N(S^{(0)};S) = [NG_N(S^{(0)};S)+1]/(N+1)$ where $NG_N(S^{(0)};S)$ is the number of simulated values which are greater than or equal to $S^{(0)}$. When $S^{(0)}, S^{(1)}, \ldots, S^{(N)}$ are all distinct [an event with probability one when the vector $(S^{(0)}, S^{(1)}, \ldots, S^{(N)})'$ has an absolutely continuous distribution], $\hat{R}_N(S^{(0)}) = N+1-NG_N(S^{(0)};S)$ is the rank of $S^{(0)}$ in the series $S^{(0)}, S^{(1)}, \ldots, S^{(N)}$.
5. The MC critical region is: $\hat{p}_N[S] \le \alpha$, $0 < \alpha < 1$. If α^* and N such that $\alpha(N+1)$ is an integer and the distribution of S is continuous under the null hypothesis, then under null, $P[\hat{p}_N[S] \le \alpha] = \alpha$.

The above algorithm is valid for any fully specified distribution of ϑ and we reject the null hypothesis

$H_0(\phi_0, \rho_0)$ at level α when $\hat{p}_N[AR^{(0)}(\phi_0, \rho_0)] \le \alpha$. If the distribution of the test statistic is not continuous, the MC test procedure can easily be adapted by using "tie-breaking" method described in Dufour (2006).[9] Correspondingly, a confidence set with level $1-\alpha$ for (ϕ, ρ) is given by the set of all values (ϕ_0, ρ_0) which are not rejected by the above MC test. More precisely, the set $C_{(\phi,\rho)}(\alpha) = \{(\phi_0, \rho_0) : \hat{p}_N[AR^{(0)}(\phi_0, \rho_0)] > \alpha\}$ is a confidence set with level $1-\alpha$ for (ϕ_0, ρ_0). For further discussion regarding MCT techniques with nuisance parameters, see Appendix D.

5 Asymptotic distributional theory

In this section, we relax the Assumptions 2.2-2.4 and 4.1, and show that under weaker distributional assumptions on X, Z_{-2} and ξ, the proposed procedures remain "asymptotically valid" . More precisely, we wish to show that if Assumption 2.2-2.4 hold jointly with a specific distributional assumption on ξ^*/σ_ξ [*e.g.*, $\xi^*/\sigma_\xi \sim \mathcal{N}(0, I_T)$] yields tests whose probability of type I error converges to the nominal level of the test as $T \to \infty$ under any parameter configuration compatible with the null hypothesis (pointwise asymptotic validity).

All our results up to now have been established for a given sample size of T. To formulate asymptotic properties, we need to consider a sequence of tests indexed by T. Consider the following sequence

$$\{S(T) := [y(T), y_{-1}(T), X(T), Z_{-2}(T), \xi(T)], T \ge T_0\} \tag{5.1}$$

and rewrite the test statistic (3.6) in the following form:

$$AR_T(\phi_0) = \kappa(T)\frac{y_T'\big(M[Q_{1T}] - M[Q_T]\big)y_T}{y_T' M[Q_T] y_T / T} \tag{5.2}$$

where $y_T = \big(y(T) - \phi_0 y_{-1}(T)\big)$, $Q_T = [Q_{1T}, Q_{2T}]$, $Q_{1T} = X(T)$, $Q_{2T} = Z_{-2}(T)$, $\kappa(T) = (T-l-k)/lT$, and k and l are the number of columns in Q_{1T} and Q_{2T}, respectively.

We examine the asymptotic distribution of $AR_T(\phi_0)$ under the following assumptions (where $\Longrightarrow$ refers to weak convergence as the sample size tends to infinity).

Assumption 5.1. *The sequence $\big(S(T), T \ge T_0\big)$ given in (5.1) belongs to a class $\mathcal{Z}$ of stochastic processes such that for each process in $\mathcal{Z}$ the following limits hold:*

1. $\frac{\xi'(T)\xi(T)}{T} \xrightarrow[T\to\infty]{p} \sigma_\xi^2 > 0$, *where σ_ξ^2 is the same for all processes in $\mathcal{Z}$;*
2. *There exists a sequence of $m \times m$, nonsingular matrices D_T such that:*

[9] Without the correction for continuity, the algorithm proposed for statistics with continuous distributions yields a conservative test, *i.e.*, the probability of rejection under the null hypothesis is not larger than the nominal level.

(A) $D_T' Q_T' Q_T D_T \xrightarrow[T\to\infty]{\text{p}} \Sigma_{QQ} = \begin{pmatrix} \Sigma_{Q_1Q_1} & \Sigma_{Q_1Q_2} \\ \Sigma_{Q_2Q_1} & \Sigma_{Q_2Q_2} \end{pmatrix}$, *where* Σ_{QQ} *and* $\Sigma_{Q_1Q_1}$ *are* $m\times m$ *and* $k\times k$ *nonsingular matrices, respectively;*

(B) $D_T' Q_T' \xi(T) \Longrightarrow q \sim \mathcal{N}\big(0, \sigma_\xi^2 \Sigma_{QQ}\big)$, *where* $q = \big(q_1', q_2'\big)'$, q_1 *and* q_2 *are* $k\times 1$ *and* $l\times 1$ *random vectors, respectively.*

It should be emphasized that Assumption 5.1 satisfies the condition

$$q_2 \mid q_1 \sim \mathcal{N}\big(\Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1} q_1, \sigma_\xi^2 \Sigma_{q_2|q_1}\big)$$

where $\Sigma_{q_2|q_1} = \Sigma_{Q_2Q_2} - \Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1}\Sigma_{Q_2Q_2}$. Thus the asymptotic distribution of $\big(q'\Sigma_{QQ}^{-1} q - q_1'\Sigma_{Q_1Q_1}^{-1} q_1\big)/\sigma_\xi^2$ is a $\chi^2_{(l)}$ distributed random variable. Note that the normality of the sub-vector of q_1 is not required, the conditional normality of q_2 given q_1 is sufficient.

Further, in the above Assumption 5.1(2), we allow both stationary and nonstationary regressors by adjusting the scaling matrix D_T, which is typical of the form, $D_T = diag\big[T^{-d_1}, \ldots, T^{-d_m}\big]$, where $d_i > 0$ for $i = 1, \ldots, m$ relying on the degree of nonstationarity of the regressors. For example, if $X(T)$ and $Z_{-2}(T)$ are stationary then $d_i = 1/2$ for $i = 1, \ldots, m$. However, if $X(T)$ and $Z_{-2}(T)$ are nonstationary and are integrated of order one, then the corresponding d_i should be one. The following proposition establishes the asymptotic validity of the AR procedure.

Proposition 5.1. ASYMPTOTIC VALIDITY OF AR-TYPE TEST. *Under the Assumption 5.1 and the null hypothesis in (3.1), the statistic* $AR_T(\phi_0)$ *in (5.2) has the same limiting distribution for all processes in* $\mathcal{Z}$, i.e., $AR_T(\phi_0) \Longrightarrow \chi^2_{(l)}/l$.

Similarly, one can show that the joint test defined in (3.12) has the null distribution of $AR_T(\phi_0, \rho_0) \Longrightarrow \chi^2_{(l)}/l$. Now we consider the test statistic of the AR-type PO procedure, which is rewritten in the following form:

$$AR_T^*(\phi_0, \rho_0, \rho_1) = T\left[\frac{y_T(\phi_0,\rho_0)' M[\hat{Q}_{1T}] y_T(\phi_0,\rho_0)}{y_T(\phi_0,\rho_1)' M[\tilde{Q}_T] y_T(\phi_0,\rho_1)} - 1\right] \tag{5.3}$$

where $y_T(\phi_0,\rho_0) = C(\rho_0)\big(y(T) - \phi_0 y_{-1}(T)\big)$, $y_T(\phi_0,\rho_1) = C(\rho_1)\big(y(T) - \phi_0 y_{-1}(T)\big)$, $\hat{Q}_{1T} = C(\rho_0)X(T)$, $\tilde{Q}_T = [\tilde{Q}_{1T}, \tilde{Q}_{2T}]$, $\tilde{Q}_{1T} = C(\rho_1)X(T)$, $\tilde{Q}_{2T} = C(\rho_1)Z_{-2}(T)$, k is the number of columns in $\hat{Q}_{1T}$ or $\tilde{Q}_{2T}$, l is the number of columns in $\tilde{Q}_{2T}$ and $m = l + k$. In order to prove the asymptotic validity of the test based on $AR_T^*(\phi_0, \rho_0, \rho_1)$ in (3.15), we need following assumption.

Assumption 5.2. *The sequence* $(S(T), T \geq T_0)$ *given in (5.1) belongs to a class* $\mathcal{Z}$ *of stochastic processes such that for each process in* $\mathcal{Z}$ *the following limits hold:*

1. $\frac{\hat{\xi}'(T)\hat{\xi}(T)}{T} \xrightarrow[T\to\infty]{\text{p}} \sigma_\xi^2 > 0$, *where* σ_ξ^2 *is the same for all processes in* $\mathcal{Z}$;

2. $\frac{\tilde{\xi}'(T)\tilde{\xi}(T)}{T} \xrightarrow[T\to\infty]{\text{p}} \sigma_\xi^2 > 0$, *where* σ_ξ^2 *is the same for all processes in* $\mathcal{Z}$;

Table 1: Size and power of asymptotic t-type test for $H_0 : \phi = \phi_0$ (nominal level: 5%)

		Panel A: Size							Panel B: Power ($H_0 : \phi_0 = 1$)						
ϕ	$T =$	100	200	300	500	1000	5000	10000	100	200	300	500	1000	5000	10000
0.10000		0.2	0.1	0.0	0.0	0.0	0.1	0.4	6.0	6.3	7.7	10.9	20.3	75.8	96.0
0.20000		0.2	0.2	0.1	0.0	0.1	1.8	2.8	7.4	13.4	20.0	33.5	61.9	99.8	100.0
0.30000		0.2	0.2	0.1	0.2	0.7	2.8	3.3	12.0	27.2	40.5	61.9	88.8	100.0	100.0
0.40000		0.2	0.2	0.3	0.9	2.2	2.8	3.0	19.5	43.0	60.0	81.1	97.6	100.0	100.0
0.50000		0.4	0.7	1.2	1.7	2.5	2.3	2.6	28.0	56.3	73.0	91.2	99.6	100.0	100.0
0.60000		1.0	1.5	1.9	2.1	2.3	2.1	2.1	35.3	65.0	81.5	96.1	100.0	100.0	100.0
0.70000		1.8	2.2	2.3	2.1	2.2	1.8	1.9	40.0	70.6	86.7	98.4	100.0	100.0	100.0
0.80000		2.7	2.5	2.7	2.1	2.2	2.3	1.9	41.4	72.7	89.2	99.1	100.0	100.0	100.0
0.90000		4.3	3.4	3.5	3.2	3.0	2.8	2.6	37.8	64.9	85.2	98.8	100.0	100.0	100.0
0.95000		7.7	5.9	5.2	4.1	3.5	3.7	3.2	31.7	48.3	69.1	94.6	100.0	100.0	100.0
0.98000		14.1	9.8	7.6	5.8	4.7	4.1	3.6	27.7	32.5	41.4	65.7	97.9	100.0	100.0
0.98500		15.8	11.5	8.8	6.8	5.0	4.4	4.0	26.9	30.4	36.3	54.8	91.5	100.0	100.0
0.99000		18.0	13.8	11.0	8.8	6.5	4.5	4.2	26.2	28.1	31.7	43.3	74.8	100.0	100.0
0.99500		20.8	17.5	15.6	13.1	9.4	5.7	4.5	25.2	26.1	27.8	34.3	48.8	100.0	100.0
0.99900		24.0	22.0	21.5	22.9	21.5	12.1	8.2	25.0	24.6	25.4	29.2	33.3	56.4	84.5
0.99950		24.5	23.1	23.1	25.4	25.5	18.2	12.7	25.0	24.3	25.0	28.8	32.5	44.2	57.5
0.99990		25.0	23.8	24.6	27.8	30.1	30.4	27.1	25.1	24.1	25.0	28.6	31.8	38.3	40.5
0.99999		25.1	24.1	24.9	28.6	31.6	36.8	36.6	25.1	24.1	25.0	28.7	31.7	37.8	38.3
1.00000		25.1	24.1	25.0	28.7	31.8	37.9	38.1	25.1	24.1	25.0	28.7	31.8	37.9	38.1

3. *There exists a sequence of $m \times m$, nonsingular matrices D_T such that:*

   (A) $D'_T \tilde{Q}'_T \tilde{Q}_T D_T \xrightarrow[T\to\infty]{\text{p}} \Sigma_{\tilde{Q}\tilde{Q}} = \begin{pmatrix} \Sigma_{\tilde{Q}_1\tilde{Q}_1} & \Sigma_{\tilde{Q}_1\tilde{Q}_2} \\ \Sigma_{\tilde{Q}_2\tilde{Q}_1} & \Sigma_{\tilde{Q}_2\tilde{Q}_2} \end{pmatrix}$, *where $\Sigma_{\tilde{Q}\tilde{Q}}$ and $\Sigma_{\tilde{Q}_1\tilde{Q}_1}$ are $m \times m$ and $k \times k$ nonsingular matrices, respectively;*

   (B) $D'_{T1} \hat{Q}'_{T1} \hat{Q}_{T1} D_{T1} \xrightarrow[T\to\infty]{\text{p}} \Sigma_{\hat{Q}_1\hat{Q}_1}$, *where $\Sigma_{\hat{Q}_1\hat{Q}_1}$ is a $k \times k$ nonsingular matrix;*

   (C) $D'_T \tilde{Q}'_T \tilde{\xi}(T) \Longrightarrow \tilde{q} \sim \mathcal{N}\big(0, \sigma^2_\xi \Sigma_{\tilde{Q}\tilde{Q}}\big)$, *where $\tilde{q} = (\tilde{q}'_1, \tilde{q}'_2)'$, $\tilde{q}_1$ and $\tilde{q}_2$ are $k \times 1$ and $l \times 1$ random vectors, respectively.*

   (D) $D'_T \hat{Q}'_{1T} \hat{\xi}(T) \Longrightarrow \hat{q}_1 \sim \mathcal{N}\big(0, \sigma^2_\xi \Sigma_{\hat{Q}_1\hat{Q}_1}\big)$, *where $\hat{q}_1$ is a $k \times 1$ random vector.*

The following proposition establishes the asymptotic validity of the AR^* optimal procedure.

Proposition 5.2. ASYMPTOTIC VALIDITY OF AR-TYPE POINT-OPTIMAL TEST. *Under the Assumption 5.2 and the null hypothesis in (3.10) against a fixed alternative $\rho = \rho_1$, the statistic $AR^*_T(\phi_0, \rho_0, \rho_1)$ in (5.3) has the same limiting distribution for all processes in $\mathcal{Z}$,* i.e., $AR^*_T(\phi_0, \rho_0, \rho_1) \Longrightarrow \chi^2_{(l)}$.

6 Simulation study

We simulate the DGP given in (2.2) with an instrument equation, which has the following compact representation:

$$y_t = \mu + \phi y_{t-1} + \xi_t, \quad \xi_t := v_t + \epsilon_t - \phi\epsilon_{t-1}, \quad v_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2_v), \quad \epsilon_t \sim \text{i.i.d. } \log(\chi^2_{(1)}) \tag{6.1}$$

$$y_{t-1} = \bar{\pi}_0 + Z'_{t-2}\bar{\pi}_1 + \eta_{t-1}, \quad \eta_{t-1} := \epsilon_{t-1} + u_{t-1}, \quad u_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2_u), \tag{6.2}$$

Table 2. Size and power comparison of joint tests with weak, low-and high-frequency instruments, nominal level: 5%

Panel A: $\mathcal{M}_1$ with weak instruments

Size

ρ	ϕ	σ_v	CP	$T=200$: $\bar{\pi}_1$	$T=200$: AR	$T=200$: AR^*	$T=300$: $\bar{\pi}_1$	$T=300$: AR	$T=300$: AR^*
0.05	0.2	3.8	0.0	0.00	5.2	5.0	0.00	4.8	4.7
		3.8	0.5	0.11	5.2	5.1	0.09	4.9	4.5
		3.8	5.0	0.35	5.3	5.3	0.29	5.2	5.0
	0.5	6.6	0.0	0.00	5.3	5.1	0.00	4.8	4.6
		6.6	0.5	0.11	5.1	5.0	0.09	4.8	4.6
		6.6	5.0	0.35	5.0	4.8	0.29	5.0	5.1
	0.9	8.9	0.0	0.00	5.2	5.0	0.00	4.8	4.6
		8.9	0.5	0.11	5.1	5.1	0.09	4.7	4.7
		8.9	5.0	0.35	4.8	4.7	0.29	4.7	4.5
	1.0	9.4	0.0	0.00	5.3	5.0	0.00	5.1	5.0
		9.4	0.5	0.11	5.3	5.1	0.09	5.2	5.1
		9.4	5.0	0.35	5.5	5.1	0.29	5.1	5.1
0.10	0.2	2.2	0.0	0.00	5.1	4.9	0.00	4.9	4.6
		2.2	0.5	0.11	5.3	5.0	0.09	5.1	4.6
		2.2	5.0	0.35	6.4	5.9	0.29	6.1	5.8
	0.5	4.3	0.0	0.00	5.2	4.8	0.00	4.9	4.4
		4.3	0.5	0.11	5.1	4.9	0.09	4.9	4.4
		4.3	5.0	0.35	5.9	5.6	0.29	6.0	5.8
	0.9	6.0	0.0	0.00	5.2	4.8	0.00	4.7	4.5
		6.0	0.5	0.11	5.0	4.7	0.09	4.6	4.4
		6.0	5.0	0.35	4.7	4.3	0.29	4.6	4.3
	1.0	6.3	0.0	0.00	5.2	4.8	0.00	5.1	4.8
		6.3	0.5	0.11	5.2	4.9	0.09	4.9	4.8
		6.3	5.0	0.35	5.2	4.5	0.29	5.1	4.6

Power ($\phi_0 = 1, \rho_0 = 0.05$)

ρ	ϕ	σ_v	CP	$T=200$: $\bar{\pi}_1$	$T=200$: AR	$T=200$: AR^*	$T=300$: $\bar{\pi}_1$	$T=300$: AR	$T=300$: AR^*
0.05	0.2	3.8	0.0	0.00	4.7	4.6	0.00	4.6	4.5
		3.8	0.5	0.11	12.7	12.1	0.09	12.2	11.9
		3.8	5.0	0.35	75.3	73.8	0.29	76.0	73.9
	0.5	6.6	0.0	0.00	4.9	4.8	0.00	4.4	4.5
		6.6	0.5	0.11	10.5	10.1	0.09	10.4	10.4
		6.6	5.0	0.35	63.5	61.7	0.29	63.8	61.6
	0.9	8.9	0.0	0.00	5.0	4.9	0.00	4.4	4.4
		8.9	0.5	0.11	6.3	6.1	0.09	5.7	5.6
		8.9	5.0	0.35	18.9	18.1	0.29	19.0	18.0
	1.0	9.4	0.0	0.00	5.3	5.0	0.00	5.1	5.0
		9.4	0.5	0.11	5.3	5.1	0.09	5.2	5.1
		9.4	5.0	0.35	5.5	5.1	0.29	5.1	5.1
0.10	0.2	2.2	0.0	0.00	4.5	100.0	0.00	4.6	100.0
		2.2	0.5	0.11	9.6	100.0	0.09	9.4	100.0
		2.2	5.0	0.35	54.1	100.0	0.29	54.5	100.0
	0.5	4.3	0.0	0.00	4.9	96.9	0.00	4.4	100.0
		4.3	0.5	0.11	9.6	97.1	0.09	9.5	100.0
		4.3	5.0	0.35	55.4	99.3	0.29	55.6	100.0
	0.9	6.0	0.0	0.00	4.9	30.3	0.00	4.4	54.2
		6.0	0.5	0.11	6.1	31.8	0.09	5.8	55.3
		6.0	5.0	0.35	17.4	45.2	0.29	16.9	65.5
	1.0	6.3	0.0	0.00	5.0	17.3	0.00	4.9	29.8
		6.3	0.5	0.11	5.1	17.2	0.09	4.9	30.0
		6.3	5.0	0.35	5.0	17.1	0.29	4.9	29.4

Panel B: $\mathcal{M}_3$ with low-frequency instruments

Size

ρ	ϕ	σ_v	$T=200$: AR	$T=200$: AR^*	$T=300$: AR	$T=300$: AR^*
0.10	0.5	4.3	4.8	3.9	4.8	4.0
	0.6	4.8	5.0	3.9	5.0	3.9
	0.7	5.2	5.3	3.8	5.2	3.7
	0.8	5.6	5.6	4.0	5.5	3.8
	0.9	6.0	6.2	4.2	6.3	4.0
	1.0	6.3	5.9	3.7	5.5	3.3
0.15	0.5	3.2	4.8	3.4	4.8	3.4
	0.6	3.6	4.9	3.4	5.1	3.2
	0.7	4.0	5.2	3.3	5.0	3.1
	0.8	4.3	5.5	3.2	5.5	3.0
	0.9	4.5	6.3	3.5	6.3	3.3
	1.0	4.8	5.5	2.7	5.2	2.5
0.25	0.5	1.9	4.9	2.6	4.7	2.4
	0.6	2.3	4.9	2.5	4.7	2.1
	0.7	2.5	5.2	2.4	4.9	2.0
	0.8	2.8	5.5	2.0	5.4	1.8
	0.9	3.0	6.4	1.9	6.1	1.9
	1.0	3.1	5.1	1.1	4.9	1.0
0.35	0.5	0.9	4.9	1.9	4.8	1.8
	0.6	1.3	4.9	1.7	4.9	1.2
	0.7	1.6	5.1	1.3	5.2	1.0
	0.8	1.8	5.4	1.1	5.2	1.0
	0.9	1.9	6.3	0.9	6.3	0.8
	1.0	2.1	5.0	0.3	5.1	0.4

Power ($\phi_0 = 1, \rho_0 = 0.1$)

ρ	ϕ	σ_v	$T=200$: AR	$T=200$: AR^*	$T=300$: AR	$T=300$: AR^*
0.10	0.5	4.3	97.8	96.1	99.9	99.8
	0.6	4.8	99.4	98.8	100.0	100.0
	0.7	5.2	99.8	99.4	100.0	100.0
	0.8	5.6	99.8	99.0	100.0	100.0
	0.9	6.0	97.9	92.6	100.0	99.5
	1.0	6.3	5.9	3.7	5.5	3.3
0.15	0.5	3.2	92.8	100.0	99.2	100.0
	0.6	3.6	97.7	100.0	99.9	100.0
	0.7	4.0	99.2	100.0	100.0	100.0
	0.8	4.3	99.3	100.0	100.0	100.0
	0.9	4.5	95.8	99.9	99.9	100.0
	1.0	4.8	4.1	20.5	3.9	44.6
0.25	0.5	1.9	67.9	100.0	85.8	100.0
	0.6	2.3	83.1	100.0	95.6	100.0
	0.7	2.5	90.7	100.0	98.8	100.0
	0.8	2.8	92.8	100.0	99.4	100.0
	0.9	3.0	86.6	100.0	98.1	100.0
	1.0	3.1	1.6	95.1	1.6	99.5
0.35	0.5	0.9	29.9	100.0	42.7	100.0
	0.6	1.3	46.6	100.0	66.0	100.0
	0.7	1.6	60.3	100.0	80.6	100.0
	0.8	1.8	66.5	100.0	86.7	100.0
	0.9	1.9	64.4	100.0	86.1	100.0
	1.0	2.1	0.1	100.0	0.3	100.0

Panel C: $\mathcal{M}_5$ with high-frequency instruments

Size

Freq.	ϕ_l	$\sigma^2_{l,v}$	ρ_l	ρ_0	$T=200$: AR	$T=200$: AR^*	$T=300$: AR	$T=300$: AR^*
1-min	0.5	0.033	0.398	0.398	5.4	1.5	5.6	1.6
	0.6	0.038	0.439	0.439	5.4	0.8	5.4	0.6
	0.7	0.043	0.467	0.467	5.5	0.1	4.9	0.1
	0.8	0.048	0.485	0.485	5.1	0.0	5.2	0.0
	0.9	0.054	0.494	0.494	5.2	0.0	5.2	0.0
	1.0	0.059	0.497	0.497	5.4	0.0	5.3	0.0
5-min	0.5	0.003	0.400	0.400	5.4	1.6	5.0	1.4
	0.6	0.003	0.441	0.441	5.3	0.9	5.2	0.7
	0.7	0.003	0.470	0.470	5.2	0.4	5.4	0.2
	0.8	0.004	0.488	0.488	5.3	0.0	5.6	0.1
	0.9	0.004	0.497	0.497	4.7	0.0	5.2	0.0
	1.0	0.005	0.500	0.500	5.4	0.0	4.9	0.0
10-min	0.5	0.003	0.400	0.400	5.0	1.5	5.2	1.4
	0.6	0.004	0.441	0.441	5.1	0.8	5.7	0.7
	0.7	0.004	0.470	0.470	5.5	0.4	5.9	0.2
	0.8	0.005	0.488	0.488	5.9	0.1	5.9	0.1
	0.9	0.005	0.497	0.497	5.5	0.0	5.7	0.0
	1.0	0.006	0.500	0.500	5.0	0.0	4.9	0.0
15-min	0.5	0.010	0.399	0.399	4.7	1.6	5.1	1.4
	0.6	0.011	0.440	0.440	4.9	0.8	5.1	0.7
	0.7	0.013	0.469	0.469	5.2	0.5	5.2	0.2
	0.8	0.014	0.487	0.487	5.5	0.0	5.4	0.0
	0.9	0.016	0.496	0.496	5.3	0.0	5.0	0.0
	1.0	0.018	0.499	0.499	4.9	0.0	4.6	0.0

Power ($\phi_l = 1, \rho_l = \rho_0$)

Freq.	ϕ_l	$\sigma^2_{l,v}$	ρ_l	ρ_0	$T=200$: AR	$T=200$: AR^*	$T=300$: AR	$T=300$: AR^*
1-min	0.5	0.033	0.398	0.398	99.7	97.5	100.0	99.7
	0.6	0.038	0.439	0.398	100.0	100.0	100.0	100.0
	0.7	0.043	0.467	0.398	100.0	100.0	100.0	100.0
	0.8	0.048	0.485	0.398	100.0	100.0	100.0	100.0
	0.9	0.054	0.494	0.398	100.0	100.0	100.0	100.0
	1.0	0.059	0.497	0.497	5.4	0.0	5.3	0.0
5-min	0.5	0.003	0.400	0.400	96.9	87.3	99.9	98.0
	0.6	0.003	0.441	0.400	97.3	100.0	99.9	100.0
	0.7	0.003	0.470	0.400	98.2	100.0	99.9	100.0
	0.8	0.004	0.488	0.400	99.7	100.0	100.0	100.0
	0.9	0.004	0.497	0.400	100.0	100.0	100.0	100.0
	1.0	0.005	0.500	0.500	5.4	0.0	4.9	0.0
10-min	0.5	0.003	0.400	0.400	96.9	85.6	99.8	97.7
	0.6	0.004	0.441	0.400	97.0	100.0	99.8	100.0
	0.7	0.004	0.470	0.400	97.3	100.0	99.8	100.0
	0.8	0.005	0.488	0.400	98.3	100.0	99.8	100.0
	0.9	0.005	0.497	0.400	99.7	100.0	100.0	100.0
	1.0	0.006	0.500	0.500	5.0	0.0	4.9	0.0
15-min	0.5	0.010	0.399	0.399	96.3	84.5	99.7	97.0
	0.6	0.011	0.440	0.399	96.8	100.0	99.8	100.0
	0.7	0.013	0.469	0.399	98.1	100.0	99.9	100.0
	0.8	0.014	0.487	0.399	99.7	100.0	100.0	100.0
	0.9	0.016	0.496	0.399	100.0	100.0	100.0	100.0
	1.0	0.018	0.499	0.499	4.9	0.0	4.6	0.0

Notes: The instrument set consists of a constant $\bar{\pi}_0 = 1$ and an instrument, $l = 1$. For $\mathcal{M}_1$ with weak instruments, based on the concentration parameter (CP), we construct first-stage coefficients $\bar{\pi}_1$ with $\sigma^2_\epsilon = \pi^2/2$ and $\sigma^2_u = 0.01$. For $\mathcal{M}_5$ with HF instruments, equal-spaced HF intraday data are considered with different frequency [1m, 5m, 10m, and 15m where 1m stands for 1-minute frequency]. We use logarithms of RV measures as instruments for high-frequency design. Details of $\mathcal{M}_5$ design are given in Appendix H.2.3. The inference procedures [AR, AR^*] are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). We use 99 Monte Carlo replications for point-optimal type procedures. For PO tests, we set the alternative to $\rho_1 = \rho$.

where $y_t = \log(s_t^2) + 1.2704$, $\bar{\pi}_1$ is an l-vector of first-stage coefficients, Z_{t-2} is an l-vector of independent $\mathcal{N}(0,1)$ variables, and the vector (ξ_t, η_{t-1}) has zero mean with $\text{Var}(\xi_t) = (1+\phi^2)\sigma_\epsilon^2 + \sigma_v^2$, $\text{Var}(\eta_{t-1}) = \sigma_\epsilon^2 + \sigma_u^2$ and $\text{Cov}(\xi_t, \eta_{t-1}) = -\phi\sigma_\epsilon^2$. Note that (6.1) is equivalent to a log-normal SV model, and in all our simulations we generate (6.1) non-linearly as given in (2.1).

We use 10,000 replications to compute the empirical levels and powers, and 99 replications for PO tests based on the MCT procedure. For all tests, the nominal level is fixed at 5%. Thus, under the null hypothesis, the rejection rates should be less than (or close to) 5% for tests to be valid.

For the DGP (6.1), we evaluate the performance of the asymptotic t-type test $[H_0 : \phi = \phi_0]$. We set $\mu = 0$, $\sigma_v = 2$ and $\phi \in [0, 1]$. Table 1 reports the size and power. The test statistic is calculated using the simple winsorized estimator of Ahsan and Dufour (2019) [see equations (3.8)-(3.9) with $J = 10$ for the estimator and Section 6.1 for the test statistic]. This estimator is more efficient than conventional methods (QMLE, GMM) and as efficient as the Bayesian procedure. In addition to this, it is extremely time-efficient, and it produces empirical estimates which are similar to the Bayesian estimates. For the details of this asymptotic t-test, see Ahsan and Dufour (2019, 2021). From the results, we can see that the asymptotic t-test (based on delta-method local approximations) fails to control the level when $\phi \longrightarrow 1$. Size distortions are severe and equal up to 38.1% when $\phi = 1$. These size distortions persist even in larger samples ($T = 5000$, 10000), particularly when $\phi > 0.999$. For theoretical discussions of this issue (the unreliability of asymptotic standard error methods), see Dufour (1997) and Dufour et al. (2024), especially when persistence is high, which appears to be common in practice.

We will now examine the performance of the tests proposed in Sections 3.1-3.2. We focus on empirically motivated misspecified model setups with weak, low- and high-frequency instruments to simplify the exposition: (1) weak instrument designs where the generated IVs are weakly correlated [based on the concentration parameter (CP)] with past lags of y_{t-1} [$\mathcal{M}_1$: (6.1)-(6.2) with $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \pi^2/2)$ and $\mathcal{M}_2$: (6.1)-(6.2)]; (2) low-frequency instrument designs where we use past lags of y_{t-1} as IVs [$\mathcal{M}_3$: (6.1)-(6.2) and $\mathcal{M}_4$: (6.1)-(6.2) with $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \pi^2/2)$]; (3) high-frequency instrument designs where we use HF realized volatility measures as IVs [$\mathcal{M}_5$: (6.1)-(6.2) with $|\phi| < 1$ and $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \sigma_\epsilon^2)$, and $\mathcal{M}_6$: (6.1)-(6.2)]. Further, model $\mathcal{M}_5$ is closed under temporal aggregation; see Appendix E for related discussion and proof. Consequently, in this design, we make inferences for the low-frequency model parameters using generated IVs from the HF series. Note that $|\phi| < 1$ is required for the identification of μ_l parameter under temporal aggregation. However, it also ensures stationarity and invertibility of both HF and LF models.

We see that models $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_6$ correspond to a log-normal SV model with nonstationary volatility from the above setups. Therefore, it is easy to see that these models ($\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_6$) are misspecified under Assumptions 2.2 and 2.4. On the other hand, in models $\mathcal{M}_1$, $\mathcal{M}_4$, and $\mathcal{M}_5$, we have Gaussian noise for ϵ_t; thus, these models are correctly specified under Assumption 2.4 but misspecified under Assumption 2.2. Note that all these models violate the independence assumption, which is in line with the property of financial returns. However, the instrument set Z_{t-2} is uncorrelated with η_{t-1}. These models are designed to broadly mimic the features of financial returns used in our empirical application.

To save space, we present only the results of models $\mathcal{M}_1$, $\mathcal{M}_3$, and $\mathcal{M}_5$ (given in Table 2) with the number of IVs $l = 1$. Additional results of models $\mathcal{M}_1$, $\mathcal{M}_3$, and $\mathcal{M}_5$ with $l = 3, 5$ and other parameter values, and the results of models $\mathcal{M}_2$, $\mathcal{M}_4$, and $\mathcal{M}_6$ are reported in Tables A2-A7 of Appendix H. The results of Table 2 confirmed the theoretical contributions of Sections 3.1-3.2 even with model misspecification. Our findings can be summarized as follows.

First, from Table 2, the levels of the proposed tests (AR, AR^*) are well controlled: rejection frequencies are less than (or close to) 5%. This result holds whether the identification is completely failed [CP = 0], weak [CP $\in (0, 0.5)$], partial [CP $\in (0.5, 5)$], moderately strong [CP = 5] (from Panel A), or strong to very strong [these are with LF and HF IVs, see Panel B and C]. This represents a substantial improvement over the asymptotic test. This result also holds whether sample sizes are different ($T = 200, 300$), or the instrument set contains a different number of IVs ($l = 3, 5$) [see Tables A2, A4 and A6]. However, as the number of IVs increases, PO tests are undersized with HF IVs when $\rho \longrightarrow 0.5$: rejection frequencies are less than 5% and close to 0%. This shows that PO tests need large samples for level control with HF IVs. In all cases, AR tests perfectly control the level.

Second, from Table 2, all tests exhibit excellent power as long as identification is not very weak. As expected, the power of these tests increases with sample size and concentration parameter (in many cases, rejection frequencies reach 100%) and decreases as the number of IVs increases [see Tables A2, A4, and A6]. Note that, in our joint tests, we have an additional restriction under the null hypothesis on the parameter of the error distribution. This restriction works as an additional source of power for the optimal tests since PO tests can gain power from the differences in covariance structure, *i.e.*, when $\rho_1 \neq \rho_0$. Hence, when $\rho_1 > \rho_0$, PO tests outperform their counterpart as expected. However, AR tests have more power in all cases compared to their counterpart AR^*. In HF design, from Panel C, in all cases of HF IVs (1-minute to 15-minute), the proposed tests have excellent power against the alternative: up to 100%, 100%, 99.2%, and 100%, respectively and the power of these tests increases with the sample size, and decreases as the number of IVs increases. All tests have excellent power across different sampling frequencies, and these tests gain power when the sampling frequency increases.

Third, from Appendix H, the empirical levels of the proposed tests are almost identical to those obtained when the model is only misspecified under Assumption 2.2 [compare Table A2 with Table A3]: rejection frequencies are similar [less than (or close to) 5% for all levels of identification] for all sample sizes considered. Further, from Table A3, the misspecification of the error distribution [$\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$] does not affect the power of these tests [compare Table A3 with Table A2]. Overall, these tests appear to be reasonably robust to a misspecification of the error distribution, even with small samples. The above results also hold for low- and high-frequency designs [compare Table A5 with Table A4 and Table A6 with Table A7]. However, from Table A7, when we simulate the $\mathcal{M}_6$ model under nonstationary volatility, results are slightly different [compare Table A7 with Table A6]: level controls are similar, but rejection frequencies for power simulations are different.

7 Application to stock prices

In this section, we consider various types of financial data, discuss a large number of IVs, and examine the strength of these IVs. The proposed tests are implemented with various IVs and confidence intervals for the volatility persistence parameter ϕ are constructed by inverting the tests.

7.1 Data description

The LF daily prices are obtained from the CRSP database. The raw series p_t is converted to returns by the transformation $r_t := 100[\log(p_t) - \log(p_{t-1})]$ and the returns are converted to residual returns by $s_t := r_t - \hat{\mu}_r$, where $\hat{\mu}_r$ is the sample average of returns. The sample period is from January 1, 2009, to December 31, 2013 (1258 trading days). The daily volatility proxy is constructed by the transformation $y_t = \log(s_t^2) + 1.2704$. Initially, we consider daily IVs of nine stocks: General Electric Company (GE), IBM Common Stock (IBM), JPMorgan Chase & Co. (JPM), The Coca-Cola Co (KO), Pfizer Inc. (PFE), Exxon Mobil Corporation (XOM), The Procter and Gamble Company (PG), AT&T Inc. (T) and Walmart Inc. (WMT). After examining the strength of daily IVs [see Appendix L], we proceed with IBM stock and consider realized measures and option implied volatilities as IVs.

IBM's tick price data are taken from the TAQ (Trade and Quote) database and option (American) data are sourced from the OptionMetrics database. The access to these databases (CRSP, TAQ, OptionMetrics) is done through the Wharton Research Data Services. Using the tick data, we construct a large number of HF IVs. Details of these HF IVs are given in Appendix J and computations are carried out using the MATLAB Oxford MFE Toolbox developed by Sheppard (2013).[10] From IBM American options, three classes of implied volatility (ImV) are considered: (1) call options; (2) put options; (3) both call and put options. For each class, we use all implied volatilities available at a given date to construct six ImV subclasses, which are mean, minimum, maximum, and three quantiles (q1, q2, q3).

[10]The Oxford MFE Toolbox can be downloaded from the GitHub: https://github.com/bashtage/mfe-toolbox.

7.2 Final instrument set, econometric model and test statistics

We consider one hundred and seventy-five IVs, which can be divided into 22 classes. The description of these IVs are given in Appendix Table A12. The HF subclass includes different sampling frequencies [tick, second and minute], sampling scheme [tick or business], and sub-sampling; these are discussed in Appendix J. We use 1-minute sub-sampling [ss] in the calculation of several HF measures.

The final instrument set also includes principal component factors (PCF) and daily log volatility of y_t. The three largest principal component factors are extracted from HF IVs. Formally, PCF-based identification-robust inference in the context of IV regressions was considered by Kapetanios et al. (2016) to deal with the problem of many IVs. Note that we use logarithms of RV-RSVP and PCF classes of IVs; see Table A12 for details about transformations.

For empirical analysis, we consider the following GSV model:

$$w_t = \mu + \phi w_{t-1} + v_t\,, \qquad y_t = w_t + \epsilon_t\,, \qquad v_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma_v^2)\,, \quad \epsilon_t \sim \text{i.i.d. } \log(\chi^2_{(1)})\,, \tag{7.1}$$

$$y_{t-1} = \bar{\pi}_0 + Z'_{t-2}\bar{\pi}_1 + \eta_{t-1}\,, \quad \eta_{t-1} := \epsilon_{t-1} + u_{t-1}\,, \quad u_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma_u^2)\,, \tag{7.2}$$

where $w_t = \log(\sigma_t^2)$, $y_t = \log(s_t^2) + 1.2704$ with $s_t := r_t - \mu_r$ is residual return of an asset with μ_r is the mean of return $r_t = 100[\log(p_t) - \log(p_{t-1})]$ and Z_{t-2} is the set of IVs.

For inference, we consider joint tests $(\phi, \rho) = (\phi_0, \rho_0)$. The inference procedures (AR, AR^*) are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). We use 99 Monte Carlo replications for PO type procedures.

7.3 Projection-based confidence sets

We discuss and build projection-based confidence sets for the volatility persistence parameter. To construct a projection-based confidence interval for the volatility persistence parameter ϕ, we first construct a confidence interval for λ with level $(1 - \alpha_1)$, denoted as $C_{\alpha_1}(\lambda)$. We parametrize the noise ratio λ rather than ρ since this is the more natural choice. We set $\alpha_1 = 0.05$, and compute λ using the simple winsorized method proposed by Ahsan and Dufour (2019). We use equations (3.8)-(3.9) with $J = 10$ of Ahsan and Dufour (2019) to estimate σ_v^2 and the corresponding standard error (SE). By setting $\sigma_\epsilon^2 = \pi^2/2$, the SE of $\hat{\lambda} = \sigma_\epsilon^2/\hat{\sigma}_v^2$ is computed using the delta method. The estimated 95% confidence interval for the nuisance parameter λ is $C_{0.05}(\lambda) = [33.943, 61.154]$ with $\hat{\lambda} = 47.548$ and $\text{SE}(\hat{\lambda}) = 6.935$. For each value of λ in the confidence interval $C_{\alpha_1}(\lambda)$, we then construct $(1 - \alpha_2)$ confidence intervals for ϕ given λ [denoted by $C_{\alpha_2}(\phi|\lambda)$] by inverting a test robust to weak IVs proposed in Sections 3.1-3.2. By Bonferroni's inequality, this confidence interval has coverage of at least $100(1 - \alpha)$%, where $\alpha = \alpha_1 + \alpha_2$. If we use $\alpha_2 = 0.05$, then a

90% confidence interval for ϕ which does not depend on λ can be obtained by

$$C_{0.10}(\phi) = \bigcup_{\lambda \in C_{0.05}(\lambda)} C_{0.05}(\phi|\lambda)\,. \tag{7.3}$$

The projection method is thoroughly discussed in Section 3.3. Note that we employ grid testing during the test inversion, in which a series of tests $[H_0 : \phi = \phi_0, \lambda = \lambda_0$, where $\phi_0 \in [0,1]$, $\lambda_0 \in C_{\alpha_1}(\lambda)]$ performed. Note that we restrict ϕ_0 in the most relevant part of the parameter space, *i.e.*, $\phi_0 \in [0,1]$. Note that these confidence intervals formed from a range of accepted values due to grid testing; thus, it is easy to get a nonparametric estimate of ϕ by applying the Hodges-Lehmann principal.

We use $\alpha_1 = \alpha_2$, which is the rule typically employed in the literature on simultaneous inference (*e.g.*, in Bonferroni-type procedures) and test combination; see Miller (1981), Savin (1984). Cavanagh et al. (1995) suggested a refinement of the Bonferroni method which makes it less conservative than the basic approach. The idea is to shrink the confidence interval for λ so that the refined interval is a subset of the original (unrefined) interval. This consequently shrinks the Bonferroni confidence interval for ϕ, achieving an exact test of the desired significance level. However, it is important to note that α should be selected a priori, not on the basis of the results yielded by different choices of α_1 for a given sample.

7.4 Precision (or informational efficiency) of instruments

We define the notions of precision (or informational efficiency) and average precision of instruments using the corresponding lengths of these identification-robust confidence sets. As pointed out by Dufour (1997), when IVs are arbitrarily weak, then confidence sets with correct coverage probability must have an infinite length with positive probability.[11] As a result, the length of a weak instrument robust confidence interval can summarize the identification strength of the corresponding instrument. Since we restrict $\phi_0 \in [0,1]$, then an irrelevant (no identification) instrument for the regressor should produce a confidence interval with length equal to 1.

From an identification-robust confidence interval, we define the precision (or informational efficiency) of an instrument set i as follows:

$$d_i := 1 - (ub_i - lb_i) \tag{7.4}$$

where ub and lb are the upper and lower bound of the confidence set, and $ub - lb$ is the length of the confidence set. The definition d_i implies that if i is a weak instrument then it will produce d_i close to 0 and if i is a strong instrument then it will produce d_i close to 1. For example, a large value of d_i implies

[11] Dufour (1997) showed that if the IVs are not correlated with the regressor [irrelevant IVs], then the corresponding parameter is not identified, and any value of the parameter is consistent with data. A valid confidence set in such a case must be infinite, at least with probability equal to the coverage. Most empirical applications use the conventional Wald confidence interval, which is always finite. As a result, the Wald confidence interval has a low coverage probability and should not be used when IVs are weak.

that the corresponding instrument set is highly informative about the parameter ϕ.

Figure A1 of Appendix G shows the precision measure d_i of different classes of IVs, where the instrument set consists of a constant and a lag of the corresponding instrument. For each class, we consider average, median, minimum, and maximum precision measures across the proposed inference methods. The following inferences emerge from Figure A1. *First*, except for JV and SJV classes, all HF classes are considered as strong instruments, *i.e.*, these classes produce very high d_i values. These results hold in all precision measures and across four inference methods. *Second*, JV and SJV classes have many weak and irrelevant (no identification) IVs because average and median precision measures of these classes are low and zero, respectively. These results suggest that JV and SJV classes have no or little predictive power regarding the latent daily volatility. However, log squared JV and SJV IVs are informative about the volatility clustering. This finding suggests that the second moment of jumps or signed jumps is correlated with the latent daily volatility proxy. *Third*, both PCF and ImV classes have some relevant IVs. However, all ImV classes include some weak IVs.

Figure A2 of Appendix G shows the precision measure of different subclasses of HF IVs. On average, all HF subclasses produce confidence intervals with similar lengths, *e.g.*, on average, both 1s and 5m produce almost similar identification-robust confidence intervals. Hence, it is easy to see that each HF subclasses contains some IVs with strong identification.

To formalize, we also define the notion of the average precision of an instrument set i over the proposed inference methods by

$$\bar{d}_{i,s} := \sum_{i=1}^{S} d_i / S \tag{7.5}$$

where $s \in S$ and S is the set of identification-robust inference methods. We use this measure to rank the information content of instruments.

7.5 Empirical Results

We construct projection-based 90% confidence intervals for ϕ using numerous types of instruments; then, using the proposed identification measures (precision and average precision), we identify several crucial empirical stylized facts. To preserve space, we present only the results with strong IVs and other complementary results are given in Tables A15-A20 of Appendix M.

7.5.1 Superior instruments

Table 3 reports the projection-based 90% confidence intervals for ϕ using strong IVs, *i.e.*, based on $\bar{d}_{i,s}$. Panel A includes superior IVs while panel B and C [see Table A15] include IVs which produce slightly larger confidence sets compared to the IVs in panel A. Panel A mostly includes HF IVs, and 70% of these are 5m

Table 3. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Strong instruments
Ticker: IBM, January 2009 - December 2013, $T = 1258$

Panel A					Panel B				
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*	No	Instruments	$\bar{d}_{i,s}$	AR	AR^*
1	RSVN-5m-ss	0.8860	[0.950, 1.0]	[0.866, 1.0]	11	ImV-C-q3	0.8805	[0.964, 1.0]	[0.843, 1.0]
2	RSVN-5m	0.8855	[0.948, 1.0]	[0.864, 1.0]	12	RV-1m	0.8800	[0.944, 1.0]	[0.857, 1.0]
3	RSVN-1m	0.8848	[0.947, 1.0]	[0.856, 1.0]	13	ImV-C-q2	0.8795	[0.958, 1.0]	[0.860, 1.0]
4	ImV-C-mean	0.8830	[0.964, 1.0]	[0.852, 1.0]	14	RRV-1m	0.8790	[0.945, 1.0]	[0.858, 1.0]
5	MinRV-5m	0.8828	[0.945, 1.0]	[0.867, 1.0]	15	MedRV-1m	0.8785	[0.944, 1.0]	[0.857, 1.0]
6	RV-5m-ss	0.8825	[0.946, 1.0]	[0.863, 1.0]	16	RV-5m	0.8783	[0.943, 1.0]	[0.858, 1.0]
7	BV-5m	0.8823	[0.945, 1.0]	[0.865, 1.0]	17	BV-1m	0.8775	[0.944, 1.0]	[0.857, 1.0]
8	BV-5m-ss	0.8823	[0.945, 1.0]	[0.865, 1.0]	18	RSVN-10m-ss	0.8775	[0.949, 1.0]	[0.858, 1.0]
9	BV-10m-ss	0.8823	[0.945, 1.0]	[0.865, 1.0]	19	RSVN-10m	0.8760	[0.946, 1.0]	[0.861, 1.0]
10	MedRV-5m	0.8823	[0.945, 1.0]	[0.866, 1.0]	20	RV-10m-ss	0.8758	[0.944, 1.0]	[0.857, 1.0]

Notes: The instrument set consists of a constant and a lag of an instrument, $l = 1$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The inference procedures $[AR, AR^*]$ are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). The confidence intervals are constructed by projection technique described in Section 3.3. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and $\text{SE}(\hat{\lambda}) = 6.935$. We use 99 Monte Carlo replications for point-optimal type procedures. The average precision of an instrument set i over the proposed inference methods is measured by $\bar{d}_{i,s} := S^{-1}\sum_{i=1}^{S} d_i$, where $s \in S$ and S is the set of identification-robust inference methods.

subclass [consistent with Liu et al. (2015)]. This finding proves that HF RV does provide an additional gain in predicting the LF volatility proxy.

Panel A of Table 3 reveals a strong relationship between negative realized semivariance (RSVN) and low-frequency volatility. The top three strong IVs are all RSVN, where 5-minute subsampled RSVN has the most predictive power. This result is related to recent studies [see Patton and Sheppard (2015), Chen and Ghysels (2011), Audrino and Hu (2016), Baillie et al. (2019), Bollerslev et al. (2020)], which showed that negative realized semivariance is crucial for asset pricing, volatility modelling, and forecasting. The average implied volatility which extracts from IBM call options is also a strong instrument. This finding is in line with Christensen and Prabhala (1998), who find that implied volatility has large explanatory power regarding past volatility.

7.5.2 Robustness to dynamics

Table 3 also gives several other conclusions. *First*, we can infer from these confidence sets that the persistence parameter lies roughly between 0.9 and 1.0 for IBM. This outcome indicates that the volatility process is highly persistent, close to unit-root, consistent with the empirical literature; see Harvey et al. (1994), Hansen (1995), Broto and Ruiz (2004). These confidence sets include $\phi = 1$, implying that these sets are also robust to nonstationarity. *Second*, in all cases, simulation-based point-optimal confidence sets are conservative compared to the corresponding AR-type confidence sets. In all tests and across instruments, we do not reject the null hypothesis of nonstationary stochastic volatility.

7.5.3 Robustness to weak instruments

Table A16 presents the projection-based 90% confidence intervals for ϕ using weak IVs, *i.e.*, based on $\bar{d}_{i,s}$. Panel A of Table A16 contains IVs with no identification. As a result, these IVs produce unbounded confidence intervals. These confidence intervals cover the entire set of $\phi \in [0,1]$. Panel A comprises mostly by JV and SJV HF classes and ImV-max subclass. Note that under no identification, all values of ϕ are observationally equivalent, which implies that the proposed test statistics yield valid confidence sets which are unbounded with a non-zero probability. Consequently, the proposed tests are robust to weak identification. From Panel C, we find that the LF daily instrument produces a valid confidence set. However, the length of this set is larger compared to HF confidence sets given in Table 3.

7.5.4 Robustness to microstructure noise

It is well-known that the market microstructure noise becomes progressively more dominant as the sampling frequency increases; see Zhang et al. (2005), Bandi and Russell (2008), and Hansen and Lunde (2006). From Table A15, we find that confidence sets with 30s RVs [Panel C: RSVN-30s, RV-30s, BV-30s, MSRV-30s] are spacious than confidence sets with 5m RVs [Panel A and B] and conclude that the effect of market microstructure noise leads to slightly wider confidence sets. Thus, our result suggests that the proposed inference methods produce valid confidence sets even with noisy RVs at a higher frequency. Further, 85% of the time, Panel A and B include IVs with frequency 1m, 5m, and 10m. These confidence sets are less sensitive to the market microstructure noise.

Further, the constant term $\bar{\pi}_0$ in the instrument equation (7.2) may captures the bias in the RV estimate due to the non-trading hours and microstructure noise. As pointed out by Takahashi et al. (2009), if the bias-correction term $\bar{\pi}_0$ is negative, RV has an upward bias which may be due to the market microstructure noise, and if $\bar{\pi}_0$ is positive, it has a downward bias due to the non-trading hours. Hence, this bias-correction term may provide an additional layer of robustness to the proposed methodology in the presence of non-trading hours and microstructure noise even with a very high sampling frequency.

7.5.5 Robustness to jumps

Table A17 presents the projection-based 90% confidence intervals for ϕ using RV and BV measures as IVs. From Panel A and B of Table A17, we see that both RV and BV measures produce almost identical confidence intervals across all 14 subclasses. This result is consistent with the fact that the proposed tests are robust to missing IVs (or instrument exclusion) [see Dufour and Taamouti (2007) for theoretical results]. When we make inferences with BV, then jump variation is considered as a missing instrument;

hence, BV produces a valid confidence set. Note that alternative methods of inference aimed at being robust to weak identification [see Wang and Zivot (1998), Kleibergen (2002), Moreira (2003), etc.] do not enjoy this type of robustness. Further, RVs produce almost identical confidence intervals as with BVs, confirming that our methodology is robust in the presence of jumps.

7.5.6 Combination of strong instruments

In Table A18, we report the estimated confidence intervals, where the instrument set includes a constant and several lags of an instrument, $l = 1, 3, 5$. In this setup, we use the first set of strong IVs [Table A15 - Panel A], ImV-C-q3, and 1-day. In most cases, we find that all confidence intervals for ϕ (AR, AR^*) are getting wider as l increases. The average length of these confidence intervals when $l = 3, 5$ are larger than the confidence intervals were when $l = 1$. Therefore, we do not see any apparent gains by adding more lags in the instrument set. The only exception is the LF daily instrument, where the average length of confidence intervals is shorter than before. This result implies that we should use more daily lags as IVs to get a smaller confidence set. We also construct several confidence sets where the instrument set includes a constant and various combinations of strong IVs. We report these confidence sets in Table A19. The conclusion is similar to Table A18, *i.e.*, no apparent gains from combining strong IVs.

7.5.7 Nonlinear relationship between jumps and LF volatility

Table A20 presents the projection-based 90% confidence intervals for ϕ using jump variation (JV) and log squared jump variation (LJV) measures as IVs. From Panel A and B of Table A20, we see that both JV and LJV measures yield completely different confidence intervals across all subclasses except 5m; JV measures produce unbounded sets, while LJV measures provide informative sets. This result suggests there may be a nonlinear relationship between jumps and low-frequency volatility.

8 Conclusion

This paper has introduced a novel class of GSV models, which can use high-frequency information content and accommodate nonstationary volatility. We employ IV methods to provide a unified framework for the analysis of GSV models. Within this framework, we have studied the problem of testing hypotheses and building confidence sets for the volatility persistence parameter. This parameter has an intrinsic interest because it measures the persistence of the latent volatility process. We proposed more reliable identification-robust finite-sample procedures, which are robust to weak IVs and/or nonstationary latent volatility. We also showed that these finite-sample procedures (based on a Gaussian assumption on the errors) remain asymptotically valid under weaker distributional assumptions. We then study the statistical properties of the proposed tests in simulation experiments. These tests outperform the asymptotic t-type test in terms of size and exhibit excellent power.

We applied these methods to IBM's price and option data and observed several empirical facts. The superior instrument set constitutes of HF realized measures and call option implied volatilities. These IVs produce confidence sets, which show that the latent volatility process of IBM is close to unit-root. We find RVs at higher frequency produce more spacious confidence intervals than RVs at slightly lower frequencies, pointing out that these confidence intervals adjust to incorporate the microstructure noise. We also find jumps and signed jumps have little information content regarding the low-frequency volatility, whereas their log squared versions have strong identification strength. When we consider irrelevant or weak instruments, the proposed procedures give unbounded confidence intervals. These confidence sets can be extended to allow for non-Gaussian error distributions [where the conditional distribution of scale transformed error has a non-Gaussian error distribution] using the MCT procedure (Section 4).

This paper focuses on testing assumptions on the persistence parameter ϕ, which are central in the present context. Of course, other hypotheses can be considered. It is important to remember that all the assumptions and restrictions which define a hypothesis are jointly tested. Error normality is a defining feature of the stochastic volatility model, which still allows the model to reproduce heteroskedasticity and heavy-tailed marginal distributions. However, we may still wish to consider other possible distributions. Given the regression framework (3.2), it is relatively simple to adapt standard specification tests to our context. The Monte Carlo test approach of Section 4 does allow one to test normality and use various non-Gaussian distributions to build confidence intervals. The inference methods developed in this paper can also be adapted to other situations, *e.g.*, measurement error in ARMA-type models, noisy realized measures in HAR volatility modeling, and multivariate models. Such extensions are topics of ongoing research.

Data availability statement

The data source is described in Section 7.1. The LF daily prices are obtained from the CRSP database. The raw series p_t is converted to returns by the transformation $r_t := 100[\log(p_t) - \log(p_{t-1})]$ and the returns are converted to residual returns by $s_t := r_t - \hat{\mu}_r$, where $\hat{\mu}_r$ is the sample average of returns. The sample period is from January 1, 2009, to December 31, 2013 (1258 trading days). The daily volatility proxy is constructed by the transformation $y_t = \log(s_t^2) + 1.2704$. Initially, we consider daily IVs of nine stocks: General Electric Company (GE), IBM Common Stock (IBM), JPMorgan Chase & Co. (JPM), The Coca-Cola Co (KO), Pfizer Inc. (PFE), Exxon Mobil Corporation (XOM) and (2) The Procter and Gamble Company (PG), AT&T Inc. (T) and Walmart Inc. (WMT). IBM's tick price data are taken from the TAQ (Trade and Quote) database and option (American) data are sourced from the OptionMetrics database. The access to these databases (CRSP, TAQ, OptionMetrics) is done through the Wharton Research Data

Services (https://wrds-www.wharton.upenn.edu). Using the tick data, we construct a large number of HF IVs. From IBM American options, three classes of implied volatility (ImV) are considered: (1) call options; (2) put options; (3) both call and put options. For each class, we use all implied volatilities available at a given date to construct six ImV subclasses, which are mean, minimum, maximum, and three quantiles (q1, q2, q3).

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High-frequency instruments and identification-robust inference for stochastic volatility models

Supplementary Appendix

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October 2024

This online Supplementary Appendix contains additional details and complementary results relevant to the paper.

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A Discussions on the importance of volatility persistence parameter

The volatility persistence parameter captures volatility clustering and plays a crucial role in many areas of financial economics. *First*, asset allocation theories have shown that this parameter can reflect the persistence in the risk premium, *e.g.*, when there is high persistence in volatility (a strong negative relation between return and volatility), a rational investor should frequently and permanently change the weighting of assets whenever a volatility shock arrives; see Bollerslev and Engle (1993), Chou (1988), So and Li (1999). *Second*, a confidence set of the volatility persistence parameter determines the conditional volatility forecast interval given the current volatility, which in turn determines the prediction interval of returns through the projection technique (risk-return trade-off). This is important for risk management, option pricing, and asset pricing:

- accurate estimation of the tails of the return distribution are of particular importance for risk management tools (Value at Risk and Expected Shortfall); see Taylor (1999);
- the volatility forecast interval is important for option pricing [see Hansen (1994)];
- accurate confidence interval estimation of volatility has consequences for the forecasts of the conditional mean (prediction interval of returns) through projection techniques; see Baillie and Bollerslev (1992), Hansen (1995), Poterba and Summers (1986).

B Discussions on nonstationarity in conditional variance

Nonstationarity in the volatility process has been well documented for macroeconomic and financial time series data; see Pagan and Schwert (1990), Loretan and Phillips (1994), McConnell and Perez-Quiros (2000), Blanchard and Simon (2001), Busetti and Taylor (2003), Sensier and Dijk (2004), Cavaliere and Taylor (2007) and Cavaliere and Taylor (2008). For instance, nonstationary volatility arises when the variance is trending (upward or downward) or undergoes structural breaks. Several studies note that the empirical estimate of the dominant root of the SV-type process is close to the unit circle; see Harvey et al. (1994), Hansen (1995), Broto and Ruiz (2004). Conditional variance nonstationarity is also important from a theoretical point of view and has broad implications for the construction of long-term volatility forecasts, which are essential in many asset-pricing models; see Poterba and Summers (1986). Inference under nonstationary stochastic volatility is rarely considered in the literature. Hansen (1995) and Boswijk et al. (2021) are the notable studies which proposed robust inference methods for the mean equation with nonstationary stochastic volatility. Compared to these studies, we consider inference on the nonstationary volatility equation.

C Proofs

PROOF OF PROPOSITION 4.1 Suppose 4.1 holds and $\phi = \phi_0$, $\rho = \rho_0$. Under Assumptions 2.1-2.3, equation (3.11) holds. Then, on multiplying the two sides of (3.11) by $M_{C_0}[X] - M_{C_0}[X, Z_{-2}]$ and $M_{C_0}[X]$, we have:

$$(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])C_0(y - \phi_0 y_{-1}) = \sigma_\xi (M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta\,, \tag{C.1}$$

$$M_{C_0}[X]C_0(y - \phi_0 y_{-1}) = \sigma_\xi M_{C_0}[X]\vartheta\,. \tag{C.2}$$

Thus, the AR-statistic in (3.12) can be rewritten as:

$$AR(\phi_0, \rho_0) = \frac{\sigma_\xi^2 \vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta/l}{\sigma_\xi^2 \vartheta' M_{C_0}[X]\vartheta/(T-l-k)} = \frac{\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta/l}{\vartheta' M_{C_0}[X]\vartheta/(T-l-k)}\,. \tag{C.3}$$

Hence, the null conditional distribution of $AR(\phi_0, \rho_0)$, given $\overline{X}$, only depends on distribution of ϑ. If normality holds conditional on $\overline{X}$, *i.e.*, $\vartheta \mid \overline{X} \sim \mathcal{N}(0, I_T)$, we have $\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta \sim \chi^2_{(l)}$ and $\vartheta' M_{C_0}[X]\vartheta \sim \chi^2_{(T-l-k)}$. Since $M_{C_0}[X, Z_{-2}](M_{C_0}[X] - M_{C_0}[X, Z_{-2}]) = 0$, hence $\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta$ and $\vartheta' M_{C_0}[X]\vartheta$ are independent conditional on $\overline{X}$. Consequently, $AR(\phi_0, \rho_0) \sim F(l, T-l-k)$. □

PROOF OF PROPOSITION 5.1 Under the null hypothesis $\phi = \phi_0$,

$$AR_T(\phi_0) = \kappa(T)\frac{\Lambda_{1T} - \Lambda_{2T}}{\Lambda_{2T}/T}, \tag{C.4}$$

where

$$\Lambda_{1T} := \xi(T)' M[Q_{1T}]\xi(T), \quad \Lambda_{2T} := \xi(T)' M[Q_T]\xi(T), \quad \kappa(T) := \frac{T-l-k}{lT}. \tag{C.5}$$

Under the Assumption (5.1), we have

$$\kappa(T) \underset{T\to\infty}{\longrightarrow} 1/l, \tag{C.6}$$

$$q_2 \mid q_1 \sim \mathcal{N}(\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1, \sigma^2_\xi \Sigma_{q_2|q_1}), \tag{C.7}$$

where $\Sigma_{q_2|q_1} = \Sigma_{Q_2Q_2} - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}$. Then

$$(q_2 - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1)'\Sigma^{-1}_{q_2|q_1}(q_2 - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1) \sim \sigma^2_\xi \chi^2_{(l)}. \tag{C.8}$$

$$\begin{aligned}
\Lambda_{1T} - \Lambda_{2T} &= \xi(T)' M[Q_{1T}]\xi(T) - \xi(T)' M[Q_T]\xi(T) \\
&= \xi(T)'(I - P[Q_{1T}])\xi(T) - \xi(T)'(I - P[Q_T])\xi(T) \\
&= \xi(T)' Q_T (Q_T' Q_T)^{-1} Q_T' \xi(T) - \xi(T)' Q_{1T}(Q_{1T}' Q_{1T})^{-1} Q_{1T}' \xi(T) \\
&= \xi(T)' Q_T D_T (D_T' Q_T' Q_T D_T)^{-1} D_T' Q_T' \xi(T) - \xi(T)' D_{1T} Q_{1T} (D_{1T}' Q_{1T}' Q_{1T} D_{1T})^{-1} D_{1T}' Q_{1T}' \xi(T) \\
&\Longrightarrow q' \Sigma^{-1}_{QQ} q - q_1' \Sigma^{-1}_{Q_1Q_1} q_1.
\end{aligned} \tag{C.9}$$

Now using standard formulas of a partitioned matrix inverse for Σ_{QQ} and setting $S = q'\Sigma^{-1}_{QQ} q - q_1'\Sigma^{-1}_{Q_1Q_1} q_1$ [see Gentle (2007, Section 3.4.1)], we have

$$\begin{aligned}
S &= q'\Sigma^{-1}_{QQ} q - q_1'\Sigma^{-1}_{Q_1Q_1} q_1 \\
&= (q_1', q_2')' \begin{bmatrix} \Sigma^{-1}_{Q_1Q_1} + \Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1}\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} & -\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1} \\ -\Sigma^{-1}_{q_2|q_1}\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} & \Sigma^{-1}_{q_2|q_1} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - q_1'\Sigma^{-1}_{Q_1Q_1} q_1 \\
&= q_1'\Sigma^{-1}_{Q_1Q_1} q_1 + q_1'\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1}\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1 - 2q_2'\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1} q_2 + q_2'\Sigma^{-1}_{q_2|q_1} q_2 - q_1'\Sigma^{-1}_{Q_1Q_1} q_1 \\
&= q_1'\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1}\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1 - 2q_2'\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1} q_2 + q_2'\Sigma^{-1}_{q_2|q_1} q_2 \\
&= (q_2 - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1)'\Sigma^{-1}_{q_2|q_1}(q_2 - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1).
\end{aligned} \tag{C.10}$$

Thus, from (C.8), (C.9), and (C.10), we get:

$$\Lambda_{1T} - \Lambda_{2T} \Longrightarrow \sigma^2_\xi \chi^2_{(l)} \quad \text{and} \quad \frac{\Lambda_{2T}}{T} \underset{T\to\infty}{\overset{p}{\longrightarrow}} \sigma^2_\xi \tag{C.11}$$

hence

$$AR_T(\phi_0) \Longrightarrow \frac{\chi^2_{(l)}}{l}. \tag{C.12}$$

□

PROOF OF PROPOSITION 5.2 Under the null hypothesis $(\phi = \phi_0, \rho = \rho_0)$,

$$AR_T^*(\phi_0, \rho_0, \rho_1) = \frac{\Lambda_{1T} - \Lambda_{2T}}{\Lambda_{2T}/T} \tag{C.13}$$

where

$$\Lambda_{1T} := \hat{\xi}(T)' M[\hat{Q}_{1T}]\hat{\xi}(T), \quad \Lambda_{2T} = \tilde{\xi}(T)' M[\tilde{Q}_T]\tilde{\xi}(T). \tag{C.14}$$

Under the Assumption 5.2, we have:

$$\begin{aligned}\Lambda_{2T}/T &= \tilde{\xi}(T)'\tilde{\xi}(T)/T - \tilde{\xi}(T)' P[\tilde{Q}_T]\tilde{\xi}(T)/T \\ &= \tilde{\xi}(T)'\tilde{\xi}(T)/T - \tilde{\xi}(T)'\tilde{Q}_T D_T (D_T'\tilde{Q}_T'\tilde{Q}_T D_T)^{-1} D_T'\tilde{Q}_T'\tilde{\xi}(T)/T \xrightarrow[T\to\infty]{\text{p}} \sigma_\xi^2\end{aligned} \tag{C.15}$$

where the last equality follows from

$$\tilde{\xi}(T)'\tilde{\xi}(T)/T \xrightarrow[T\to\infty]{\text{p}} \sigma_\xi^2, \tag{C.16}$$

$$\tilde{\xi}(T)'\tilde{Q}_T D_T (D_T'\tilde{Q}_T'\tilde{Q}_T D_T)^{-1} D_T'\tilde{Q}_T'\tilde{\xi}(T)/T \Longrightarrow \frac{\sigma_\xi^2 \chi^2_{(l+k)}}{T} \xrightarrow[T\to\infty]{\text{p}} 0. \tag{C.17}$$

Now using restrictions under the null and alternative that $\hat{\xi}(T) = \tilde{\xi}(T) := \xi_T^* \sim \mathcal{N}(0, I_T)$, we have

$$\begin{aligned}\Lambda_{1T} - \Lambda_{2T} &= \hat{\xi}(T)' M[\hat{Q}_{1T}]\hat{\xi}(T) - \tilde{\xi}(T)' M[\tilde{Q}_T]\tilde{\xi}(T) \\ &= \xi_T^{*\prime} M[\hat{Q}_{1T}]\xi_T^* - \xi_T^{*\prime} M[\tilde{Q}_T]\xi_T^* \\ &= \left[\xi_T^{*\prime}\xi_T^* - \xi_T^{*\prime}\xi_T^*\right] + \left[\xi_T^{*\prime} P[\tilde{Q}_T]\xi_T^* - \xi_T^{*\prime} P[\hat{Q}_{1T}]\xi_T^*\right] \\ &= \xi_T^{*\prime} P[\tilde{Q}_T]\xi_T^* - \xi_T^{*\prime} P[\hat{Q}_{1T}]\xi_T^* \\ &= \xi_T^{*\prime}\tilde{Q}_T(\tilde{Q}_T'\tilde{Q}_T)^{-1}\tilde{Q}_T'\xi_T^* - \xi_T^{*\prime}\hat{Q}_{1T}(\hat{Q}_{1T}'\hat{Q}_{1T})^{-1}\hat{Q}_{1T}'\xi_T^* \\ &= \xi_T^{*\prime} Q_T[Q_T'\Sigma(\rho_1)^{-1}Q_T]^{-1}Q_T'\Sigma(\rho_1)^{-1}\xi_T^* - \xi_T^{*\prime} Q_{1T}[Q_{1T}'\Sigma(\rho_0)^{-1}Q_{1T}]^{-1}Q_{1T}'\Sigma(\rho_0)^{-1}\xi_T^* \\ &= \xi_T^{*\prime} Q_T D_T[D_T' Q_T'\Sigma(\rho_1)^{-1}Q_T D_T]^{-1}D_T' Q_T'\Sigma(\rho_1)^{-1}\xi_T^* \\ &\quad - \xi_T^{*\prime} Q_{1T}D_{1T}[D_{1T}'Q_{1T}'\Sigma(\rho_0)^{-1}Q_{1T}D_{1T}]^{-1}D_{1T}'Q_{1T}'\Sigma(\rho_0)^{-1}\xi_T^* \\ &= \xi_T^{*\prime}\overline{\Lambda}_1\xi_T^* - \xi_T^{*\prime}\overline{\Lambda}_0\xi_T^* \\ &= \overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0,\end{aligned} \tag{C.18}$$

where $Q_T = [Q_{1T} \vdots Q_{2T}]$, $Q_{1T} = X(T)$, $Q_{2T} = Z_{-2}(T)$, and

$$\overline{\Lambda}_1 := Q_T D_T[D_T' Q_T'\Sigma(\rho_1)^{-1}Q_T D_T]^{-1}D_T' Q_T'\Sigma(\rho_1)^{-1}, \tag{C.19}$$

$$\overline{\Lambda}_0 := Q_{1T}D_{1T}[D_{1T}'Q_{1T}'\Sigma(\rho_0)^{-1}Q_{1T}D_{1T}]^{-1}D_{1T}'Q_{1T}'\Sigma(\rho_0)^{-1}, \tag{C.20}$$

$$\overline{\overline{\Lambda}}_1 := \xi_T^{*\prime}\overline{\Lambda}_1\xi_T^*, \qquad \overline{\overline{\Lambda}}_0 := \xi_T^{*\prime}\overline{\Lambda}_0\xi_T^*. \tag{C.21}$$

Under the Assumption 5.2, we have

$$\overline{\overline{\Lambda}}_1 = \xi_T^{*\prime}\overline{\Lambda}_1\xi_T^* \Longrightarrow \sigma_\xi^2\chi^2_{(l+k)}, \quad \overline{\overline{\Lambda}}_0 = \xi_T^{*\prime}\overline{\Lambda}_0\xi_T^* \Longrightarrow \sigma_\xi^2\chi^2_{(k)}. \tag{C.22}$$

Further, from the properties of quadratic forms [see Hogg and Craig (1958)], if $\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 \geq 0$, then

$$\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 \Longrightarrow \sigma_\xi^2\chi^2_{(l)}. \tag{C.23}$$

Since $\overline{\Lambda}_1$ is a projection onto $[D_{1T}X(T),\ D_{2T}Z_{-2}(T)]$ plane and $\overline{\Lambda}_0$ is a projection onto $D_{1T}X(T)$, $\overline{\Lambda}_1 - \overline{\Lambda}_0$ is a projection onto $D_{2T}Z_{-2}(T)$, *i.e.*, it is a projection onto the orthogonal complement of $D_{1T}X(T)$ within $[D_{1T}X(T),\ D_{2T}Z_{-2}(T)]$. As a result, $\overline{\Lambda}_1 - \overline{\Lambda}_0$ is an idempotent and positive-semidefinite matrix. This implies

$$\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 = \xi_T^{*\prime}(\overline{\Lambda}_1 - \overline{\Lambda}_0)\xi_T^* \geq 0\,, \tag{C.24}$$

and therefore

$$\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 \Longrightarrow \sigma_\xi^2 \chi_{(l)}^2\,. \tag{C.25}$$

Hence from (C.15) and (C.25), we have

$$AR_T^*(\phi_0, \rho_0, \rho_1) \Longrightarrow \chi_{(l)}^2\,. \tag{C.26}$$

□

D Monte Carlo tests with nuisance parameters

In this section, we discuss Monte Carlo tests when the distribution of the test statistic depends on nuisance parameters. Consider now the case where the distribution of ϑ involves a nuisance parameter υ and $\upsilon \in \Phi_0$.

1. Let $S^{(0)}$ be the observed test statistic (based on data).
2. For each $\upsilon \in \Phi_0$, by Monte Carlo methods, draw N i.i.d. replications of $\vartheta : \vartheta_{(j)} = \left[\vartheta_1^{(j)}, \ldots, \vartheta_T^{(j)}\right]$, $j = 1, \ldots, N$ and compute the statistics, $S^{(j)}(\upsilon) = \bar{S}(\vartheta_{(j)}(\upsilon), X)$, $j = 1, \ldots, N$.
3. Using these simulations we compute the MC p-value $\hat{p}_N[S] := p_N(S^{(0)}; S)$, where

$$\hat{p}_N\left[x; S \mid \upsilon\right] := \frac{N\hat{G}_N\left[x; S \mid \upsilon\right] + 1}{N+1}. \tag{D.1}$$

4. The p-value function $\hat{p}_N[S \mid \upsilon]$ as a function of υ is maximized over the parameter values compatible with the Φ_0, and H_0 is rejected if

$$\sup_{\upsilon \in \Phi_0} \hat{p}_N[S \mid \upsilon] \leq \alpha. \tag{D.2}$$

   If the number of simulated statistics N is chosen such that $\alpha(N+1)$ is an integer, then we have under H_0:

$$P\left[\sup_{\upsilon \in \Phi_0} \left\{\hat{p}_N[S \mid \upsilon]\right\} \leq \alpha\right] \leq \alpha, \tag{D.3}$$

   The test defined by $\hat{p}_N[S \mid \upsilon] \leq \alpha$ has size α for known υ . Treating υ as a nuisance parameter and Φ_0 is a nuisance parameter set consistent with null, the test is *exact at level* α; for a proof, see Dufour (2006).

Because of the maximization in the critical region (D.2) the test is called a *maximized Monte Carlo* (MMC) test. MMC tests provide valid inference under general regularity conditions such as almost-unidentified models or time series processes involving unit-roots. In particular, even though the moment conditions defining the estimator are derived under the stationarity assumption, this does not question in any way the validity of maximized MC tests, unlike the parametric bootstrap whose distributional theory is based on strong regularity conditions. Only the power of MMC tests may be affected. However, the simulated p-value function is not continuous, thus standard gradient-based methods cannot be used to maximize it. But search methods applicable to non-differentiable functions are applicable, *e.g.*, simulated annealing [see Goffe et al. (1994)]. A simplified approximate version of the MMC procedure can alleviate its computational load whenever a consistent point or set estimate of υ is available; for further discussion, see Dufour (2006).

E Temporal aggregation of stochastic volatility models

In this section, we consider the following HF SV model:

$$w_t = \mu_h + \phi_h w_{t-1} + v_t\,, \qquad y_t = w_t + \epsilon_t\,, \qquad y_t := \log(s_t^2) - \mu_{h,z}\,, \tag{E.1}$$

where $\mu_{h,z} = \mathbb{E}\left[\log(z_t^2)\right]$. Further, the model satisfy $|\phi| < 1$ and $(v_t, \epsilon_t)' \sim$ i.i.d. $\mathcal{N}\big(0, \mathrm{diag}[\sigma_v^2, \sigma_\epsilon^2]\big)$. This model is a modified version of the log-normal SV model where $\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$ is replaced by $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \sigma_\epsilon^2)$.

Since we assume stationarity of the latent HF volatility process ($|\phi_h| < 1$), the HF process y_t given in (E.1) admits an ARMA(1, 1) representation [see Ahsan and Dufour (2019, Proposition 3.1), Granger and Morris (1976)], which is given by

$$(1 - \phi_h B)\, y_t = \mu_h + (1 - \theta_h B)\varsigma_t\,, \tag{E.2}$$

where $\varsigma_t - \theta_h \varsigma_{t-1} = v_t + \epsilon_t - \phi_h \epsilon_{t-1}$. The moving average parameter θ_h and the white noise variance $\sigma^2_{h,\varsigma}$ are related to ϕ_h, $\sigma^2_{h,v}$ and $\sigma^2_{h,\epsilon}$ through non-linear equations:

$$(1 + \theta_h^2)\sigma^2_{h,\varsigma} = \sigma^2_{h,v} + (1 + \phi_h^2)\sigma^2_{h,\epsilon}\,, \qquad -\theta_h \sigma^2_{h,\varsigma} = -\phi_h \sigma^2_{h,\epsilon}\,. \tag{E.3}$$

Equating coefficients and making substitutions leads to $\sigma^2_{h,\varsigma} = \sigma^2_{h,\epsilon}\phi_h/\theta_h$ and θ_h is a solution to the quadratic equation

$$\theta_h^2 - \theta_h \tilde{k} + 1 = 0\,, \quad \text{where} \quad \tilde{k} = (\sigma^2_{h,v} + \sigma^2_{h,\epsilon}(1 + \phi_h^2))/(\sigma^2_{h,\epsilon}\phi_h).$$

It can be shown that $\tilde{k}^2 - 4 = (\tilde{k} - 2)(\tilde{k} + 2)$ is positive since $\tilde{k} > 2$ is equivalent to $\sigma^2_{h,v} + \sigma^2_{h,\epsilon}(1 - \phi_h)^2 > 0$. The induced model (E.2) is invertible if $|\theta_h| < 1$ which after some algebra is shown to be true for the root $(\tilde{k} - (\tilde{k}^2 - 4)^{1/2})/2$ when $0 < \phi_h < 1$ and for the root $(\tilde{k} + (\tilde{k}^2 - 4)^{1/2})/2$ when $-1 < \phi_h < 0$. So, given $0 < \phi_h < 1$, we have

$$\theta_h = (\tilde{k} - (\tilde{k}^2 - 4)^{1/2})/2\,, \qquad \sigma^2_{h,\varsigma} = \frac{\phi_h}{\theta_h}\sigma^2_{h,\epsilon}\,. \tag{E.4}$$

The following Proposition establishes temporal aggregation for model (E.2) by exploiting several well-known results for ARMA processes.[1]

Proposition E.1. Temporal Aggregation of High-frequency Model. *Under Assumptions of the model (E.1), the process y_t [given in (E.2)] is closed under temporal aggregation and the m-period nonoverlapping aggregates of y_t, denoted by y_T^*, has the following ARMA(1, 1) representation:*

$$(1 - \phi_l \mathcal{B})\, y_T^* = \mu_l + (1 - \theta_l \mathcal{B})\varsigma_T^* \tag{E.5}$$

[1] Temporal aggregation of the family of linear ARMA models has been widely studied; see for example, Amemiya and Wu (1972), Brewer (1973), Stram and Wei (1986), Silvestrini and Veredas (2008) and Teles and Sousa (2018), among many others. In general, we call a basic model "closed" under temporal aggregation if the aggregated model belongs to the same family of processes as the basic model, with possibly different orders and parameter values, for any order of aggregation m. It can be shown that the ARMA model is closed under temporal aggregation. The orders and parameters of the aggregated ARMA process can be derived from the autocovariance links between the basic and aggregated series; see Wei (2006, Chapter 20) and references therein.

where T is the aggregate (LF) temporal index such that $T = (m, 2m, \ldots)$, $\mathcal{B}$ is the backshift operator on the aggregate time unit T such that $y^*_T \mathcal{B}^j = y^*_{T-j}$ and

$$y^*_T = y^*_{tm} = \mathcal{W}(B) y_{tm}, \quad \phi_l = \phi_h^m, \quad \mu_l = m\Big(\big(1-\phi_h^m\big)/\big(1-\phi_h\big)\Big)\mu_h, \tag{E.6}$$

where the m-period temporal aggregation operator $\mathcal{W}(B) = \sum_{j=0}^{m-1} B^j = (1-B^m)/(1-B)$, B is the disaggregate backshift operator and θ_l is the root of the quadratic equation:

$$\theta_l^2 + \bar{\psi}\theta_l + 1 = 0 \tag{E.7}$$

where

$$\bar{\psi} = \psi_1/\psi_2, \tag{E.8}$$

$$\psi_1 = \sum_{i=0}^{m-1}\Big(1 + (\phi_h - \theta_h)\sum_{j=0}^{i-1}\phi_h^j\Big)^2 + \sum_{i=m}^{2(m-1)}\Big((\phi_h - \theta_h)\sum_{j=i-m}^{m-2}\phi_h^j - \theta_h\phi_h^{m-1}\Big)^2 + \Big(\theta_h\phi_h^{m-1}\Big)^2, \tag{E.9}$$

$$\psi_2 = \sum_{i=0}^{m-2}\Big(1 + (\phi_h - \theta_h)\sum_{j=0}^{i-1}\phi_h^j\Big)\Big((\phi_h - \theta_h)\sum_{j=i}^{m-2}\phi_h^j - \theta_h\phi_h^{m-1}\Big) - \Big(1 + (\phi_h - \theta_h)\sum_{j=0}^{m-2}\phi_h^j\Big)\theta_h\phi_h^{m-1}, \tag{E.10}$$

and $\theta_l = (-\bar{\psi} \pm (\bar{\psi}^2 - 4)^{1/2})/2$ such that $|\theta_l| < 1$ to ensure invertibility of the LF model. Further, $\sigma^2_{l,\varsigma} = \psi_2\sigma^2_{h,\varsigma}/(1+\theta_l^2)$.

Note that if we have $0 < \phi_h < 1$ in the HF SV model, then both HF and LF ARMA models are invertible with the following MA parameters:

$$\theta_h = (\tilde{k} - (\tilde{k}^2 - 4)^{1/2})/2, \qquad \theta_l = (-\bar{\psi} - (\bar{\psi}^2 - 4)^{1/2})/2. \tag{E.11}$$

Further, given the LF ARMA parameters $\Theta_l = \big(\mu_l, \phi_l, \theta_l, \sigma^2_{l,\varsigma}\big)$, we can also recuperate the parameters of the following LF SV model

$$w^*_T = \mu_l + \phi_l w^*_{T-1} + v^*_T, \qquad y^*_T = w^*_T + \epsilon^*_T, \qquad y^*_T := \log(s^{*2}_T) - \mu_{l,z}, \tag{E.12}$$

where $\mu_{l,z} = \mathbb{E}\left[\log(z^{*2}_T)\right] = m\mu_{h,z}$ and LF SV parameters $\Theta_{sv,l} = \Big(\mu_{sv,l}, \phi_{sv,l}, \sigma^2_{sv,l,v}, \sigma^2_{sv,l,\epsilon}\Big)$ are given by the following non-linear equations [similar to (E.3)]:

$$\mu_{sv,l} = \mu_l, \quad \phi_{sv,l} = \phi_l, \quad \sigma^2_{sv,l,\epsilon} = \frac{\theta_l}{\phi_l}\sigma^2_{l,\varsigma}, \quad \sigma^2_{sv,l,v} = (1+\theta_l^2)\sigma^2_{l,\varsigma} - (1+\phi_l^2)\sigma^2_{sv,l,\epsilon}. \tag{E.13}$$

It is easy to see that the LF parameter $\rho_l = \frac{\theta_l}{(1+\theta_l^2)}$. Finally, due to the above temporal aggregation results, we can consider the joint test $\phi_l = \phi_0$, $\rho_l = \rho_0$ in the LF model with HF instruments.

PROOF OF PROPOSITION E.1 Under Assumptions of model (E.1), the HF model is given by

$$(1 - \phi_h B) y_t = \mu_h + (1 - \theta_h B)\varsigma_t, \tag{E.14}$$

where the HF ARMA parameters are $\Theta_h = \big(\mu_h, \phi_h, \theta_h, \sigma^2_{h,\varsigma}\big)$. We define an m-period nonoverlapping aggregates of y_t as

$$y^*_t = \mathcal{W}(B) y_t = \sum_{j=0}^{m-1} B^j y_t, \tag{E.15}$$

where m is the fixed order of aggregation, y_t and y^*_t are basic HF and aggregate LF time series. The m-period *temporal aggregation operator* $\mathcal{W}(B) = \sum_{j=0}^{m-1} B^j = (1-B^m)/(1-B)$ transforms a HF process to a LF process under this flow scheme [note that $m = 1$ implies no aggregation]. Now re-writing model (E.14) as

$$\phi_h(B)\tilde{y}_t = \theta_h(B)\varsigma_t, \tag{E.16}$$

where $\phi_h(B) = 1-\phi_h B$, $\theta_h(B) = 1-\theta_h B$, $\tilde{y}_t = y_t - \mathbb{E}[y_t]$ and $\mathbb{E}[y_t] = \mu_h/(1-\phi_h)$. In an ARMA context, HF and LF models are linked via a polynomial operator $\mathcal{T}(B)$. This polynomial is a function of the roots of $\phi_h(B)$ and of the temporal aggregation operator $\mathcal{W}(B)$. This function drives us from one model to the other. In general, the AR and MA polynomials of the disaggregate model expressed in terms of their roots are multiplied by $\mathcal{T}(B)$:

$$\mathcal{T}(B)\phi_h(B)\tilde{y}_t = \mathcal{T}(B)\theta_h(B)\varsigma_t. \tag{E.17}$$

The resulting AR polynomial, $\mathcal{T}(B)\phi_h(B)$, has roots only divisible by $B^m = \mathcal{B}$ [i.e., at the aggregate frequency], and this way $\tilde{y}_t$ is transformed into $\tilde{y}^*_T$. Furthermore, from Brewer (1973) [also see Silvestrini and Veredas (2008)], it is well-known that the temporal aggregation of an ARMA(p,q) model can be represented by an ARMA(p,r) process where r, the maximum order of the aggregate moving average polynomial, is equal to $r = \lfloor m^{-1}\big((p+1)(m-1)+q\big)\rfloor$ with $\lfloor b \rfloor$ indicating the integer part of a real number b. As a result, since the HF time series y_t in (E.14) follows an ARMA(1, 1) model, the LF series y^*_T in (E.15) follows an ARMA(1, 1) model and the LF parameters $\Theta_l = \big(\mu_l, \phi_l, \theta_l, \sigma^2_{l,\varsigma}\big)$ are functions of HF parameters $\Theta_h = \big(\mu_h, \phi_h, \theta_h, \sigma^2_{h,\varsigma}\big)$. For an ARMA(1, 1) model, the $\mathcal{T}(B)$ operator takes the following form:

$$\mathcal{T}(B) = \left[\frac{1-\phi_h^m B^m}{1-\phi_h B}\right]\mathcal{W}(B) = \left[\frac{1-\phi_h^m B^m}{1-\phi_h B}\right]\left[\frac{1-B^m}{1-B}\right]. \tag{E.18}$$

Using the form of ARMA(1, 1) $-\,\mathcal{T}(B)$ in (E.17), we have

$$\left[\frac{1-\phi_h^m B^m}{1-\phi_h B}\right]\left[\frac{1-B^m}{1-B}\right]\tilde{y}_t = \left[\frac{1-\phi_h^m B^m}{1-\phi_h B}\right]\left[\frac{1-B^m}{1-B}\right]\theta_h(B)\varsigma_t, \tag{E.19}$$

$$\implies \left[\frac{1-\phi_h^m B^m}{1-\phi_h B}\right]\mathcal{W}(B)\phi_h(B)\tilde{y}_t = \left[\frac{1-\phi_h^m B^m}{1-\phi_h B}\right]\theta_h(B)\mathcal{W}(B)\varsigma_t, \tag{E.20}$$

$$\implies \big(1-\phi_h^m B^m\big)\mathcal{W}(B)\tilde{y}_t = \sum_{j=0}^{m-1}(\phi_h B)^j\mathcal{W}(B)(1-\theta_h B)\varsigma_t, \tag{E.21}$$

$$\implies \big(1-\phi_h^m B^m\big)\tilde{y}^*_t = \sum_{j=0}^{m-1}(\phi_h B)^j(1-\theta_h B)\varsigma^*_t, \tag{E.22}$$

Let now $\mathcal{B} = B^m$ to operate on the aggregate time unit T. The temporal index $T = m, 2m, \ldots$ is in the low-frequency. Then, the aggregate series in (E.22) may be represented by the process

$$\big(1-\phi_l\mathcal{B}\big)\tilde{y}^*_T = \big(1-\theta_l\mathcal{B}\big)\varsigma^*_T, \tag{E.23}$$

or,

$$\big(1-\phi_l\mathcal{B}\big)y^*_T = \mu_l + \big(1-\theta_l\mathcal{B}\big)\varsigma^*_T. \tag{E.24}$$

where

$$\phi_l = \phi_h^m, \quad \mu_l = m\Big(\big(1-\phi_h^m\big)/\big(1-\phi_h\big)\Big)\mu_h. \tag{E.25}$$

The expected value of y^*_T is also a function of past values of y^*_T and past values of ς^*_T. However, what differs now with respect to y_t is that $y^*_T = y^*_{tm} = \mathcal{W}(B)y_{tm}$. That is, the aggregate data are a function of the disaggregated data. Therefore aggregate parameters θ_l and $\sigma^2_{l,\varsigma}$ are, through y^*_T, functions of the autocovariance structure of y_t. As a result, to compute LF parameters $(\theta_l, \sigma^2_{l,\varsigma})'$, we define $Y_{tm} := \big(1-\phi_h^m B^m\big)\mathcal{W}(B)\tilde{y}_{tm}$, then it is easily seen that $\mathbb{E}(Y_{tm}) = 0$ from (E.21) since ς_t's are i.i.d. variables with mean zero and autocovariances of Y_{tm} are given by

$$\mathrm{Cov}(Y_{tm},\ Y_{tm+km}) = \mathbb{E}(Y_{tm},\ Y_{tm+km}) = \begin{cases} \psi_1\sigma^2_{h,\varsigma} & \text{if } k=0 \\ \psi_2\sigma^2_{h,\varsigma} & \text{if } k=1 \\ 0 & \text{if } k\geq 2, \end{cases} \tag{E.26}$$

where

$$\psi_1 = \sum_{i=0}^{m-1}\Big(1+(\phi_h-\theta_h)\sum_{j=0}^{i-1}\phi_h^j\Big)^2 + \sum_{i=m}^{2(m-1)}\Big((\phi_h-\theta_h)\sum_{j=i-m}^{m-2}\phi_h^j - \theta_h\phi_h^{m-1}\Big)^2 + \Big(\theta_h\phi_h^{m-1}\Big)^2, \tag{E.27}$$

$$\psi_2 = \sum_{i=0}^{m-2}\Big(1+(\phi_h-\theta_h)\sum_{j=0}^{i-1}\phi_h^j\Big)\Big((\phi_h-\theta_h)\sum_{j=i}^{m-2}\phi_h^j - \theta_h\phi_h^{m-1}\Big) - \Big(1+(\phi_h-\theta_h)\sum_{j=0}^{m-2}\phi_h^j\Big)\theta_h\phi_h^{m-1}. \tag{E.28}$$

The expression given in (E.26) is also derived in Teles and Sousa (2018, Section 2.2) and Wei (2006, Chapter 20). Therefore, from (E.26), Y_{tm} is an MA(1) model, i.e., $Y_{tm} = \big(1-\theta_l\mathcal{B}\big)\varsigma_T^*$ and θ_l is the root of the quadratic equation:

$$\theta_l^2 + \bar{\psi}\theta_l + 1 = 0 \tag{E.29}$$

where $\bar{\psi} = \psi_1/\psi_2$ and $\theta_l = (-\bar{\psi} \pm (\bar{\psi}^2-4)^{1/2})/2$ such that $|\theta_l| < 1$ to ensure invertibility of the LF model. Further, $\sigma^2_{l,\varsigma} = \psi_1\sigma^2_{h,\varsigma}/(1+\theta_l^2)$. □

F Testable null values for joint hypothesis

Table A1. Testable null values for joint hypothesis (ϕ_0, ρ_0) and corresponding values of λ_0

ϕ_0 \ ρ_0	-0.5	-0.4999	-0.45	-0.4	-0.35	-0.3	-0.25	-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.4999	0.5
-1	-	2499.50	4.50	2.00	1.17	0.75	0.50	0.33	0.21	0.13	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.9999	-	2499.81	4.50	2.00	1.17	0.75	0.50	0.33	0.21	0.13	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.95	-	-	4.79	2.12	1.23	0.79	0.53	0.35	0.23	0.13	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.9	-	-	5.26	2.27	1.31	0.84	0.56	0.37	0.24	0.14	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.8	-	-	7.26	2.78	1.55	0.97	0.64	0.42	0.27	0.16	0.07	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.7	-	-	15.25	3.85	1.96	1.19	0.76	0.50	0.31	0.18	0.08	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.6	-	-	-	7.14	2.82	1.56	0.96	0.61	0.38	0.22	0.09	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.5	-	-	-	-	5.60	2.40	1.33	0.80	0.48	0.27	0.11	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.4	-	-	-	-	-	5.77	2.27	1.19	0.66	0.35	0.15	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.3	-	-	-	-	-	-	9.09	2.44	1.10	0.52	0.20	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.2	-	-	-	-	-	-	-	-	3.41	1.04	0.34	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.1	-	-	-	-	-	-	-	-	-	-	1.01	0.00	-	-	-	-	-	-	-	-	-	-	-
0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
0.1	-	-	-	-	-	-	-	-	-	-	-	0.00	1.01	-	-	-	-	-	-	-	-	-	-
0.2	-	-	-	-	-	-	-	-	-	-	-	0.00	0.34	1.04	3.41	-	-	-	-	-	-	-	-
0.3	-	-	-	-	-	-	-	-	-	-	-	0.00	0.20	0.52	1.10	2.44	9.09	-	-	-	-	-	-
0.4	-	-	-	-	-	-	-	-	-	-	-	0.00	0.15	0.35	0.66	1.19	2.27	5.77	-	-	-	-	-
0.5	-	-	-	-	-	-	-	-	-	-	-	0.00	0.11	0.27	0.48	0.80	1.33	2.40	5.60	∞	-	-	-
0.6	-	-	-	-	-	-	-	-	-	-	-	0.00	0.09	0.22	0.38	0.61	0.96	1.56	2.82	7.14	-	-	-
0.7	-	-	-	-	-	-	-	-	-	-	-	0.00	0.08	0.18	0.31	0.50	0.76	1.19	1.96	3.85	15.25	-	-
0.8	-	-	-	-	-	-	-	-	-	-	-	0.00	0.07	0.16	0.27	0.42	0.64	0.97	1.55	2.78	7.26	-	-
0.9	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.14	0.24	0.37	0.56	0.84	1.31	2.27	5.26	-	-
0.95	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.13	0.23	0.35	0.53	0.79	1.23	2.12	4.79	-	-
0.9999	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.13	0.21	0.33	0.50	0.75	1.17	2.00	4.50	2499.81	-
1	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.13	0.21	0.33	0.50	0.75	1.17	2.00	4.50	2499.50	∞

Note: These are corresponding values for $\lambda_0 = \rho_0/[\phi_0 - \rho_0(1+\phi_0)^2] \in [0, \infty)$, under the joint null hypothesis given by

$$\bar{H}_0(\phi_0, \rho_0) : \left(\phi = \phi_0 \in [-1, 1]\,,\ \rho = \rho_0 \in [-1/2, 1/2]\right).$$

G Figures

Average precision

Classes of instruments	AR	AR*
RV	0.9357	0.8475
RVbr	0.9302	0.8451
RVMS	0.9347	0.8467
RKcub	0.9390	0.8582
RKbart	0.9376	0.8604
RKth2	0.9378	0.8616
RKnfp	0.9372	0.8584
RRV	0.9453	0.8543
BV	0.9369	0.8518
MedRV	0.9417	0.8502
MinRV	0.9420	0.8518
RSVN	0.9378	0.8477
RSVP	0.9339	0.8435
JV	0.2478	0.2009
SJV	0.1198	0.0864
LJV	0.9235	0.8096
LSJV	0.9188	0.8003
PCF	0.9197	0.8217
ImV-C	0.7983	0.7048
ImV-P	0.7887	0.6887
ImV-A	0.7942	0.7013
1-day	0.9050	0.7500

Median precision

Classes of instruments	AR	AR*
RV	0.9380	0.8480
RVbr	0.9300	0.8460
RVMS	0.9360	0.8550
RKcub	0.9390	0.8580
RKbart	0.9380	0.8600
RKth2	0.9380	0.8610
RKnfp	0.9370	0.8580
RRV	0.9450	0.8550
BV	0.9410	0.8510
MedRV	0.9415	0.8495
MinRV	0.9420	0.8540
RSVN	0.9390	0.8480
RSVP	0.9370	0.8450
JV	0.0000	0.0000
SJV	0.0000	0.0000
LJV	0.9200	0.8150
LSJV	0.9200	0.8000
PCF	0.9230	0.8310
ImV-C	0.9550	0.8415
ImV-P	0.9460	0.8235
ImV-A	0.9520	0.8400
1-day	0.9050	0.7500

Maximum precision

Classes of instruments	AR	AR*
RV	0.9460	0.8630
RVbr	0.9350	0.8530
RVMS	0.9410	0.8620
RKcub	0.9400	0.8600
RKbart	0.9380	0.8620
RKth2	0.9390	0.8660
RKnfp	0.9380	0.8640
RRV	0.9460	0.8580
BV	0.9450	0.8650
MedRV	0.9450	0.8660
MinRV	0.9450	0.8670
RSVN	0.9500	0.8660
RSVP	0.9420	0.8560
JV	0.9250	0.7700
SJV	0.8180	0.7000
LJV	0.9370	0.8380
LSJV	0.9370	0.8410
PCF	0.9240	0.8440
ImV-C	0.9640	0.8600
ImV-P	0.9490	0.8450
ImV-A	0.9580	0.8570
1-day	0.9050	0.7500

Minimum precision

Classes of instruments	AR	AR*
RV	0.9220	0.8310
RVbr	0.9250	0.8340
RVMS	0.9260	0.8120
RKcub	0.9380	0.8560
RKbart	0.9370	0.8580
RKth2	0.9370	0.8590
RKnfp	0.9360	0.8540
RRV	0.9450	0.8500
BV	0.9240	0.8320
MedRV	0.9380	0.8340
MinRV	0.9390	0.8350
RSVN	0.9230	0.8270
RSVP	0.9230	0.8320
JV	0.0000	0.0000
SJV	0.0000	0.0000
LJV	0.9070	0.7500
LSJV	0.8910	0.7500
PCF	0.9120	0.7900
ImV-C	0.0000	0.0000
ImV-P	0.0000	0.0000
ImV-A	0.0000	0.0000
1-day	0.9050	0.7500

Figure A1. IBM: 2009-2013: Precision of different classes of instruments.

Note: The instrument set consists of a constant and a lag of instrument, $l = 1$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The precision of an instrument set i is defined as $d_i = 1 - (ub_i - lb_i)$. For each class, we consider the average, median, minimum, and maximum precision measure across the proposed inference methods $[AR,\ AR^*]$. These inference procedures are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). We use 99 Monte Carlo replications for point-optimal type procedures.

Average precision

Inference methods

Subclasses of instruments	AR	AR*
1s	0.9360	0.8415
5s	0.9356	0.8361
30s	0.9399	0.8485
1m	0.9313	0.8424
5m	0.9312	0.8404
10m	0.9336	0.8491
5m-ss	0.9360	0.8313
10m-ss	0.9237	0.8148
1t	0.9314	0.8483
5t	0.9314	0.8492
10t	0.9324	0.8537
20t	0.9336	0.8528
50t	0.9339	0.8502

Median precision

Inference methods

Subclasses of instruments	AR	AR*
1s	0.9380	0.8410
5s	0.9380	0.8345
30s	0.9415	0.8470
1m	0.9310	0.8360
5m	0.9300	0.8380
10m	0.9300	0.8480
5m-ss	0.9370	0.8445
10m-ss	0.9315	0.8330
1t	0.9340	0.8540
5t	0.9335	0.8530
10t	0.9330	0.8555
20t	0.9330	0.8525
50t	0.9340	0.8510

Maximum precision

Inference methods

Subclasses of instruments	AR	AR*
1s	0.9420	0.8580
5s	0.9390	0.8410
30s	0.9440	0.8580
1m	0.9420	0.8580
5m	0.9410	0.8560
10m	0.9440	0.8560
5m-ss	0.9450	0.8660
10m-ss	0.9420	0.8530
1t	0.9400	0.8660
5t	0.9410	0.8600
10t	0.9390	0.8610
20t	0.9400	0.8620
50t	0.9400	0.8590

Minimum precision

Inference methods

Subclasses of instruments	AR	AR*
1s	0.9250	0.8190
5s	0.9260	0.8320
30s	0.9310	0.8450
1m	0.9220	0.8270
5m	0.9230	0.8330
10m	0.9260	0.8440
5m-ss	0.9250	0.7700
10m-ss	0.8900	0.7400
1t	0.9220	0.8270
5t	0.9230	0.8330
10t	0.9260	0.8440
20t	0.9290	0.8420
50t	0.9290	0.8340

Figure A2. IBM: 2009-2013: Precision of different subclasses of HF instruments.

Note: The instrument set consists of a constant and a lag of instrument, $l = 1$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The precision of an instrument set i is defined as $d_i = 1 - (ub_i - lb_i)$. For each class, we consider the average, median, minimum, and maximum precision measure across the proposed inference methods $[AR, AR^*]$. These inference procedures are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). We use 99 Monte Carlo replications for point-optimal type procedures.

H Extended simulation study

We simulate the DGP given in (2.2) with an instrument equation, which has the following compact representation:

$$y_t = \mu + \phi y_{t-1} + \xi_t, \quad \xi_t := v_t + \epsilon_t - \phi\epsilon_{t-1}, \quad v_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma_v^2), \quad \epsilon_t \sim \text{i.i.d. } \log(\chi^2_{(1)}) \tag{H.1}$$

$$y_{t-1} = \bar{\pi}_0 + Z'_{t-2}\bar{\pi}_1 + \eta_{t-1}, \quad \eta_{t-1} := \epsilon_{t-1} + u_{t-1}, \quad u_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma_u^2), \tag{H.2}$$

where $y_t = \log(s_t^2) + 1.2704$, $\bar{\pi}_1$ is an l-vector of first-stage coefficients, Z_{t-2} is an l-vector of independent $\mathcal{N}(0,1)$ variables, and the vector (ξ_t, η_{t-1}) has zero mean with $\text{Var}(\xi_t) = (1+\phi^2)\sigma_\epsilon^2 + \sigma_v^2$, $\text{Var}(\eta_{t-1}) = \sigma_\epsilon^2 + \sigma_u^2$ and $\text{Cov}(\xi_t, \eta_{t-1}) = -\phi\sigma_\epsilon^2$. Note that (H.1) is equivalent to a log-normal SV model, and in all our simulations we generate (H.1) non-linearly as given in (2.1).

We use 10,000 replications to compute the empirical level and powers, and 99 replications for PO tests based on the MCT procedure. For all tests, the nominal level is fixed at 5%. Thus, under the null hypothesis, the rejection rates should be less than (or close to) 5% for tests to be valid. Except for the analysis of asymptotic tests (Section H.1), the sample sizes are $T = 200,\ 300$.

H.1 Test performance of asymptotic t-test

In this section, we evaluate the performance of the asymptotic t-type test of $H_0 : \phi = \phi_0$. The simulated DGP is (H.1) with $\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$ [it is the log-normal SV model]. We set $\mu = 0$, $\sigma_v = 2$ and $\phi \in [0, 1]$. For sample sizes, $T \in (100, 10{,}000)$ are used.

Table 1 reports the size and power of asymptotic t-type tests for $H_0(\phi) : \phi = \phi_0$. The test statistic is calculated using the simple winsorized estimator of Ahsan and Dufour (2019) [equations (3.8)-(3.9) with $J = 10$]. This estimator is more efficient than conventional methods (QMLE, GMM) and as efficient as the Bayesian procedure. In addition, it is extremely time-efficient, and it produces empirical estimates which are similar to the Bayesian estimates. For the details of this asymptotic t-test, see Section 6.1 of Ahsan and Dufour (2019).

We can see from the results that the t-test (which is based on the asymptotic standard error) fails to control the level when $\phi \longrightarrow 1$. Size distortions are severe and equal up to 38.1% when $\phi = 1$. These size distortions do not go away even in larger samples ($T = 5000,\ 10000$), especially when $\phi > 0.999$, *i.e.*, ϕ is close to the unit circle.

H.2 Performance of the proposed tests

We will now examine the performance of the tests proposed in Sections 3.1-3.2. We focus on empirically motivated misspecified model setups with weak, low- and high-frequency instruments to simplify the exposition.

- Models with weak instruments where the generated instrument set Z_{t-2} includes weak IVs which are weakly correlated with past lags of the LF volatility proxy y_{t-1}.
  - $\mathcal{M}_1$: (H.1)-(H.2) with $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \pi^2/2)$, and $\mathcal{M}_2$: (H.1)-(H.2) with $\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$.
- Models with low-frequency instruments where we use past lags of the observed volatility proxy y_{t-1} as IVs.
  - $\mathcal{M}_3$: (H.1)-(H.2) with $\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$ and $\mathcal{M}_4$: (H.1)-(H.2) with $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \pi^2/2)$.
- Models with high-frequency instruments where we use HF realized volatility measures as IVs.

– $\mathcal{M}_5$: (H.1)-(H.2) with $|\phi| < 1$ and $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \sigma_\epsilon^2)$, and $\mathcal{M}_6$: (H.1)-(H.2) with $\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$.

From the above setups, we see that models $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_6$ correspond to a log-normal SV model with nonstationary volatility. It is easy to see that these models ($\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_6$) are misspecified under Assumptions 2.2 and 2.4. On the other hand, in models $\mathcal{M}_1$, $\mathcal{M}_4$, and $\mathcal{M}_5$, we have Gaussian noise for ϵ_t; thus, these models are correctly specified under Assumption 2.4 but misspecified under Assumption 2.2. Note that all these models violate the independence assumption, which is in line with the property of financial returns. However, the instrument set Z_{t-2} is uncorrelated with η_{t-1}. These models are designed to broadly mimic the features of financial returns used in our empirical application. For all models $\mathcal{M}_1$-$\mathcal{M}_6$, we consider the joint tests $[H_0 : (\phi,\ \rho) = (\phi_0,\ \rho_0)]$. The test statistics ($AR$, AR^*, SS, SS^*) are given in equations (3.12) and (3.15).

The results of models $\mathcal{M}_1$-$\mathcal{M}_6$ are reported in Tables A2-A7 and additional results of models $\mathcal{M}_1$-$\mathcal{M}_2$ are reported in Appendix I.

H.2.1 Test performance under $\mathcal{M}_1 - \mathcal{M}_2$ with weak instruments

For the weak IVs robustness check, we simulate models $\mathcal{M}_1 - \mathcal{M}_2$ and construct first-stage coefficients $\bar{\pi}_1$ as $\frac{||\bar{\lambda}||\sqrt{(\sigma_\epsilon^2+\sigma_u^2)}}{\sqrt{Tl}}\iota_l$, where ι_l is an l-vector of ones. Since $\mathrm{Var}(Z_{t-2}) = I_l$ and $\mathrm{Var}(\eta_{t-1}) = \sigma_\epsilon^2 + \sigma_u^2$, so that $||\bar{\lambda}^2|| = \frac{T\bar{\pi}_1'\bar{\pi}_1}{\sigma_\epsilon^2+\sigma_u^2}$ is the concentration parameter (CP) in this model. We consider the number of IVs $l = (1, 3, 5)$, CP $= (0, 0.5, 5)$, $\bar{\pi}_0 = 1$, $\sigma_\epsilon^2 = \pi^2/2$ (this also holds for $\mathcal{M}_2$, since $\mathrm{Var}\big(\log(\chi^2_{(1)}\big) = \pi^2/2)$ and $\sigma_u^2 = 0.01$. Thus, given $CP = (0, 0.5, 5)$, the corresponding values of the first stage coefficients for $l = 1$ are $\bar{\pi}_1 = (0, 0.11, 0.35)$ for $T = 200$ and $\bar{\pi}_1 = (0, 0.09, 0.29)$ for $T = 300$.

The simulated models use $\mu_z = -1.2704$, $\mu = 2.5$ and different values of ϕ and ρ. These values are $\phi \in (0.1, 1)$ and $\rho = (0.05, 0.1)$. Thus, given $\rho = 0.05$ and $\phi = (0.1, 0.2, 0.5, 0.8, 0.9, 1)$, the corresponding values of λ [$= \rho/(\phi - \rho(1+\phi)^2)$] are $(1.01, 0.34, 0.11, 0.07, 0.06, 0.06)$. Since we fix $\sigma_\epsilon^2 = \pi^2/2$, so $\lambda = (1.010, 0.338, 0.114, 0.070, 0.062, 0.056)$ the corresponding values of σ_v [$= \sigma_\epsilon/\sqrt{\lambda}$] are $(2.21, 3.82, 6.57, 8.42, 8.94, 9.42)$. Similarly, for $\rho = 0.1$, we have different set of values for λ and σ_v. As a result, a restriction on ρ implies a restriction on λ or σ_v. For example, a joint null $(\phi_0, \rho_0) = (1, 0.05)$ is same as $(\phi_0, \lambda_0) = (1, 0.06)$ or $(\phi_0, \sigma_{v0}) = (1, 9.42)$. For PO tests, we set the alternative ρ_1 to the simulated ρ value.

The results of $\mathcal{M}_1 - \mathcal{M}_2$ confirmed the theoretical contributions of Sections 3.1-3.2 even with model misspecification. Our findings can be summarized as follows.

First, from Table A2, the levels of the proposed tests (AR, AR^*) are well controlled: rejection frequencies are less than (or close to) 5%. This result holds whether the identification is completely failed [CP = 0], weak [CP $\in (0, 0.5)$], partial [CP $\in (0.5, 5)$], or moderately strong [CP = 5]. This represents a substantial improvement over the asymptotic test; AR and AR^* tests perfectly control the level.

Second, from Table A2, all tests exhibit excellent power as long as identification is not very weak. Note that, in our joint tests, we have an additional restriction under the null hypothesis on the parameter of the error distribution. This restriction works as an additional source of power for the optimal tests. In all cases [weak or strong IVs], the AR^* tests have more power than the AR tests. As expected, the power of these tests increases with sample size (in many cases, rejection frequencies reach 100%) and concentration parameter and decreases as the number of IVs increases.

Third, from Table A3, the empirical levels of the proposed tests are almost identical to those obtained when the model is only misspecified under Assumption 2.2 [compare Table A2 with Table A3]: rejection frequencies are less than (or close to) 5%, whether identification is completely failed [CP = 0], weak [CP $\in (0, 0.5)$], partial [CP $\in (0.5, 5)$], or moderately strong [CP = 5], for all sample sizes considered.

Fourth, from Table A3, the misspecification of the error distribution [$\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$] does not affect the power of these tests [compare Table A3 with Table A2]. Overall, these tests appear to be reasonably robust to a misspecification of the error distribution, even with small samples.

H.2.2 Test performance under $\mathcal{M}_3 - \mathcal{M}_4$ with low-frequency instruments

We simulate $\mathcal{M}_3 - \mathcal{M}_4$ models with $\mu_z = -1.2704$, $\mu = 2.5$, $\phi \in (0.5,\ 1]$ and $\rho \in (0.1,\ 0.35)$. We use past lags of y_{t-1} as IVs (Z_{t-2}) with a constant $\bar{\pi}_0 = 1$, so the instrument set Z_{t-2} is not independent of the error distributions of v and ϵ. In this setting, for PO tests, we set the alternative to $\rho_1 = \rho$. The results appear in Tables A4 and A5, and the main findings are the following.

First, in both samples ($T = 200, 300$), the levels of the proposed tests (AR, AR^*) are well controlled, even when $\phi = 1$.

Second, all these tests exhibit excellent power (see from the second part of Table A4). Note that PO tests can gain power from the differences in covariance structure, *i.e.*, when $\rho_1 \neq \rho_0$. Hence, when $\rho_1 \in (0.15, 0.35)$, PO tests outperform their counterpart as expected. However, AR tests have more power in all cases compared to their counterpart AR^* when $l = 1$ and $\rho = 0.1$. Again, as expected, the power of these tests increases with sample size and decreases as the number of IVs increases.

Third, from Table A5, when we simulate the same DGP with $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \pi^2/2)$, results are almost identical [compare Table A5 with Table A4]: rejection frequencies are similar.

H.2.3 Test performance under $\mathcal{M}_5 - \mathcal{M}_6$ with high-frequency instruments

The model (H.1) with $|\phi| < 1$ and $\epsilon_t \sim$ i.i.d. $N(0, \sigma_v^2)$ is closed under temporal aggregation; see Appendix E for related discussion and proof. Both y_t and y_T^* [the m-period nonoverlapping aggregates of y_t, defined as $y_T^* := \sum_{j=0}^{m-1} B^j y_t$] have an ARMA(1, 1) representation, and the following equations relate LF and HF parameters:

$$\phi_l = \phi_h^m, \ \ \mu_{l,z} = m\mu_{h,z}, \ \ \mu_l = m\Big(\big(1-\phi_h^m\big)/\big(1-\phi_h\big)\Big)\mu_h, \ \ \sigma_{l,\epsilon}^2 = \frac{\theta_l}{\phi_l}\sigma_{l,\varsigma}^2, \ \ \sigma_{l,v}^2 = (1+\theta_l^2)\sigma_{l,\varsigma}^2 - (1+\phi_l^2)\sigma_{l,\epsilon}^2, \tag{H.3}$$

and if we have $0 < \phi_h < 1$ then $\theta_l = (-\bar{\psi} - (\bar{\psi}^2 - 4)^{1/2})/2$ where $\bar{\psi} = \psi_1/\psi_2$,

$$\psi_1 = \sum_{i=0}^{m-1}\Big(1 + (\phi_h - \theta_h)\sum_{j=0}^{i-1}\phi_h^j\Big)^2 + \sum_{i=m}^{2(m-1)}\Big((\phi_h - \theta_h)\sum_{j=i-m}^{m-2}\phi_h^j - \theta_h\phi_h^{m-1}\Big)^2 + \Big(\theta_h\phi_h^{m-1}\Big)^2, \tag{H.4}$$

$$\psi_2 = \sum_{i=0}^{m-2}\Big(1 + (\phi_h - \theta_h)\sum_{j=0}^{i-1}\phi_h^j\Big)\Big((\phi_h - \theta_h)\sum_{j=i}^{m-2}\phi_h^j - \theta_h\phi_h^{m-1}\Big) - \Big(1 + (\phi_h - \theta_h)\sum_{j=0}^{m-2}\phi_h^j\Big)\theta_h\phi_h^{m-1}, \tag{H.5}$$

$$\theta_h = (\tilde{k} - (\tilde{k}^2 - 4)^{1/2})/2, \quad \tilde{k} = (\sigma_{h,v}^2 + \sigma_{h,\epsilon}^2(1+\phi_h^2))/(\sigma_{h,\epsilon}^2\phi_h), \tag{H.6}$$

$$\sigma_{l,\varsigma}^2 = \psi_2\sigma_{h,\varsigma}^2/(1+\theta_l^2), \quad \sigma_{h,\varsigma}^2 = \sigma_{h,\epsilon}^2\phi_h/\theta_h. \tag{H.7}$$

For model $\mathcal{M}_5$, we simulate (H.1) with $|\phi| < 1$ and $\epsilon_t \sim$ i.i.d. $\mathcal{N}(0, \sigma_\epsilon^2)$] at a higher frequency and use these HF observations to construct RV estimates. The instrument set contains a constant $\bar{\pi}_0 = 1$ and log of lagged RVs in this setup. Consequently, we make inferences for the low-frequency model parameters using generated IVs from the HF series. Note that $|\phi| < 1$ is required for the identification of μ_l parameter under temporal aggregation. However, it also ensures stationarity and invertibility of both HF and LF models.

Equal-spaced HF intraday data are considered with frequency set = (1m, 5m, 10m, 15m) where 1m stands for 1-minute frequency. Therefore, within a day (trading hours = 6.5) the number of HF observations are $m = (390, 78, 39, 26)$. The HF sample size T_{hf} is equal to $T \times m$, where T is the LF sample size. Given the frequency set m, we generate data from the HF model (H.1 with μ_h = (1e−4, 5e−4, 1e−3, 5e−3), $\sigma_{h,v}^2$ = (1e−9, 1e−8, 1e−7, 1e−6), $\sigma_{h,\epsilon}^2 = (0.01268, 0.06328, 0.12655, 0.1900)$, $\phi_{hf} = \phi_{lf}^{1/m}$ with $\phi_{lf} = (0.5, 0.6, 0.7, 0.8, 0.9, 0.99999)$ and $\mu_{h,z} = \mu_{l,z}/m$ with $\mu_{l,z} = -1.2704$. For each of this four settings leads to $\sigma^2_{l,\epsilon} \approx \pi^2/2$. Note that, to generate a nearly nonstationary LF volatility process, we use large values of ϕ_{hf}, *e.g.*, in case of 1-min frequency, $\phi_{hf} = 0.99999997$ is corresponds to $\phi_{lf} = 0.99999$.

For model $\mathcal{M}_6$, in all frequency, we set $\mu_{h,z} = -1.2704$, $\sigma^2_{h,v} = 1\text{e}{-7}$, $\sigma^2_{h,\epsilon} = \pi^2/2$, and $\phi_{hf} = \phi_{lf}^{1/m}$ with $\phi_{lf} = (0.5, 0.6, 0.7, 0.8, 0.9, 1)$. Compared to $\mathcal{M}_5$, we allow nonstationary volatility in $\mathcal{M}_6$ with $\epsilon_t \sim$ i.i.d. $\log(\chi^2_{(1)})$. Hence, $\mathcal{M}_6$ does not have the same error distributions under temporal aggregation. However, (H.3) is still valid for the HF and LF parameters of $\mathcal{M}_6$ model. For the identification of the μ_l parameter when $\phi_{hf} = \phi_{lf} = 1$, we put an identification restriction such that $\mu_l = m\Big(\big(1-\tilde{\phi}_h^m\big)/\big(1-\tilde{\phi}_h\big)\Big)\mu_h$, where $\tilde{\phi}_h = \phi_h - \hat{d}$ with $\hat{d} = 1\text{e}{-15}$.

The simulation results for model $M5 - M6$ are displayed in Tables A6-A7. The following conclusions emerge from these tables.

First, we see from Table A6 that in all cases of HF IVs (these are the logarithms of RVs), the proposed tests (AR, AR^*) controls the levels very well: rejection frequencies are less than (or close to) 5%. This result holds whether sample sizes are different ($T = 200, 300$), or the instrument set contains a different number of IVs ($l = 1, 3, 5$). However, PO tests are undersized with HF IVs: rejection frequencies are less than 5% and close to 0% when $\rho \longrightarrow 0.5$. This shows that PO tests need large samples for level control in these cases.

Second, from Table A6, in all cases of HF IVs (1-minute to 15-minute), the proposed tests have excellent power against the alternative: up to 100%, and the power of these tests increases with the sample size, and decreases as the number of IVs increases.

Third, all tests have excellent power across different sampling frequencies, and these tests gain power when the sampling frequency increases.

Fourth, from Table A7, when we simulate the $\mathcal{M}_6$ model under nonstationary volatility, results are slightly different [compare Table A7 with Table A6]: level controls are similar, but rejection frequencies for power simulations are slightly different.

Table A2. Size and power comparison of joint tests under $\mathcal{M}_1$ with weak instruments, nominal level: 5%

Size

Panel A: Size					T = 200									T = 300								
					$l=1$			$l=3$			$l=5$			$l=1$			$l=3$			$l=5$		
ρ	ϕ	σ_v	λ	CP	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	5.1	5.1	0.00	4.8	4.6	0.00	4.8	4.7	0.00	5.0	4.8	0.00	5.3	5.0	0.00	5.1	4.9
		2.21	1.010	0.5	0.11	5.1	5.3	0.06	4.9	4.7	0.05	4.8	4.7	0.09	5.1	4.9	0.05	5.2	5.0	0.04	5.0	4.9
		2.21	1.010	5.0	0.35	5.5	5.3	0.20	5.4	5.1	0.16	4.9	5.0	0.29	5.3	5.3	0.17	5.4	5.0	0.13	5.2	5.1
	0.2	3.82	0.338	0.0	0.00	5.2	5.0	0.00	4.9	4.8	0.00	4.9	4.8	0.00	4.8	4.7	0.00	5.0	4.8	0.00	5.2	5.0
		3.82	0.338	0.5	0.11	5.2	5.1	0.06	4.9	4.9	0.05	4.9	5.0	0.09	4.9	4.5	0.05	4.9	4.8	0.04	5.1	5.0
		3.82	0.338	5.0	0.35	5.3	5.3	0.20	5.1	5.0	0.16	4.9	5.1	0.29	5.2	5.0	0.17	5.1	5.0	0.13	5.5	5.1
	0.5	6.57	0.114	0.0	0.00	5.3	5.1	0.00	5.2	5.2	0.00	4.9	4.8	0.00	4.8	4.6	0.00	5.1	5.0	0.00	5.0	5.0
		6.57	0.114	0.5	0.11	5.1	5.0	0.06	5.1	5.0	0.05	4.9	4.9	0.09	4.8	4.6	0.05	5.2	4.9	0.04	5.0	4.9
		6.57	0.114	5.0	0.35	5.0	4.8	0.20	5.1	5.0	0.16	4.9	4.9	0.29	5.0	5.1	0.17	5.1	5.0	0.13	5.2	4.9
	0.8	8.42	0.070	0.0	0.00	5.2	5.0	0.00	5.1	5.1	0.00	5.0	4.8	0.00	4.7	4.6	0.00	5.1	5.0	0.00	5.0	4.8
		8.42	0.070	0.5	0.11	5.0	5.0	0.06	5.1	5.0	0.05	5.0	4.8	0.09	4.7	4.5	0.05	5.0	4.9	0.04	5.0	4.8
		8.42	0.070	5.0	0.35	4.9	4.7	0.20	5.1	4.8	0.16	4.9	4.8	0.29	4.8	4.6	0.17	4.9	4.9	0.13	5.0	4.9
	0.9	8.94	0.062	0.0	0.00	5.2	5.0	0.00	5.0	5.1	0.00	5.1	4.8	0.00	4.8	4.6	0.00	5.1	5.0	0.00	5.0	4.8
		8.94	0.062	0.5	0.11	5.1	5.1	0.06	5.1	4.9	0.05	5.0	4.8	0.09	4.7	4.7	0.05	5.1	4.9	0.04	5.0	4.7
		8.94	0.062	5.0	0.35	4.8	4.7	0.20	5.0	4.8	0.16	4.9	4.9	0.29	4.7	4.5	0.17	4.8	4.8	0.13	4.9	4.8
	1.0	9.42	0.056	0.0	0.00	5.3	5.0	0.00	4.9	5.1	0.00	5.1	4.9	0.00	5.1	5.0	0.00	5.2	5.0	0.00	4.8	4.7
		9.42	0.056	0.5	0.11	5.3	5.1	0.06	5.0	4.9	0.05	4.9	4.9	0.09	5.2	5.1	0.05	5.2	5.2	0.04	4.7	4.8
		9.42	0.056	5.0	0.35	5.5	5.1	0.20	5.1	4.9	0.16	5.2	4.9	0.29	5.1	5.1	0.17	5.1	5.0	0.13	4.9	4.7
0.10	0.2	2.18	1.042	0.0	0.00	5.1	4.9	0.00	4.7	4.4	0.00	4.7	4.4	0.00	4.9	4.6	0.00	5.1	4.8	0.00	5.1	4.6
		2.18	1.042	0.5	0.11	5.3	5.0	0.06	4.9	4.5	0.05	4.7	4.4	0.09	5.1	4.6	0.05	5.2	4.7	0.04	5.1	4.6
		2.18	1.042	5.0	0.35	6.4	5.9	0.20	5.4	5.2	0.16	5.2	5.0	0.29	6.1	5.8	0.17	5.6	5.1	0.13	5.4	4.9
	0.5	4.30	0.267	0.0	0.00	5.2	4.8	0.00	5.2	4.7	0.00	4.7	4.4	0.00	4.9	4.4	0.00	4.9	4.6	0.00	5.1	4.5
		4.30	0.267	0.5	0.11	5.1	4.9	0.06	5.3	4.8	0.05	4.8	4.6	0.09	4.9	4.4	0.05	5.0	4.7	0.04	5.2	4.5
		4.30	0.267	5.0	0.35	5.9	5.6	0.20	5.7	5.3	0.16	5.3	4.8	0.29	6.0	5.8	0.17	5.5	5.1	0.13	5.6	4.9
	0.8	5.60	0.157	0.0	0.00	5.3	4.8	0.00	5.1	4.8	0.00	5.0	4.5	0.00	4.7	4.4	0.00	5.0	4.6	0.00	4.8	4.4
		5.60	0.157	0.5	0.11	5.1	4.7	0.06	5.2	4.9	0.05	4.9	4.5	0.09	4.8	4.4	0.05	5.0	4.6	0.04	4.9	4.4
		5.60	0.157	5.0	0.35	5.1	4.7	0.20	5.1	4.6	0.16	4.9	4.7	0.29	5.1	4.6	0.17	5.1	4.6	0.13	5.0	4.6
	0.9	5.96	0.139	0.0	0.00	5.2	4.8	0.00	5.1	4.9	0.00	5.0	4.6	0.00	4.7	4.5	0.00	5.0	4.7	0.00	4.9	4.3
		5.96	0.139	0.5	0.11	5.0	4.7	0.06	5.2	4.9	0.05	4.9	4.6	0.09	4.6	4.4	0.05	5.0	4.6	0.04	4.9	4.3
		5.96	0.139	5.0	0.35	4.7	4.3	0.20	5.1	4.6	0.16	4.8	4.4	0.29	4.6	4.3	0.17	4.9	4.6	0.13	5.0	4.4
	1.0	6.28	0.125	0.0	0.00	5.2	4.8	0.00	5.1	4.9	0.00	4.9	4.5	0.00	5.1	4.8	0.00	5.2	4.9	0.00	4.8	4.5
		6.28	0.125	0.5	0.11	5.2	4.9	0.06	5.1	4.7	0.05	4.9	4.5	0.09	4.9	4.8	0.05	5.2	4.9	0.04	4.7	4.5
		6.28	0.125	5.0	0.35	5.2	4.5	0.20	5.1	4.5	0.16	5.0	4.5	0.29	5.1	4.6	0.17	5.1	4.7	0.13	4.7	4.3

Panel B: Power ($H_0: \phi_0 = 1, \rho_0 = 0.05$)					T = 200									T = 300								
					$l=1$			$l=3$			$l=5$			$l=1$			$l=3$			$l=5$		
ρ	ϕ	σ_v	λ	CP	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	4.5	4.3	0.00	4.3	4.3	0.00	4.4	4.3	0.00	4.5	4.4	0.00	4.4	4.3	0.00	4.3	4.0
		2.21	1.010	0.5	0.11	10.3	9.8	0.06	7.2	6.9	0.05	6.7	6.4	0.09	10.1	9.9	0.05	7.3	7.0	0.04	6.1	6.2
		2.21	1.010	5.0	0.35	61.5	60.3	0.20	40.9	39.8	0.16	31.1	30.4	0.29	62.1	60.5	0.17	40.5	39.5	0.13	32.0	30.8
	0.2	3.82	0.338	0.0	0.00	4.7	4.6	0.00	4.2	4.4	0.00	4.5	4.5	0.00	4.6	4.5	0.00	4.3	4.5	0.00	4.4	4.1
		3.82	0.338	0.5	0.11	12.7	12.1	0.06	8.3	8.2	0.05	7.6	7.1	0.09	12.2	11.9	0.05	8.0	7.8	0.04	7.2	6.8
		3.82	0.338	5.0	0.35	75.3	73.8	0.20	53.6	51.9	0.16	42.5	41.1	0.29	76.0	73.9	0.17	54.7	53.3	0.13	44.1	42.5
	0.4	5.81	0.146	0.0	0.00	4.9	4.8	0.00	4.4	4.6	0.00	4.5	4.5	0.00	4.4	4.4	0.00	4.6	4.5	0.00	4.5	4.3
		5.81	0.146	0.5	0.11	11.5	11.0	0.06	7.8	7.6	0.05	7.3	6.9	0.09	11.4	11.3	0.05	7.6	7.4	0.04	7.0	6.7
		5.81	0.146	5.0	0.35	70.1	68.2	0.20	47.8	46.1	0.16	37.9	36.5	0.29	70.4	68.4	0.17	49.1	47.2	0.13	39.3	37.8
	0.5	6.57	0.114	0.0	0.00	4.9	4.8	0.00	4.4	4.6	0.00	4.5	4.4	0.00	4.4	4.5	0.00	4.6	4.6	0.00	4.6	4.5
		6.57	0.114	0.5	0.11	10.5	10.1	0.06	7.4	7.3	0.05	6.7	6.5	0.09	10.4	10.4	0.05	7.4	7.1	0.04	6.5	6.3
		6.57	0.114	5.0	0.35	63.5	61.7	0.20	41.5	39.8	0.16	32.3	31.5	0.29	63.8	61.6	0.17	42.4	40.8	0.13	33.6	32.2
	0.8	8.42	0.070	0.0	0.00	5.1	4.8	0.00	4.6	4.8	0.00	4.8	4.8	0.00	4.3	4.3	0.00	4.8	4.8	0.00	4.9	4.7
		8.42	0.070	0.5	0.11	7.5	7.4	0.06	5.8	5.7	0.05	5.7	5.6	0.09	7.0	6.8	0.05	5.7	5.7	0.04	5.7	5.5
		8.42	0.070	5.0	0.35	32.2	30.5	0.20	19.4	18.3	0.16	14.7	14.3	0.29	32.1	30.4	0.17	18.7	18.3	0.13	15.0	14.4
	0.9	8.94	0.062	0.0	0.00	5.0	4.9	0.00	4.9	5.1	0.00	5.0	4.7	0.00	4.4	4.4	0.00	4.9	4.9	0.00	4.9	4.7
		8.94	0.062	0.5	0.11	6.3	6.1	0.06	5.4	5.4	0.05	5.4	5.1	0.09	5.7	5.6	0.05	5.5	5.4	0.04	5.4	5.2
		8.94	0.062	5.0	0.35	18.9	18.1	0.20	11.9	11.3	0.16	9.7	9.4	0.29	19.0	18.0	0.17	11.6	11.2	0.13	9.8	9.3
	1.0	9.42	0.056	0.0	0.00	5.3	5.0	0.00	4.9	5.1	0.00	5.1	4.9	0.00	5.1	5.0	0.00	5.2	5.0	0.00	4.8	4.7
		9.42	0.056	0.5	0.11	5.3	5.1	0.06	5.0	4.9	0.05	4.9	4.9	0.09	5.2	5.1	0.05	5.2	5.2	0.04	4.7	4.8
		9.42	0.056	5.0	0.35	5.5	5.1	0.20	5.1	4.9	0.16	5.2	4.9	0.29	5.1	5.1	0.17	5.1	5.0	0.13	4.9	4.7
0.10	0.2	2.18	1.042	0.0	0.00	4.5	100.0	0.00	4.4	95.7	0.00	4.5	75.2	0.00	4.6	100.0	0.00	4.4	100.0	0.00	4.3	99.7
		2.18	1.042	0.5	0.11	9.6	100.0	0.06	6.8	96.2	0.05	6.4	78.1	0.09	9.4	100.0	0.05	7.0	100.0	0.04	5.9	99.7
		2.18	1.042	5.0	0.35	54.1	100.0	0.20	34.3	99.0	0.16	26.0	92.2	0.29	54.5	100.0	0.17	34.4	100.0	0.13	26.9	99.9
	0.5	4.30	0.267	0.0	0.00	4.9	96.9	0.00	4.5	62.5	0.00	4.6	41.4	0.00	4.4	100.0	0.00	4.5	96.3	0.00	4.4	81.8
		4.30	0.267	0.5	0.11	9.6	97.1	0.06	7.0	66.4	0.05	6.4	45.9	0.09	9.5	100.0	0.05	6.8	96.7	0.04	6.1	83.8
		4.30	0.267	5.0	0.35	55.4	99.3	0.20	35.2	88.3	0.16	26.6	74.1	0.29	55.6	100.0	0.17	35.7	99.0	0.13	27.6	94.6
	0.8	5.60	0.157	0.0	0.00	4.9	51.1	0.00	4.6	24.0	0.00	4.6	18.0	0.00	4.4	80.8	0.00	4.7	46.9	0.00	4.6	32.4
		5.60	0.157	0.5	0.11	7.2	53.5	0.06	5.6	26.0	0.05	5.5	19.5	0.09	6.7	82.1	0.05	5.6	49.1	0.04	5.4	34.5
		5.60	0.157	5.0	0.35	28.6	71.0	0.20	17.0	43.5	0.16	13.1	32.8	0.29	28.5	89.8	0.17	16.8	65.3	0.13	13.4	50.2
	0.9	5.96	0.139	0.0	0.00	4.9	30.3	0.00	4.9	15.9	0.00	5.0	13.2	0.00	4.4	54.2	0.00	4.7	28.2	0.00	4.7	21.1
		5.96	0.139	0.5	0.11	6.1	31.8	0.06	5.5	17.3	0.05	5.3	13.9	0.09	5.8	55.3	0.05	5.1	29.2	0.04	5.2	21.7
		5.96	0.139	5.0	0.35	17.4	45.2	0.20	10.9	26.0	0.16	9.0	20.2	0.29	16.9	65.5	0.17	10.5	39.0	0.13	8.7	29.0
	1.0	6.28	0.125	0.0	0.00	5.0	17.3	0.00	4.9	11.0	0.00	4.7	9.6	0.00	4.9	29.8	0.00	5.1	16.6	0.00	4.6	13.5
		6.28	0.125	0.5	0.11	5.1	17.2	0.06	5.0	11.1	0.05	4.7	9.8	0.09	4.9	30.0	0.05	5.1	16.7	0.04	4.6	13.4
		6.28	0.125	5.0	0.35	5.0	17.1	0.20	4.9	11.1	0.16	4.8	9.6	0.29	4.9	29.4	0.17	5.0	16.6	0.13	4.5	13.6

Table A3. Size and power comparison of joint tests under $\mathcal{M}_2$ with weak instruments, nominal level: 5%

Size

Panel A: Size					T = 200									T = 300								
					$l=1$			$l=3$			$l=5$			$l=1$			$l=3$			$l=5$		
ρ	ϕ	σ_v	λ	CP	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	4.8	4.7	0.00	5.0	4.8	0.00	5.2	5.0	0.00	4.8	4.9	0.00	5.0	4.9	0.00	5.1	4.9
		2.21	1.010	0.5	0.11	4.7	4.8	0.06	4.8	4.8	0.05	5.2	5.0	0.09	4.8	4.7	0.05	4.9	5.1	0.04	5.2	4.9
		2.21	1.010	5.0	0.35	5.0	4.9	0.20	4.9	5.1	0.16	5.2	5.2	0.29	5.0	5.1	0.17	5.2	5.2	0.13	5.2	5.1
	0.2	3.82	0.338	0.0	0.00	4.8	5.0	0.00	5.0	4.8	0.00	5.2	5.0	0.00	4.8	4.6	0.00	4.9	4.9	0.00	5.3	5.1
		3.82	0.338	0.5	0.11	4.8	4.9	0.06	5.0	4.9	0.05	5.1	4.9	0.09	4.8	4.6	0.05	4.9	4.9	0.04	5.2	5.2
		3.82	0.338	5.0	0.35	5.0	5.0	0.20	5.1	4.9	0.16	5.1	5.0	0.29	4.9	4.9	0.17	5.1	5.0	0.13	5.5	5.4
	0.5	6.57	0.114	0.0	0.00	5.0	5.3	0.00	5.3	5.1	0.00	5.3	5.1	0.00	4.7	4.6	0.00	5.0	5.0	0.00	5.2	5.0
		6.57	0.114	0.5	0.11	5.0	5.2	0.06	5.2	5.0	0.05	5.3	5.1	0.09	4.9	4.6	0.05	5.0	5.0	0.04	5.1	5.1
		6.57	0.114	5.0	0.35	4.9	4.9	0.20	5.0	4.9	0.16	5.1	5.1	0.29	4.9	4.8	0.17	5.0	4.9	0.13	5.3	5.1
	0.8	8.42	0.070	0.0	0.00	5.1	5.3	0.00	5.4	5.1	0.00	5.3	5.0	0.00	4.9	4.7	0.00	5.1	5.0	0.00	5.1	5.0
		8.42	0.070	0.5	0.11	5.1	5.3	0.06	5.2	5.0	0.05	5.3	5.1	0.09	4.8	4.6	0.05	5.1	4.9	0.04	5.2	4.9
		8.42	0.070	5.0	0.35	4.6	4.7	0.20	5.0	4.8	0.16	5.1	5.0	0.29	4.7	4.3	0.17	5.0	4.8	0.13	5.2	5.0
	0.9	8.94	0.062	0.0	0.00	5.1	5.3	0.00	5.4	5.1	0.00	5.4	5.1	0.00	4.9	4.7	0.00	5.1	5.0	0.00	5.1	5.0
		8.94	0.062	0.5	0.11	5.2	5.2	0.06	5.3	5.1	0.05	5.4	5.1	0.09	4.9	4.7	0.05	5.1	5.0	0.04	5.0	5.0
		8.94	0.062	5.0	0.35	4.9	4.8	0.20	5.0	4.9	0.16	5.1	5.0	0.29	4.6	4.4	0.17	4.9	4.9	0.13	5.0	4.9
	1.0	9.42	0.056	0.0	0.00	5.1	5.2	0.00	5.2	5.0	0.00	5.3	5.0	0.00	5.3	5.3	0.00	5.2	5.1	0.00	4.9	4.8
		9.42	0.056	0.5	0.11	5.2	5.1	0.06	5.2	5.1	0.05	5.3	5.0	0.09	5.2	5.2	0.05	5.1	5.0	0.04	4.9	4.7
		9.42	0.056	5.0	0.35	5.5	5.4	0.20	5.4	5.2	0.16	5.3	5.0	0.29	5.2	5.1	0.17	5.2	5.0	0.13	4.9	4.8
0.10	0.2	2.18	1.042	0.0	0.00	4.7	4.6	0.00	4.9	4.6	0.00	5.1	4.8	0.00	4.8	4.6	0.00	5.0	4.8	0.00	5.2	4.6
		2.18	1.042	0.5	0.11	4.8	4.7	0.06	4.8	4.6	0.05	5.2	4.9	0.09	5.0	4.6	0.05	5.0	4.9	0.04	5.2	4.7
		2.18	1.042	5.0	0.35	6.0	5.6	0.20	5.4	5.0	0.16	5.5	5.1	0.29	5.9	5.7	0.17	5.7	5.4	0.13	5.7	5.3
	0.5	4.30	0.267	0.0	0.00	4.9	4.8	0.00	5.2	4.8	0.00	5.3	4.9	0.00	4.7	4.4	0.00	4.8	4.7	0.00	5.2	4.8
		4.30	0.267	0.5	0.11	5.0	4.8	0.06	5.1	4.7	0.05	5.2	4.9	0.09	4.7	4.4	0.05	5.0	4.8	0.04	5.2	4.9
		4.30	0.267	5.0	0.35	6.1	5.6	0.20	5.5	4.9	0.16	5.4	5.2	0.29	5.9	5.4	0.17	5.7	5.4	0.13	5.9	5.3
	0.8	5.60	0.157	0.0	0.00	4.9	5.1	0.00	5.3	4.9	0.00	5.5	4.8	0.00	4.8	4.6	0.00	5.1	4.8	0.00	5.1	4.7
		5.60	0.157	0.5	0.11	5.1	4.9	0.06	5.2	4.7	0.05	5.4	4.8	0.09	4.6	4.4	0.05	5.1	4.9	0.04	5.2	4.8
		5.60	0.157	5.0	0.35	5.1	4.8	0.20	5.2	4.7	0.16	5.2	4.9	0.29	5.0	4.6	0.17	5.3	4.8	0.13	5.4	4.8
	0.9	5.96	0.139	0.0	0.00	5.0	5.1	0.00	5.3	4.9	0.00	5.5	4.8	0.00	4.9	4.6	0.00	5.0	4.8	0.00	5.1	4.7
		5.96	0.139	0.5	0.11	5.1	5.0	0.06	5.2	4.8	0.05	5.4	4.8	0.09	4.7	4.4	0.05	5.1	4.8	0.04	5.0	4.6
		5.96	0.139	5.0	0.35	4.7	4.3	0.20	5.0	4.6	0.16	5.2	4.7	0.29	4.6	4.2	0.17	5.0	4.7	0.13	5.0	4.5
	1.0	6.28	0.125	0.0	0.00	5.1	5.1	0.00	5.2	4.8	0.00	5.4	4.9	0.00	5.2	4.9	0.00	5.2	5.0	0.00	5.0	4.7
		6.28	0.125	0.5	0.11	5.2	5.0	0.06	5.2	4.7	0.05	5.4	4.8	0.09	5.2	4.8	0.05	5.1	5.0	0.04	5.0	4.5
		6.28	0.125	5.0	0.35	5.2	4.7	0.20	5.1	4.9	0.16	5.3	4.8	0.29	5.1	4.9	0.17	5.1	4.8	0.13	4.9	4.5

Power

Panel B: Power ($H_0: \phi_0 = 1, \rho_0 = 0.05$)					T = 200									T = 300								
					$l=1$			$l=3$			$l=5$			$l=1$			$l=3$			$l=5$		
ρ	ϕ	σ_v	λ	CP	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	4.4	4.5	0.00	4.4	4.2	0.00	4.1	4.0	0.00	4.3	4.2	0.00	4.4	4.1	0.00	4.4	4.2
		2.21	1.010	0.5	0.11	10.0	10.0	0.06	7.0	7.1	0.05	6.4	6.1	0.09	10.3	9.7	0.05	7.0	6.9	0.04	6.1	6.0
		2.21	1.010	5.0	0.35	62.2	60.0	0.20	39.2	38.2	0.16	30.6	29.7	0.29	62.3	60.6	0.17	41.4	40.2	0.13	33.0	31.8
	0.2	3.82	0.338	0.0	0.00	4.7	4.6	0.00	4.5	4.5	0.00	4.4	4.4	0.00	4.2	4.1	0.00	4.4	4.0	0.00	4.3	4.3
		3.82	0.338	0.5	0.11	12.2	11.9	0.06	8.0	8.1	0.05	7.3	7.0	0.09	12.4	12.0	0.05	7.9	7.7	0.04	6.9	6.5
		3.82	0.338	5.0	0.35	75.6	73.7	0.20	52.9	51.2	0.16	42.6	40.6	0.29	76.1	74.1	0.17	54.3	53.0	0.13	44.2	42.5
	0.5	6.57	0.114	0.0	0.00	4.7	4.6	0.00	4.3	4.6	0.00	4.6	4.6	0.00	4.3	4.2	0.00	4.7	4.4	0.00	4.6	4.5
		6.57	0.114	0.5	0.11	10.6	10.2	0.06	7.1	7.3	0.05	6.7	6.4	0.09	10.3	10.4	0.05	7.0	6.9	0.04	6.6	6.2
		6.57	0.114	5.0	0.35	63.6	61.4	0.20	41.6	40.1	0.16	32.1	31.0	0.29	63.9	61.8	0.17	43.0	41.7	0.13	34.1	32.9
	0.8	8.42	0.070	0.0	0.00	5.0	5.0	0.00	4.9	4.9	0.00	4.9	4.8	0.00	4.3	4.3	0.00	4.7	4.7	0.00	4.8	4.8
		8.42	0.070	0.5	0.11	7.2	6.9	0.06	6.1	6.0	0.05	5.6	5.6	0.09	7.3	7.0	0.05	5.9	5.7	0.04	5.8	5.6
		8.42	0.070	5.0	0.35	31.8	30.0	0.20	18.8	17.9	0.16	14.7	14.2	0.29	31.9	30.4	0.17	19.6	18.7	0.13	15.3	14.8
	0.9	8.94	0.062	0.0	0.00	5.0	5.2	0.00	5.0	5.1	0.00	5.2	5.0	0.00	4.6	4.5	0.00	4.8	4.8	0.00	4.8	4.8
		8.94	0.062	0.5	0.11	6.1	6.1	0.06	5.7	5.6	0.05	5.6	5.4	0.09	6.1	5.8	0.05	5.5	5.3	0.04	5.4	5.4
		8.94	0.062	5.0	0.35	19.1	18.3	0.20	11.4	11.2	0.16	9.6	9.2	0.29	18.7	17.9	0.17	11.9	11.3	0.13	9.7	9.4
	1.0	9.42	0.056	0.0	0.00	5.1	5.2	0.00	5.2	5.0	0.00	5.3	5.0	0.00	5.3	5.3	0.00	5.2	5.1	0.00	4.9	4.8
		9.42	0.056	0.5	0.11	5.2	5.1	0.06	5.2	5.1	0.05	5.3	5.0	0.09	5.2	5.2	0.05	5.1	5.0	0.04	4.9	4.7
		9.42	0.056	5.0	0.35	5.5	5.4	0.20	5.4	5.2	0.16	5.3	5.0	0.29	5.2	5.1	0.17	5.2	5.0	0.13	4.9	4.8
0.10	0.2	2.18	1.042	0.0	0.00	4.4	100.0	0.00	4.5	95.6	0.00	4.2	75.6	0.00	4.2	100.0	0.00	4.3	100.0	0.00	4.4	99.7
		2.18	1.042	0.5	0.11	8.9	100.0	0.06	6.5	95.9	0.05	6.1	78.1	0.09	9.4	100.0	0.05	6.7	100.0	0.04	5.9	99.7
		2.18	1.042	5.0	0.35	54.4	100.0	0.20	33.0	99.0	0.16	25.5	91.9	0.29	54.7	100.0	0.17	35.1	100.0	0.13	27.5	99.9
	0.5	4.30	0.267	0.0	0.00	4.8	96.6	0.00	4.5	62.4	0.00	4.6	40.9	0.00	4.4	100.0	0.00	4.4	96.4	0.00	4.6	82.0
		4.30	0.267	0.5	0.11	9.4	97.3	0.06	6.6	66.9	0.05	6.2	45.3	0.09	9.3	100.0	0.05	6.7	96.9	0.04	5.9	84.2
		4.30	0.267	5.0	0.35	55.7	99.3	0.20	34.4	88.1	0.16	26.3	73.7	0.29	55.7	100.0	0.17	35.9	99.1	0.13	28.8	94.5
	0.8	5.60	0.157	0.0	0.00	4.9	50.3	0.00	4.8	23.6	0.00	5.0	17.5	0.00	4.4	80.8	0.00	4.6	47.4	0.00	4.8	33.0
		5.60	0.157	0.5	0.11	6.9	53.3	0.06	5.8	25.6	0.05	5.6	19.0	0.09	6.9	82.1	0.05	5.7	49.6	0.04	5.5	35.2
		5.60	0.157	5.0	0.35	28.5	71.1	0.20	16.4	43.4	0.16	13.3	32.3	0.29	28.6	90.1	0.17	17.4	65.3	0.13	13.5	50.3
	0.9	5.96	0.139	0.0	0.00	5.0	30.2	0.00	4.9	15.9	0.00	5.1	12.8	0.00	4.5	54.0	0.00	4.9	28.8	0.00	4.8	21.4
		5.96	0.139	0.5	0.11	6.0	31.8	0.06	5.5	16.8	0.05	5.4	13.4	0.09	6.1	54.8	0.05	5.3	30.3	0.04	5.4	22.5
		5.96	0.139	5.0	0.35	16.9	45.1	0.20	10.5	25.1	0.16	8.9	19.5	0.29	17.2	65.3	0.17	11.1	39.6	0.13	9.1	30.2
	1.0	6.28	0.125	0.0	0.00	5.1	17.7	0.00	5.2	11.3	0.00	5.1	9.7	0.00	5.2	29.8	0.00	5.2	17.1	0.00	4.9	14.2
		6.28	0.125	0.5	0.11	5.0	17.5	0.06	5.0	11.4	0.05	5.1	9.7	0.09	5.0	29.9	0.05	5.2	17.1	0.04	4.8	14.0
		6.28	0.125	5.0	0.35	5.0	16.9	0.20	5.0	11.1	0.16	5.1	9.5	0.29	4.9	29.2	0.17	5.1	16.8	0.13	4.6	13.9

Table A4. Size and power comparison of joint tests under $\mathcal{M}_3$ with low-frequency (past lags) instruments, nominal level: 5%

Panel A: Size				$T = 200$						$T = 300$					
				$l = 1$		$l = 3$		$l = 5$		$l = 1$		$l = 3$		$l = 5$	
ρ	ϕ	λ	σ_v	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
0.10	0.50	0.27	4.30	4.8	3.9	4.8	4.2	4.7	4.5	4.8	4.0	5.1	4.4	4.6	4.2
	0.60	0.22	4.79	5.0	3.9	4.9	4.3	4.9	4.5	5.0	3.9	5.0	4.3	4.8	4.1
	0.70	0.18	5.21	5.3	3.8	5.1	4.2	5.0	4.3	5.2	3.7	5.1	4.1	4.9	4.3
	0.80	0.16	5.60	5.6	4.0	5.2	4.1	5.1	4.4	5.5	3.8	5.1	4.2	5.0	4.2
	0.90	0.14	5.96	6.2	4.2	5.3	4.3	5.3	4.5	6.3	4.0	5.4	4.2	5.3	4.4
	1.00	0.13	6.28	5.9	3.7	5.4	4.0	5.1	4.1	5.5	3.3	5.4	4.1	5.1	4.2
0.15	0.50	0.48	3.21	4.8	3.4	4.8	3.8	4.7	3.9	4.8	3.4	5.1	3.8	4.5	3.9
	0.60	0.38	3.61	4.9	3.4	4.9	3.9	4.8	3.8	5.1	3.2	5.1	3.8	4.5	3.8
	0.70	0.31	3.96	5.2	3.3	5.0	3.7	4.7	3.7	5.0	3.1	5.1	3.6	4.7	3.7
	0.80	0.27	4.27	5.5	3.2	5.0	3.7	4.9	3.9	5.5	3.0	5.2	3.6	4.7	3.8
	0.90	0.24	4.55	6.3	3.5	5.3	3.7	5.1	3.9	6.3	3.3	5.3	3.4	5.0	3.8
	1.00	0.21	4.80	5.5	2.7	5.1	3.3	4.9	3.4	5.2	2.5	5.5	3.5	5.0	3.5
0.20	0.50	0.80	2.48	4.7	2.9	4.8	3.3	4.8	3.4	4.8	3.0	4.9	3.6	4.8	3.6
	0.60	0.61	2.84	4.9	2.9	4.9	3.4	4.8	3.3	4.7	2.7	4.8	3.3	4.6	3.5
	0.70	0.50	3.15	5.2	2.8	4.9	3.2	4.8	3.3	4.9	2.5	4.9	3.1	4.7	3.4
	0.80	0.42	3.41	5.5	2.6	5.0	3.0	4.8	3.3	5.3	2.4	5.0	2.9	4.6	3.1
	0.90	0.37	3.64	6.3	2.7	5.3	3.0	5.2	3.1	6.2	2.6	5.2	2.7	4.8	3.2
	1.00	0.33	3.85	5.2	1.7	5.0	2.5	4.9	2.9	5.4	1.7	5.3	2.8	5.0	3.0
0.25	0.50	1.33	1.92	4.9	2.6	4.7	3.0	4.8	3.1	4.7	2.4	4.9	3.1	4.8	3.4
	0.60	0.96	2.27	4.9	2.5	4.8	2.9	4.7	2.8	4.7	2.1	5.0	2.9	4.7	3.1
	0.70	0.76	2.54	5.2	2.4	4.9	2.8	4.8	2.8	4.9	2.0	5.0	2.6	4.7	2.7
	0.80	0.64	2.77	5.5	2.0	5.0	2.4	4.9	2.6	5.4	1.8	5.0	2.5	4.7	2.6
	0.90	0.56	2.97	6.4	1.9	5.2	2.4	5.0	2.6	6.1	1.9	5.1	2.4	4.9	2.4
	1.00	0.50	3.14	5.1	1.1	4.8	1.9	4.7	2.3	4.9	1.0	5.2	2.2	4.8	2.1
0.30	0.50	2.40	1.43	4.8	2.2	5.1	2.7	4.9	2.6	4.7	2.0	5.0	2.7	4.7	2.8
	0.60	1.56	1.78	4.9	2.0	4.8	2.4	4.8	2.4	4.8	1.5	4.8	2.4	4.8	2.6
	0.70	1.19	2.04	5.1	1.8	5.0	2.1	4.9	2.3	5.0	1.4	4.9	2.0	4.6	2.2
	0.80	0.97	2.25	5.5	1.6	5.0	2.0	4.9	2.1	5.3	1.3	4.9	1.9	4.8	1.9
	0.90	0.84	2.42	6.4	1.4	5.3	1.8	5.0	2.1	6.2	1.3	5.3	1.6	4.8	1.8
	1.00	0.75	2.57	5.1	0.6	4.8	1.3	5.0	1.7	5.0	0.6	5.1	1.3	4.8	1.4
0.35	0.50	5.60	0.94	4.9	1.9	5.3	2.3	4.8	2.2	4.8	1.8	5.0	2.3	4.8	2.1
	0.60	2.82	1.32	4.9	1.7	5.3	1.9	5.0	2.0	4.9	1.2	5.0	2.1	4.8	2.0
	0.70	1.96	1.59	5.1	1.3	5.1	1.6	4.8	1.7	5.2	1.0	4.9	1.7	4.8	1.8
	0.80	1.55	1.79	5.4	1.1	5.1	1.6	5.0	1.5	5.2	1.0	5.1	1.4	4.8	1.4
	0.90	1.31	1.94	6.3	0.9	5.3	1.3	5.2	1.5	6.3	0.8	5.1	1.1	4.8	1.2
	1.00	1.17	2.06	5.0	0.3	5.0	0.8	5.1	1.1	5.1	0.4	5.1	0.8	4.7	0.8
Panel B: Power ($H_0 : \phi_0 = 1, \rho_0 = 0.1$)				$T = 200$						$T = 300$					
				$l = 1$		$l = 3$		$l = 5$		$l = 1$		$l = 3$		$l = 5$	
ρ	ϕ	λ	σ_v	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
0.10	0.50	0.27	4.30	97.8	96.1	90.8	88.0	82.0	78.8	99.9	99.8	99.1	98.6	97.2	95.8
	0.60	0.22	4.79	99.4	98.8	96.4	94.5	90.7	87.6	100.0	100.0	99.9	99.8	99.3	98.7
	0.70	0.18	5.21	99.8	99.4	97.8	96.3	93.4	90.3	100.0	100.0	100.0	99.9	99.7	99.4
	0.80	0.16	5.60	99.8	99.0	97.0	94.6	90.2	86.2	100.0	100.0	99.9	99.8	99.4	98.8
	0.90	0.14	5.96	97.9	92.6	82.1	74.0	66.4	58.4	100.0	99.5	97.7	95.1	91.3	87.0
	1.00	0.13	6.28	5.9	3.7	5.4	4.0	5.1	4.1	5.5	3.3	5.4	4.1	5.1	4.2
0.15	0.50	0.48	3.21	92.8	100.0	80.2	100.0	67.9	100.0	99.2	100.0	96.1	100.0	90.6	100.0
	0.60	0.38	3.61	97.7	100.0	91.0	100.0	81.1	100.0	99.9	100.0	99.2	100.0	97.1	100.0
	0.70	0.31	3.96	99.2	100.0	94.7	100.0	86.1	100.0	100.0	100.0	99.7	100.0	98.6	100.0
	0.80	0.27	4.27	99.3	100.0	93.4	100.0	82.8	99.7	100.0	100.0	99.7	100.0	98.2	100.0
	0.90	0.24	4.55	95.8	99.9	74.7	98.0	57.5	89.1	99.9	100.0	95.3	100.0	85.8	99.9
	1.00	0.21	4.80	4.1	20.5	4.6	13.1	4.5	11.0	3.9	44.6	4.9	25.5	4.6	19.6
0.20	0.50	0.80	2.48	83.4	100.0	63.8	100.0	51.7	100.0	95.4	100.0	87.3	100.0	77.6	100.0
	0.60	0.61	2.84	92.9	100.0	79.7	100.0	67.0	100.0	99.3	100.0	96.3	100.0	90.9	100.0
	0.70	0.50	3.15	96.7	100.0	87.3	100.0	74.8	100.0	99.9	100.0	98.6	100.0	95.3	100.0
	0.80	0.42	3.41	97.2	100.0	85.9	100.0	71.9	100.0	99.9	100.0	98.5	100.0	94.7	100.0
	0.90	0.37	3.64	92.2	100.0	65.5	100.0	48.2	99.7	99.5	100.0	90.9	100.0	77.5	100.0
	1.00	0.33	3.85	2.8	73.3	4.3	55.7	4.3	43.4	2.7	92.1	4.6	84.3	4.7	75.4
0.25	0.50	1.33	1.92	67.9	100.0	46.2	100.0	35.9	100.0	85.8	100.0	70.2	100.0	57.4	100.0
	0.60	0.96	2.27	83.1	100.0	63.6	100.0	50.4	100.0	95.6	100.0	87.7	100.0	77.3	100.0
	0.70	0.76	2.54	90.7	100.0	73.8	100.0	59.3	100.0	98.8	100.0	94.5	100.0	87.1	100.0
	0.80	0.64	2.77	92.8	100.0	73.7	100.0	57.9	100.0	99.4	100.0	95.1	100.0	87.0	100.0
	0.90	0.56	2.97	86.6	100.0	54.7	100.0	39.1	100.0	98.1	100.0	82.6	100.0	65.3	100.0
	1.00	0.50	3.14	1.6	95.1	4.0	90.9	4.1	85.8	1.6	99.5	4.6	99.0	4.4	98.2
0.30	0.50	2.40	1.43	48.2	100.0	30.4	100.0	23.7	100.0	67.1	100.0	46.4	100.0	36.5	100.0
	0.60	1.56	1.78	67.1	100.0	44.8	100.0	35.0	100.0	85.7	100.0	70.1	100.0	56.4	100.0
	0.70	1.19	2.04	79.0	100.0	56.4	100.0	43.3	100.0	93.7	100.0	83.1	100.0	70.3	100.0
	0.80	0.97	2.25	83.1	100.0	57.7	100.0	43.2	100.0	96.3	100.0	85.7	100.0	72.0	100.0
	0.90	0.84	2.42	77.4	100.0	43.1	100.0	30.0	100.0	94.5	100.0	70.1	100.0	51.7	100.0
	1.00	0.75	2.57	0.7	99.6	4.2	99.3	4.3	98.9	0.8	100.0	4.8	100.0	4.6	100.0
0.35	0.50	5.60	0.94	29.9	100.0	18.4	100.0	15.3	100.0	42.7	100.0	26.3	100.0	21.0	100.0
	0.60	2.82	1.32	46.6	100.0	28.8	100.0	22.5	100.0	66.0	100.0	45.0	100.0	35.0	100.0
	0.70	1.96	1.59	60.3	100.0	38.2	100.0	29.4	100.0	80.6	100.0	61.8	100.0	47.7	100.0
	0.80	1.55	1.79	66.5	100.0	40.4	100.0	30.5	100.0	86.7	100.0	67.2	100.0	51.6	100.0
	0.90	1.31	1.94	64.4	100.0	32.1	100.0	22.6	100.0	86.1	100.0	54.0	100.0	37.8	100.0
	1.00	1.17	2.06	0.1	100.0	4.8	100.0	4.8	99.9	0.3	100.0	4.7	100.0	5.1	100.0

Table A5. Size and power comparison of joint tests under $\mathcal{M}_4$ with low-frequency instruments, nominal level: 5%

Panel A: Size				T = 200						T = 300					
				$l=1$		$l=3$		$l=5$		$l=1$		$l=3$		$l=5$	
ρ	ϕ	λ	σ_v	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
0.10	0.50	0.27	4.30	5.1	4.0	5.0	4.2	4.7	4.2	5.3	4.2	4.9	4.6	5.1	4.5
	0.60	0.22	4.79	5.0	3.9	5.1	4.2	4.8	4.1	5.2	4.0	5.0	4.4	5.1	4.5
	0.70	0.18	5.21	5.2	3.8	5.3	4.4	4.8	4.0	5.4	3.9	5.2	4.4	5.0	4.3
	0.80	0.16	5.60	5.4	3.9	5.4	4.2	4.8	4.0	5.8	4.1	5.3	4.3	5.2	4.3
	0.90	0.14	5.96	5.9	4.0	5.5	4.6	5.2	4.3	6.3	4.1	5.5	4.6	5.3	4.5
	1.00	0.13	6.28	5.8	3.6	5.3	4.2	5.4	4.4	5.5	3.5	5.5	4.1	5.0	3.9
0.15	0.50	0.48	3.21	4.8	3.4	4.9	3.8	4.9	3.8	5.1	3.6	5.0	4.1	4.9	3.9
	0.60	0.38	3.61	5.0	3.2	4.9	3.7	4.8	3.7	5.1	3.4	5.0	4.0	5.0	3.9
	0.70	0.31	3.96	5.1	3.3	5.0	3.8	4.9	3.7	5.4	3.2	5.2	3.9	5.0	3.8
	0.80	0.27	4.27	5.5	3.2	5.2	3.8	4.9	3.8	5.6	3.2	5.3	3.7	5.1	3.8
	0.90	0.24	4.55	6.0	3.2	5.6	3.8	5.2	3.9	6.2	3.4	5.5	3.7	5.2	3.7
	1.00	0.21	4.80	5.4	2.5	5.2	3.3	5.3	3.7	5.2	2.5	5.2	3.5	5.0	3.5
0.20	0.50	0.80	2.48	4.7	2.8	5.0	3.6	4.8	3.4	4.9	2.9	4.8	3.6	4.7	3.4
	0.60	0.61	2.84	4.8	2.6	5.0	3.4	5.0	3.2	5.2	2.8	4.9	3.4	4.7	3.3
	0.70	0.50	3.15	5.0	2.7	4.9	3.3	4.9	3.1	5.2	2.6	5.1	3.3	4.8	3.1
	0.80	0.42	3.41	5.3	2.6	5.0	3.3	5.1	3.2	5.6	2.5	5.3	3.0	5.0	3.1
	0.90	0.37	3.64	5.9	2.6	5.6	3.4	5.3	3.2	6.1	2.6	5.4	3.1	5.1	3.3
	1.00	0.33	3.85	5.3	1.7	5.0	2.7	5.2	3.0	5.2	1.7	5.2	2.7	5.0	2.9
0.25	0.50	1.33	1.92	4.8	2.3	4.8	3.0	4.8	2.9	4.7	2.5	4.8	3.0	4.6	3.0
	0.60	0.96	2.27	4.9	2.1	5.0	2.9	4.9	2.8	5.1	2.2	4.9	3.0	4.8	2.7
	0.70	0.76	2.54	5.0	2.1	5.1	2.8	4.9	2.6	5.2	2.1	5.0	2.8	4.7	2.7
	0.80	0.64	2.77	5.4	1.9	5.1	2.6	5.2	2.5	5.5	2.0	5.3	2.4	4.9	2.6
	0.90	0.56	2.97	6.0	2.0	5.8	2.6	5.5	2.6	6.1	1.9	5.4	2.3	5.0	2.7
	1.00	0.50	3.14	5.0	1.1	4.9	2.0	5.0	2.6	5.0	1.0	5.1	2.0	4.9	2.3
0.30	0.50	2.40	1.43	4.9	2.1	4.8	2.6	4.6	2.5	4.7	2.2	4.7	2.6	4.7	2.7
	0.60	1.56	1.78	4.8	1.7	4.9	2.3	4.7	2.2	5.0	1.9	4.7	2.5	4.7	2.5
	0.70	1.19	2.04	4.9	1.5	5.3	2.2	4.8	2.0	5.1	1.6	5.0	2.2	4.8	2.3
	0.80	0.97	2.25	5.4	1.4	5.2	2.0	5.1	2.1	5.5	1.4	5.2	1.9	4.8	2.1
	0.90	0.84	2.42	6.0	1.4	5.7	2.0	5.4	1.9	6.0	1.4	5.4	1.7	5.0	2.0
	1.00	0.75	2.57	4.9	0.7	4.9	1.4	5.0	1.8	5.1	0.5	4.9	1.3	4.9	1.5
0.35	0.50	5.60	0.94	4.7	1.7	4.7	2.0	4.6	2.1	4.8	1.7	4.8	2.2	4.7	2.1
	0.60	2.82	1.32	4.9	1.3	4.8	1.9	4.6	1.8	4.8	1.4	4.8	1.8	4.8	1.9
	0.70	1.96	1.59	4.8	1.1	5.2	1.6	4.6	1.6	5.2	1.1	4.9	1.7	4.8	1.7
	0.80	1.55	1.79	5.3	1.0	5.2	1.5	4.9	1.4	5.4	0.8	5.2	1.4	4.9	1.5
	0.90	1.31	1.94	6.1	0.9	5.9	1.2	5.4	1.3	6.1	0.9	5.3	1.1	5.0	1.2
	1.00	1.17	2.06	4.9	0.3	4.9	0.8	4.9	1.0	5.1	0.3	4.7	0.8	4.7	1.0

Panel B: Power ($H_0: \phi_0 = 1, \rho_0 = 0.1$)				T = 200						T = 300					
				$l=1$		$l=3$		$l=5$		$l=1$		$l=3$		$l=5$	
ρ	ϕ	λ	σ_v	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
0.10	0.50	0.27	4.30	97.6	96.2	90.9	88.1	81.5	78.4	99.9	99.8	99.1	98.6	97.0	96.0
	0.60	0.22	4.79	99.4	98.8	96.3	94.5	90.8	88.0	100.0	100.0	99.9	99.8	99.3	98.9
	0.70	0.18	5.21	99.8	99.3	97.8	96.3	93.6	90.9	100.0	100.0	100.0	99.9	99.7	99.4
	0.80	0.16	5.60	99.8	99.0	97.0	94.5	90.5	86.7	100.0	100.0	100.0	99.8	99.4	98.9
	0.90	0.14	5.96	97.8	93.0	82.2	74.7	66.1	59.1	100.0	99.6	97.7	95.1	91.3	87.1
	1.00	0.13	6.28	5.8	3.6	5.3	4.2	5.4	4.4	5.5	3.5	5.5	4.1	5.0	3.9
0.15	0.50	0.48	3.21	93.0	100.0	80.5	100.0	67.8	100.0	99.1	100.0	96.2	100.0	90.2	100.0
	0.60	0.38	3.61	97.4	100.0	90.5	100.0	80.8	100.0	99.9	100.0	99.2	100.0	97.1	100.0
	0.70	0.31	3.96	99.0	100.0	94.3	100.0	86.4	100.0	100.0	100.0	99.8	100.0	98.8	100.0
	0.80	0.27	4.27	99.2	100.0	93.1	100.0	82.9	99.8	100.0	100.0	99.7	100.0	98.2	100.0
	0.90	0.24	4.55	95.8	99.9	74.8	97.8	57.7	89.9	99.9	100.0	95.4	100.0	85.4	99.8
	1.00	0.21	4.80	4.2	20.4	4.7	13.4	4.9	11.0	4.1	44.8	4.6	25.8	4.7	19.6
0.20	0.50	0.80	2.48	83.2	100.0	64.2	100.0	52.3	100.0	95.6	100.0	87.0	100.0	76.9	100.0
	0.60	0.61	2.84	93.0	100.0	80.2	100.0	66.8	100.0	99.2	100.0	96.7	100.0	90.2	100.0
	0.70	0.50	3.15	96.7	100.0	87.2	100.0	75.1	100.0	99.8	100.0	98.6	100.0	95.2	100.0
	0.80	0.42	3.41	97.3	100.0	86.0	100.0	72.0	100.0	99.9	100.0	98.8	100.0	94.5	100.0
	0.90	0.37	3.64	92.4	100.0	65.7	100.0	48.3	99.5	99.5	100.0	90.7	100.0	77.1	100.0
	1.00	0.33	3.85	2.8	73.4	4.3	56.3	4.5	44.0	2.7	91.7	4.5	84.2	4.5	75.5
0.25	0.50	1.33	1.92	68.3	100.0	46.5	100.0	36.4	100.0	85.5	100.0	69.0	100.0	57.6	100.0
	0.60	0.96	2.27	83.0	100.0	64.1	100.0	50.9	100.0	95.8	100.0	87.6	100.0	76.3	100.0
	0.70	0.76	2.54	90.6	100.0	74.6	100.0	60.0	100.0	98.7	100.0	94.8	100.0	86.1	100.0
	0.80	0.64	2.77	92.5	100.0	74.6	100.0	58.6	100.0	99.4	100.0	95.4	100.0	86.0	100.0
	0.90	0.56	2.97	86.6	100.0	55.2	100.0	39.2	100.0	98.2	100.0	82.4	100.0	65.2	100.0
	1.00	0.50	3.14	1.7	94.2	4.1	90.6	4.3	84.7	1.5	99.4	4.4	98.8	4.6	98.1
0.30	0.50	2.40	1.43	49.2	100.0	30.1	100.0	24.1	100.0	67.0	100.0	46.7	100.0	37.2	100.0
	0.60	1.56	1.78	67.5	100.0	45.4	100.0	35.5	100.0	85.2	100.0	69.2	100.0	56.6	100.0
	0.70	1.19	2.04	78.8	100.0	56.7	100.0	44.1	100.0	93.6	100.0	82.7	100.0	69.8	100.0
	0.80	0.97	2.25	82.9	100.0	58.5	100.0	44.2	100.0	96.6	100.0	85.4	100.0	71.1	100.0
	0.90	0.84	2.42	77.3	100.0	43.6	100.0	30.6	100.0	94.3	100.0	70.0	100.0	52.4	100.0
	1.00	0.75	2.57	0.7	99.4	4.2	98.9	4.5	98.3	0.6	100.0	4.7	100.0	4.8	100.0
0.35	0.50	5.60	0.94	30.2	100.0	18.2	100.0	15.2	100.0	42.3	100.0	26.4	100.0	21.8	100.0
	0.60	2.82	1.32	47.5	100.0	28.7	100.0	22.9	100.0	65.6	100.0	45.0	100.0	35.6	100.0
	0.70	1.96	1.59	61.3	100.0	38.3	100.0	29.7	100.0	80.4	100.0	61.5	100.0	48.3	100.0
	0.80	1.55	1.79	67.3	100.0	41.0	100.0	31.0	100.0	86.9	100.0	66.7	100.0	52.1	100.0
	0.90	1.31	1.94	64.7	100.0	32.5	100.0	23.2	100.0	85.6	100.0	54.4	100.0	39.2	100.0
	1.00	1.17	2.06	0.3	100.0	4.3	99.9	4.7	99.9	0.2	100.0	4.8	100.0	4.9	100.0

Table A6. Size and power comparison of joint tests under $\mathcal{M}_5$ with high-frequency instruments, nominal level: 5%

Panel A: Size

											$T=200$						$T=300$					
											$l=1$		$l=3$		$l=5$		$l=1$		$l=3$		$l=5$	
Freq.	ϕ_h	μ_h	$\sigma^2_{h,\epsilon}$	$\sigma^2_{h,v}$	ϕ_l	μ_l	$\sigma^2_{l,\epsilon}$	$\sigma^2_{l,v}$	ρ_l	ρ_0	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
1-min	0.99822428	0.0001	0.0127	1.00E-09	0.50000	11.0	4.94	0.033	0.398	0.398	5.4	1.5	4.7	1.7	4.7	1.7	5.6	1.6	5.0	1.9	5.0	1.9
	0.99869105	0.0001	0.0127	1.00E-09	0.60000	11.9	4.94	0.038	0.439	0.439	5.4	0.8	5.1	1.0	4.8	0.9	5.4	0.6	5.0	1.0	4.9	1.2
	0.99908587	0.0001	0.0127	1.00E-09	0.70000	12.8	4.94	0.043	0.467	0.467	5.5	0.1	5.1	0.4	4.7	0.3	4.9	0.1	4.9	0.4	5.2	0.4
	0.99942800	0.0001	0.0127	1.00E-09	0.80000	13.6	4.94	0.048	0.485	0.485	5.1	0.0	5.1	0.0	4.8	0.1	5.2	0.0	5.1	0.1	5.2	0.1
	0.99972988	0.0001	0.0127	1.00E-09	0.90000	14.4	4.94	0.054	0.494	0.494	5.2	0.0	5.1	0.0	5.1	0.0	5.2	0.0	5.0	0.0	5.4	0.0
	0.99999997	0.0001	0.0127	1.00E-09	0.99999	15.2	4.94	0.059	0.497	0.497	5.4	0.0	5.2	0.0	5.2	0.0	5.3	0.0	5.1	0.0	5.3	0.0
5-min	0.99115287	0.0005	0.0633	1.00E-08	0.50000	2.20	4.94	0.003	0.400	0.400	5.4	1.6	4.9	1.9	4.5	1.5	5.0	1.4	5.2	1.6	5.1	1.8
	0.99347235	0.0005	0.0633	1.00E-08	0.60000	2.39	4.94	0.003	0.441	0.441	5.3	0.9	4.9	1.1	4.6	0.8	5.2	0.7	5.3	0.9	4.9	1.0
	0.99543768	0.0005	0.0633	1.00E-08	0.70000	2.56	4.94	0.003	0.470	0.470	5.2	0.4	5.1	0.5	4.8	0.3	5.4	0.2	5.4	0.4	4.9	0.4
	0.99714327	0.0005	0.0633	1.00E-08	0.80000	2.73	4.94	0.004	0.488	0.488	5.3	0.0	5.3	0.1	5.1	0.0	5.6	0.1	5.5	0.1	5.3	0.1
	0.99865014	0.0005	0.0633	1.00E-08	0.90000	2.89	4.94	0.004	0.497	0.497	4.7	0.0	5.1	0.0	5.1	0.0	5.2	0.0	5.4	0.0	5.3	0.0
	0.99999987	0.0005	0.0633	1.00E-08	0.99999	3.04	4.94	0.005	0.500	0.500	5.4	0.0	5.5	0.0	5.4	0.0	4.9	0.0	5.2	0.0	5.2	0.0
10-min	0.98238400	0.001	0.1266	1.00E-07	0.50000	1.11	4.93	0.003	0.400	0.400	5.0	1.5	5.0	1.9	4.8	1.6	5.2	1.4	5.0	1.5	5.0	1.6
	0.98698731	0.001	0.1266	1.00E-07	0.60000	1.20	4.93	0.004	0.441	0.441	5.1	0.8	5.1	1.0	5.0	0.9	5.7	0.7	5.1	0.9	4.9	0.8
	0.99089618	0.001	0.1266	1.00E-07	0.70000	1.29	4.93	0.004	0.470	0.470	5.5	0.4	5.5	0.4	4.9	0.4	5.9	0.2	5.0	0.4	5.0	0.4
	0.99429471	0.001	0.1266	1.00E-07	0.80000	1.37	4.93	0.005	0.488	0.488	5.9	0.1	5.5	0.2	5.1	0.1	5.9	0.1	5.1	0.1	5.1	0.1
	0.99730209	0.001	0.1266	1.00E-07	0.90000	1.45	4.93	0.005	0.497	0.497	5.5	0.0	5.6	0.0	5.3	0.0	5.7	0.0	5.3	0.0	5.1	0.0
	0.99999974	0.001	0.1266	1.00E-07	0.99999	1.52	4.93	0.006	0.500	0.500	5.0	0.0	5.3	0.0	5.0	0.0	4.9	0.0	5.0	0.0	5.1	0.0
15-min	0.97369272	0.005	0.1900	1.00E-06	0.50000	2.47	4.94	0.010	0.399	0.399	4.7	1.6	4.7	1.7	4.9	1.7	5.1	1.4	4.9	1.8	4.9	1.7
	0.98054461	0.005	0.1900	1.00E-06	0.60000	2.67	4.94	0.011	0.440	0.440	4.9	0.8	4.9	1.0	5.0	0.8	5.1	0.7	4.9	0.8	4.9	0.8
	0.98637540	0.005	0.1900	1.00E-06	0.70000	2.86	4.94	0.013	0.469	0.469	5.2	0.5	4.8	0.4	5.1	0.4	5.2	0.2	5.0	0.4	5.0	0.4
	0.99145428	0.005	0.1900	1.00E-06	0.80000	3.04	4.94	0.014	0.487	0.487	5.5	0.0	5.2	0.1	5.4	0.1	5.4	0.0	4.9	0.1	5.1	0.0
	0.99595587	0.005	0.1900	1.00E-06	0.90000	3.21	4.94	0.016	0.496	0.496	5.3	0.0	5.2	0.0	5.4	0.0	5.0	0.0	4.8	0.0	5.0	0.0
	0.99999962	0.005	0.1900	1.00E-06	0.99999	3.38	4.94	0.018	0.499	0.499	4.9	0.0	5.1	0.0	5.3	0.0	4.6	0.0	5.0	0.0	4.7	0.0

Panel B: Power ($H_0 : \phi_0 = 1, \rho = \rho_0$)

											$T=200$						$T=300$					
											$l=1$		$l=3$		$l=5$		$l=1$		$l=3$		$l=5$	
Freq.	ϕ_h	μ_h	$\sigma^2_{h,\epsilon}$	$\sigma^2_{h,v}$	ϕ_l	μ_l	$\sigma^2_{l,\epsilon}$	$\sigma^2_{l,v}$	ρ_l	ρ_0	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
1-min	0.99822428	0.0001	0.0127	1.00E-09	0.50000	11.0	4.94	0.033	0.398	0.398	99.7	97.5	91.2	75.2	81.8	61.2	100.0	99.7	99.3	94.9	97.1	87.8
	0.99869105	0.0001	0.0127	1.00E-09	0.60000	11.9	4.94	0.038	0.439	0.398	100.0	100.0	94.6	100.0	83.1	100.0	100.0	100.0	99.6	100.0	97.4	100.0
	0.99908587	0.0001	0.0127	1.00E-09	0.70000	12.8	4.94	0.043	0.467	0.398	100.0	100.0	99.7	100.0	89.7	100.0	100.0	100.0	100.0	100.0	98.8	100.0
	0.99942800	0.0001	0.0127	1.00E-09	0.80000	13.6	4.94	0.048	0.485	0.398	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	0.99972988	0.0001	0.0127	1.00E-09	0.90000	14.4	4.94	0.054	0.494	0.398	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	0.99999997	0.0001	0.0127	1.00E-09	0.99999	15.2	4.94	0.059	0.497	0.497	5.4	0.0	5.2	0.0	5.2	0.0	5.3	0.0	5.1	0.0	5.3	0.0
5-min	0.99115287	0.0005	0.0633	1.00E-08	0.50000	2.20	4.94	0.003	0.400	0.400	96.9	87.3	89.3	71.3	79.9	58.3	99.9	98.0	98.8	92.6	96.4	85.9
	0.99347235	0.0005	0.0633	1.00E-08	0.60000	2.39	4.94	0.003	0.441	0.400	97.3	100.0	89.5	100.0	80.0	100.0	99.9	100.0	98.9	100.0	96.4	100.0
	0.99543768	0.0005	0.0633	1.00E-08	0.70000	2.56	4.94	0.003	0.470	0.400	98.2	100.0	90.3	100.0	80.2	100.0	99.9	100.0	99.0	100.0	96.5	100.0
	0.99714327	0.0005	0.0633	1.00E-08	0.80000	2.73	4.94	0.004	0.488	0.400	99.7	100.0	93.2	100.0	82.3	100.0	100.0	100.0	99.4	100.0	97.1	100.0
	0.99865014	0.0005	0.0633	1.00E-08	0.90000	2.89	4.94	0.004	0.497	0.400	100.0	100.0	99.8	100.0	96.4	100.0	100.0	100.0	100.0	100.0	99.6	100.0
	0.99999987	0.0005	0.0633	1.00E-08	0.99999	3.04	4.94	0.005	0.500	0.500	5.4	0.0	5.5	0.0	5.4	0.0	4.9	0.0	5.2	0.0	5.2	0.0
10-min	0.98238400	0.001	0.1266	1.00E-07	0.50000	1.11	4.93	0.003	0.400	0.400	96.9	85.6	88.2	69.1	78.5	56.5	99.8	97.7	98.7	92.0	95.8	84.5
	0.98698731	0.001	0.1266	1.00E-07	0.60000	1.20	4.93	0.004	0.441	0.400	97.0	100.0	88.2	100.0	78.5	100.0	99.8	100.0	98.8	100.0	95.9	100.0
	0.99089618	0.001	0.1266	1.00E-07	0.70000	1.29	4.93	0.004	0.470	0.400	97.3	100.0	88.5	100.0	78.6	100.0	99.8	100.0	98.8	100.0	95.9	100.0
	0.99429471	0.001	0.1266	1.00E-07	0.80000	1.37	4.93	0.005	0.488	0.400	98.3	100.0	89.6	100.0	79.2	100.0	99.8	100.0	98.9	100.0	96.0	100.0
	0.99730209	0.001	0.1266	1.00E-07	0.90000	1.45	4.93	0.005	0.497	0.400	99.7	100.0	91.8	100.0	80.9	100.0	100.0	100.0	99.1	100.0	96.4	100.0
	0.99999974	0.001	0.1266	1.00E-07	0.99999	1.52	4.93	0.006	0.500	0.500	5.0	0.0	5.3	0.0	5.0	0.0	4.9	0.0	5.0	0.0	5.1	0.0
15-min	0.97369272	0.005	0.1900	1.00E-06	0.50000	2.47	4.94	0.010	0.399	0.399	96.3	84.5	86.1	66.3	76.0	53.7	99.7	97.0	97.9	90.5	95.0	82.4
	0.98054461	0.005	0.1900	1.00E-06	0.60000	2.67	4.94	0.011	0.440	0.399	96.8	100.0	86.3	100.0	76.1	100.0	99.8	100.0	97.9	100.0	95.0	100.0
	0.98637540	0.005	0.1900	1.00E-06	0.70000	2.86	4.94	0.013	0.469	0.399	98.1	100.0	87.5	100.0	76.5	100.0	99.9	100.0	98.4	100.0	95.1	100.0
	0.99145428	0.005	0.1900	1.00E-06	0.80000	3.04	4.94	0.014	0.487	0.399	99.7	100.0	92.1	100.0	79.5	100.0	100.0	100.0	99.2	100.0	95.9	100.0
	0.99595587	0.005	0.1900	1.00E-06	0.90000	3.21	4.94	0.016	0.496	0.399	100.0	100.0	100.0	100.0	97.4	100.0	100.0	100.0	100.0	100.0	99.8	100.0
	0.99999962	0.005	0.1900	1.00E-06	0.99999	3.38	4.94	0.018	0.499	0.499	4.9	0.0	5.1	0.0	5.3	0.0	4.6	0.0	5.0	0.0	4.7	0.0

Table A7. Size and power comparison of joint tests under $\mathcal{M}_6$ with high-frequency instruments, nominal level: 5%

Panel A: Size											T = 200						T = 300					
											$l=1$		$l=3$		$l=5$		$l=1$		$l=3$		$l=5$	
Freq.	ϕ_h	μ_h	$\sigma^2_{h,\epsilon}$	$\sigma^2_{h,v}$	ϕ_l	μ_l	$\sigma^2_{l,\epsilon}$	$\sigma^2_{l,v}$	ρ_l	ρ_0	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
1-min	0.9982	0.0001	4.93	1.00E-07	0.50	10.98	1923.558	3.35	0.399	0.399	5.0	1.5	5.1	1.8	5.0	1.6	5.6	1.5	5.0	1.7	5.2	1.5
	0.9987	0.0001	4.93	1.00E-07	0.60	11.92	1923.57	3.80	0.441	0.441	5.0	0.7	5.1	0.7	5.0	0.8	5.2	0.7	4.9	0.8	5.3	0.8
	0.9991	0.0001	4.93	1.00E-07	0.70	12.80	1923.577	4.29	0.469	0.469	5.1	0.3	4.9	0.3	4.9	0.2	5.0	0.2	4.8	0.2	5.1	0.3
	0.9994	0.0001	4.93	1.00E-07	0.80	13.64	1923.581	4.81	0.487	0.487	4.8	0.1	5.0	0.1	4.7	0.1	4.8	0.0	5.0	0.0	4.9	0.1
	0.9997	0.0001	4.93	1.00E-07	0.90	14.44	1923.583	5.35	0.496	0.496	4.7	0.0	5.1	0.0	4.8	0.0	4.8	0.0	4.9	0.0	5.1	0.0
	1.0000	0.0001	4.93	1.00E-07	1.00	15.21	1923.584	5.93	0.499	0.499	5.0	0.0	5.3	0.0	5.3	0.0	5.4	0.0	5.4	0.0	5.3	0.0
5-min	0.9912	0.0005	4.93	1.00E-07	0.50	2.20	384.9064	0.03	0.400	0.400	5.1	1.6	5.5	1.8	5.0	1.5	5.1	1.5	5.2	1.5	4.9	1.7
	0.9935	0.0005	4.93	1.00E-07	0.60	2.39	384.9065	0.03	0.441	0.441	5.0	0.8	5.4	1.0	5.0	0.8	5.1	0.7	5.1	0.7	5.0	0.9
	0.9954	0.0005	4.93	1.00E-07	0.70	2.56	384.9066	0.03	0.470	0.470	5.2	0.3	5.4	0.5	5.1	0.4	5.4	0.2	5.1	0.3	5.1	0.3
	0.9971	0.0005	4.93	1.00E-07	0.80	2.73	384.9066	0.04	0.488	0.488	5.3	0.2	5.4	0.1	5.3	0.1	5.7	0.0	5.1	0.0	5.1	0.1
	0.9987	0.0005	4.93	1.00E-07	0.90	2.89	384.9066	0.04	0.497	0.497	5.8	0.0	5.4	0.0	5.4	0.0	5.8	0.0	5.4	0.0	5.1	0.0
	1.0000	0.0005	4.93	1.00E-07	1.00	3.04	384.9067	0.05	0.500	0.500	5.2	0.0	5.4	0.0	5.4	0.0	5.2	0.0	5.6	0.0	5.2	0.0
10-min	0.9824	0.001	4.93	1.00E-07	0.50	1.11	192.4563	0.00	0.400	0.400	5.1	1.6	4.7	1.7	4.5	1.5	4.9	1.4	5.4	2.0	5.2	1.8
	0.9870	0.001	4.93	1.00E-07	0.60	1.20	192.4563	0.00	0.441	0.441	5.2	0.8	4.8	0.7	4.7	0.7	4.9	0.8	5.6	1.0	5.2	0.8
	0.9909	0.001	4.93	1.00E-07	0.70	1.29	192.4563	0.00	0.470	0.470	5.3	0.3	4.9	0.3	4.8	0.3	5.0	0.3	5.5	0.3	5.1	0.3
	0.9943	0.001	4.93	1.00E-07	0.80	1.37	192.4563	0.00	0.488	0.488	5.3	0.1	4.9	0.1	4.9	0.1	5.2	0.1	5.5	0.1	5.1	0.1
	0.9973	0.001	4.93	1.00E-07	0.90	1.45	192.4563	0.01	0.497	0.497	5.6	0.0	5.3	0.0	5.0	0.0	5.4	0.0	5.7	0.0	5.4	0.0
	1.0000	0.001	4.93	1.00E-07	1.00	1.52	192.4563	0.01	0.500	0.500	4.8	0.0	4.9	0.0	5.0	0.0	5.1	0.0	5.1	0.0	5.1	0.0
15-min	0.9737	0.005	4.93	1.00E-07	0.50	2.47	128.3045	0.00	0.400	0.400	5.0	1.6	5.3	1.8	5.1	1.6	4.9	1.5	5.2	1.7	5.0	1.7
	0.9805	0.005	4.93	1.00E-07	0.60	2.67	128.3046	0.00	0.441	0.441	5.2	0.7	5.1	1.0	5.2	0.9	5.0	0.7	5.1	0.9	5.0	0.9
	0.9864	0.005	4.93	1.00E-07	0.70	2.86	128.3046	0.00	0.470	0.470	5.3	0.3	5.3	0.3	5.4	0.3	5.2	0.3	5.1	0.3	4.9	0.3
	0.9915	0.005	4.93	1.00E-07	0.80	3.04	128.3046	0.00	0.488	0.488	5.4	0.1	5.6	0.1	5.6	0.1	5.1	0.1	5.1	0.1	5.0	0.0
	0.9960	0.005	4.93	1.00E-07	0.90	3.21	128.3046	0.00	0.497	0.497	5.6	0.0	5.6	0.0	5.6	0.0	5.1	0.0	5.1	0.0	5.1	0.0
	1.0000	0.005	4.93	1.00E-07	1.00	3.38	128.3046	0.00	0.500	0.500	4.6	0.0	4.7	0.0	5.4	0.0	5.0	0.0	5.3	0.0	5.1	0.0

Panel B: Power ($H_0: \phi_0 = 1, \rho = \rho_0$)											T = 200						T = 300					
											$l=1$		$l=3$		$l=5$		$l=1$		$l=3$		$l=5$	
Freq.	ϕ_h	μ_h	$\sigma^2_{h,\epsilon}$	$\sigma^2_{h,v}$	ϕ_l	μ_l	$\sigma^2_{l,\epsilon}$	$\sigma^2_{l,v}$	ρ_l	ρ_0	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*	AR	AR^*
1-min	0.9982	0.0001	4.93	1.00E-07	0.50	10.98	1923.558	3.35	0.399	0.399	65.0	36.7	41.4	22.1	32.5	16.2	84.2	60.1	64.5	40.4	52.4	30.5
	0.9987	0.0001	4.93	1.00E-07	0.60	11.92	1923.57	3.80	0.441	0.399	65.2	100.0	41.5	100.0	32.5	100.0	84.4	100.0	64.7	100.0	52.5	100.0
	0.9991	0.0001	4.93	1.00E-07	0.70	12.80	1923.577	4.29	0.469	0.399	65.8	100.0	41.3	100.0	32.4	100.0	84.7	100.0	64.7	100.0	52.3	100.0
	0.9994	0.0001	4.93	1.00E-07	0.80	13.64	1923.581	4.81	0.487	0.399	66.5	100.0	41.1	100.0	32.2	100.0	85.5	100.0	64.4	100.0	51.9	100.0
	0.9997	0.0001	4.93	1.00E-07	0.90	14.44	1923.583	5.35	0.496	0.399	63.5	100.0	37.3	100.0	30.3	100.0	85.2	100.0	59.7	100.0	49.1	100.0
	1.0000	0.0001	4.93	1.00E-07	1.00	15.21	1923.584	5.93	0.499	0.499	5.0	0.0	5.3	0.0	5.3	0.0	5.4	0.0	5.4	0.0	5.3	0.0
5-min	0.9912	0.0005	4.93	1.00E-07	0.50	2.20	384.9064	0.03	0.400	0.400	64.9	37.1	42.3	22.5	32.9	16.1	84.7	59.8	65.0	40.7	52.8	30.3
	0.9935	0.0005	4.93	1.00E-07	0.60	2.39	384.9065	0.03	0.441	0.400	65.0	100.0	42.3	100.0	32.9	100.0	84.7	100.0	65.0	100.0	52.7	100.0
	0.9954	0.0005	4.93	1.00E-07	0.70	2.56	384.9066	0.03	0.470	0.400	65.3	100.0	42.3	100.0	32.9	100.0	84.8	100.0	65.1	100.0	52.7	100.0
	0.9971	0.0005	4.93	1.00E-07	0.80	2.73	384.9066	0.04	0.488	0.400	65.3	100.0	42.4	100.0	33.0	100.0	84.8	100.0	65.0	100.0	52.8	100.0
	0.9987	0.0005	4.93	1.00E-07	0.90	2.89	384.9066	0.04	0.497	0.400	62.8	100.0	39.9	100.0	31.7	100.0	84.0	100.0	62.8	100.0	51.4	100.0
	1.0000	0.0005	4.93	1.00E-07	1.00	3.04	384.9067	0.05	0.500	0.500	5.2	0.0	5.4	0.0	5.4	0.0	5.2	0.0	5.6	0.0	5.2	0.0
10-min	0.9824	0.001	4.93	1.00E-07	0.50	1.11	192.4563	0.00	0.400	0.400	65.5	37.7	42.4	22.2	32.8	15.9	85.0	61.4	65.8	41.7	53.6	30.9
	0.9870	0.001	4.93	1.00E-07	0.60	1.20	192.4563	0.00	0.441	0.400	65.6	100.0	42.5	100.0	32.9	100.0	85.0	100.0	65.8	100.0	53.6	100.0
	0.9909	0.001	4.93	1.00E-07	0.70	1.29	192.4563	0.00	0.470	0.400	65.7	100.0	42.6	100.0	32.8	100.0	85.0	100.0	65.8	100.0	53.6	100.0
	0.9943	0.001	4.93	1.00E-07	0.80	1.37	192.4563	0.00	0.488	0.400	65.7	100.0	42.7	100.0	32.7	100.0	85.1	100.0	66.0	100.0	53.6	100.0
	0.9973	0.001	4.93	1.00E-07	0.90	1.45	192.4563	0.01	0.497	0.400	64.4	100.0	41.1	100.0	32.1	100.0	84.5	100.0	64.7	100.0	52.7	100.0
	1.0000	0.001	4.93	1.00E-07	1.00	1.52	192.4563	0.01	0.500	0.500	4.8	0.0	4.9	0.0	5.0	0.0	5.1	0.0	5.1	0.0	5.1	0.0
15-min	0.9737	0.005	4.93	1.00E-07	0.50	2.47	128.3045	0.00	0.400	0.400	66.5	38.6	43.8	22.7	34.1	16.7	85.5	62.2	65.7	42.3	54.4	31.5
	0.9805	0.005	4.93	1.00E-07	0.60	2.67	128.3046	0.00	0.441	0.400	66.7	100.0	43.9	100.0	34.0	100.0	85.5	100.0	65.8	100.0	54.5	100.0
	0.9864	0.005	4.93	1.00E-07	0.70	2.86	128.3046	0.00	0.470	0.400	67.1	100.0	43.9	100.0	34.1	100.0	85.8	100.0	65.9	100.0	54.4	100.0
	0.9915	0.005	4.93	1.00E-07	0.80	3.04	128.3046	0.00	0.488	0.400	68.1	100.0	44.0	100.0	34.1	100.0	86.5	100.0	66.1	100.0	54.5	100.0
	0.9960	0.005	4.93	1.00E-07	0.90	3.21	128.3046	0.00	0.497	0.400	64.6	100.0	39.8	100.0	32.2	100.0	85.9	100.0	63.1	100.0	52.3	100.0
	1.0000	0.005	4.93	1.00E-07	1.00	3.38	128.3046	0.00	0.500	0.500	4.6	0.0	4.7	0.0	5.4	0.0	5.0	0.0	5.3	0.0	5.1	0.0

I Additional simulation results

Table A8. Size comparison of joint tests ($H_0 : \phi = \phi_0, \rho = \rho_0$) under $\mathcal{M}_1$ with weak instruments, nominal level: 5%

					$T = 200$									$T = 300$								
					$l = 1$			$l = 3$			$l = 5$			$l = 1$			$l = 3$			$l = 5$		
ρ	ϕ	σ_v	λ	CP	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	5.1	5.1	0.00	4.8	4.6	0.00	4.8	4.7	0.00	5.0	4.8	0.00	5.3	5.0	0.00	5.1	4.9
		2.21	1.010	0.5	0.11	5.1	5.3	0.06	4.9	4.7	0.05	4.8	4.7	0.09	5.1	4.9	0.05	5.2	5.0	0.04	5.0	4.9
		2.21	1.010	5.0	0.35	5.5	5.3	0.20	5.4	5.1	0.16	4.9	5.0	0.29	5.3	5.3	0.17	5.4	5.0	0.13	5.2	5.1
	0.2	3.82	0.338	0.0	0.00	5.2	5.0	0.00	4.9	4.8	0.00	4.9	4.8	0.00	4.8	4.7	0.00	5.0	4.8	0.00	5.2	5.0
		3.82	0.338	0.5	0.11	5.2	5.1	0.06	4.9	4.9	0.05	4.9	5.0	0.09	4.9	4.5	0.05	4.9	4.8	0.04	5.1	5.0
		3.82	0.338	5.0	0.35	5.3	5.3	0.20	5.1	5.0	0.16	4.9	5.1	0.29	5.2	5.0	0.17	5.1	5.0	0.13	5.5	5.1
	0.3	4.92	0.204	0.0	0.00	5.3	5.0	0.00	5.1	4.9	0.00	4.7	4.7	0.00	4.8	4.8	0.00	5.1	4.9	0.00	5.1	4.9
		4.92	0.204	0.5	0.11	5.0	5.1	0.06	5.2	5.0	0.05	4.6	4.7	0.09	4.8	4.6	0.05	5.0	4.8	0.04	5.1	4.9
		4.92	0.204	5.0	0.35	5.2	5.1	0.20	5.2	5.1	0.16	4.8	5.0	0.29	5.1	5.2	0.17	5.1	4.9	0.13	5.4	5.1
	0.4	5.81	0.146	0.0	0.00	5.4	5.1	0.00	5.1	5.1	0.00	4.9	4.8	0.00	4.8	4.7	0.00	5.0	4.9	0.00	5.1	5.0
		5.81	0.146	0.5	0.11	5.1	5.0	0.06	5.0	5.0	0.05	4.8	4.9	0.09	4.8	4.6	0.05	5.1	4.9	0.04	5.0	4.9
		5.81	0.146	5.0	0.35	5.1	5.1	0.20	5.2	5.1	0.16	4.8	5.0	0.29	5.1	5.1	0.17	5.2	4.9	0.13	5.3	5.0
	0.5	6.57	0.114	0.0	0.00	5.3	5.1	0.00	5.2	5.2	0.00	4.9	4.8	0.00	4.8	4.6	0.00	5.1	5.0	0.00	5.0	5.0
		6.57	0.114	0.5	0.11	5.1	5.0	0.06	5.1	5.0	0.05	4.9	4.9	0.09	4.8	4.6	0.05	5.2	4.9	0.04	5.0	4.9
		6.57	0.114	5.0	0.35	5.0	4.8	0.20	5.1	5.0	0.16	4.9	4.9	0.29	5.0	5.1	0.17	5.1	5.0	0.13	5.2	4.9
	0.6	7.25	0.094	0.0	0.00	5.3	5.1	0.00	5.2	5.2	0.00	5.0	4.8	0.00	4.7	4.6	0.00	5.1	5.0	0.00	5.1	4.9
		7.25	0.094	0.5	0.11	5.1	5.0	0.06	5.1	5.0	0.05	4.9	4.7	0.09	4.8	4.6	0.05	5.1	5.0	0.04	5.0	4.9
		7.25	0.094	5.0	0.35	4.9	4.8	0.20	5.2	5.0	0.16	4.9	4.8	0.29	5.0	4.9	0.17	5.1	4.9	0.13	5.0	4.9
	0.7	7.86	0.080	0.0	0.00	5.2	5.0	0.00	5.1	5.1	0.00	5.0	4.8	0.00	4.8	4.6	0.00	5.1	5.0	0.00	5.0	4.8
		7.86	0.080	0.5	0.11	5.0	5.0	0.06	5.1	5.0	0.05	4.9	4.7	0.09	4.8	4.5	0.05	5.1	5.0	0.04	5.0	4.8
		7.86	0.080	5.0	0.35	4.9	4.8	0.20	5.2	5.0	0.16	4.9	4.9	0.29	5.0	4.6	0.17	5.0	4.9	0.13	5.0	4.9
	0.8	8.42	0.070	0.0	0.00	5.2	5.0	0.00	5.1	5.1	0.00	5.0	4.8	0.00	4.7	4.6	0.00	5.1	5.0	0.00	5.0	4.8
		8.42	0.070	0.5	0.11	5.0	5.0	0.06	5.1	5.0	0.05	5.0	4.8	0.09	4.7	4.5	0.05	5.0	4.9	0.04	5.0	4.8
		8.42	0.070	5.0	0.35	4.9	4.7	0.20	5.1	4.8	0.16	4.9	4.8	0.29	4.8	4.6	0.17	4.9	4.9	0.13	5.0	4.9
	0.9	8.94	0.062	0.0	0.00	5.2	5.0	0.00	5.0	5.1	0.00	5.1	4.8	0.00	4.8	4.6	0.00	5.1	5.0	0.00	5.0	4.8
		8.94	0.062	0.5	0.11	5.1	5.1	0.06	5.1	4.9	0.05	5.0	4.8	0.09	4.7	4.7	0.05	5.1	4.9	0.04	5.0	4.7
		8.94	0.062	5.0	0.35	4.8	4.7	0.20	5.0	4.8	0.16	4.9	4.9	0.29	4.7	4.5	0.17	4.8	4.8	0.13	4.9	4.8
	1.0	9.42	0.056	0.0	0.00	5.3	5.0	0.00	4.9	5.1	0.00	5.1	4.9	0.00	5.1	5.0	0.00	5.2	5.0	0.00	4.8	4.7
		9.42	0.056	0.5	0.11	5.3	5.1	0.06	5.0	4.9	0.05	4.9	4.9	0.09	5.2	5.1	0.05	5.2	5.2	0.04	4.7	4.8
		9.42	0.056	5.0	0.35	5.5	5.1	0.20	5.1	4.9	0.16	5.2	4.9	0.29	5.1	5.1	0.17	5.1	5.0	0.13	4.9	4.7
0.10	0.2	2.18	1.042	0.0	0.00	5.1	4.9	0.00	4.7	4.4	0.00	4.7	4.4	0.00	4.9	4.6	0.00	5.1	4.8	0.00	5.1	4.6
		2.18	1.042	0.5	0.11	5.3	5.0	0.06	4.9	4.5	0.05	4.7	4.4	0.09	5.1	4.6	0.05	5.2	4.7	0.04	5.1	4.6
		2.18	1.042	5.0	0.35	6.4	5.9	0.20	5.4	5.2	0.16	5.2	5.0	0.29	6.1	5.8	0.17	5.6	5.1	0.13	5.4	4.9
	0.3	3.07	0.524	0.0	0.00	5.1	4.8	0.00	4.9	4.5	0.00	4.7	4.6	0.00	4.8	4.5	0.00	4.9	4.6	0.00	5.2	4.5
		3.07	0.524	0.5	0.11	5.2	5.1	0.06	5.1	4.6	0.05	4.8	4.6	0.09	5.0	4.6	0.05	5.0	4.7	0.04	5.2	4.5
		3.07	0.524	5.0	0.35	6.4	6.0	0.20	5.6	5.1	0.16	5.1	5.0	0.29	6.2	5.9	0.17	5.6	5.2	0.13	5.6	5.0
	0.4	3.74	0.352	0.0	0.00	5.2	4.8	0.00	5.1	4.7	0.00	4.8	4.5	0.00	4.8	4.4	0.00	4.9	4.6	0.00	5.2	4.5
		3.74	0.352	0.5	0.11	5.1	5.1	0.06	5.2	4.8	0.05	4.9	4.6	0.09	4.9	4.5	0.05	5.1	4.7	0.04	5.2	4.5
		3.74	0.352	5.0	0.35	6.3	5.7	0.20	5.7	5.3	0.16	5.1	4.9	0.29	6.3	5.9	0.17	5.6	5.2	0.13	5.7	5.0
	0.5	4.30	0.267	0.0	0.00	5.2	4.8	0.00	5.2	4.7	0.00	4.7	4.4	0.00	4.9	4.4	0.00	4.9	4.6	0.00	5.1	4.5
		4.30	0.267	0.5	0.11	5.1	4.9	0.06	5.3	4.8	0.05	4.8	4.6	0.09	4.9	4.4	0.05	5.0	4.7	0.04	5.2	4.5
		4.30	0.267	5.0	0.35	5.9	5.6	0.20	5.7	5.3	0.16	5.3	4.8	0.29	6.0	5.8	0.17	5.5	5.1	0.13	5.6	4.9
	0.6	4.79	0.216	0.0	0.00	5.3	4.8	0.00	5.1	4.8	0.00	4.9	4.5	0.00	4.8	4.5	0.00	5.1	4.6	0.00	5.0	4.4
		4.79	0.216	0.5	0.11	5.0	4.8	0.06	5.2	4.8	0.05	4.8	4.5	0.09	4.8	4.4	0.05	5.1	4.6	0.04	5.1	4.4
		4.79	0.216	5.0	0.35	5.7	5.3	0.20	5.5	5.2	0.16	5.2	4.8	0.29	5.8	5.4	0.17	5.4	5.0	0.13	5.4	4.7
	0.7	5.21	0.181	0.0	0.00	5.3	4.8	0.00	5.1	4.8	0.00	4.8	4.4	0.00	4.7	4.4	0.00	5.1	4.6	0.00	4.9	4.4
		5.21	0.181	0.5	0.11	5.1	4.8	0.06	5.1	4.9	0.05	4.8	4.5	0.09	4.8	4.4	0.05	5.1	4.6	0.04	5.0	4.4
		5.21	0.181	5.0	0.35	5.4	5.1	0.20	5.3	4.9	0.16	5.1	4.7	0.29	5.4	5.0	0.17	5.3	4.7	0.13	5.3	4.6
	0.8	5.60	0.157	0.0	0.00	5.3	4.8	0.00	5.1	4.8	0.00	5.0	4.5	0.00	4.7	4.4	0.00	5.0	4.6	0.00	4.8	4.4
		5.60	0.157	0.5	0.11	5.1	4.7	0.06	5.2	4.9	0.05	4.9	4.5	0.09	4.8	4.4	0.05	5.0	4.6	0.04	4.9	4.4
		5.60	0.157	5.0	0.35	5.1	4.7	0.20	5.1	4.6	0.16	4.9	4.7	0.29	5.1	4.6	0.17	5.1	4.6	0.13	5.0	4.6
	0.9	5.96	0.139	0.0	0.00	5.2	4.8	0.00	5.1	4.9	0.00	5.0	4.6	0.00	4.7	4.5	0.00	5.0	4.7	0.00	4.9	4.3
		5.96	0.139	0.5	0.11	5.0	4.7	0.06	5.2	4.9	0.05	4.9	4.6	0.09	4.6	4.4	0.05	5.0	4.6	0.04	4.9	4.3
		5.96	0.139	5.0	0.35	4.7	4.3	0.20	5.1	4.6	0.16	4.8	4.4	0.29	4.6	4.3	0.17	4.9	4.6	0.13	5.0	4.4
	1.0	6.28	0.125	0.0	0.00	5.2	4.8	0.00	5.1	4.9	0.00	4.9	4.5	0.00	5.1	4.8	0.00	5.2	4.9	0.00	4.8	4.5
		6.28	0.125	0.5	0.11	5.2	4.9	0.06	5.1	4.7	0.05	4.9	4.5	0.09	4.9	4.8	0.05	5.2	4.9	0.04	4.7	4.5
		6.28	0.125	5.0	0.35	5.2	4.5	0.20	5.1	4.5	0.16	5.0	4.5	0.29	5.1	4.6	0.17	5.1	4.7	0.13	4.7	4.3

Table A9. Power comparison of joint tests ($H_0 : \phi = 1, \rho = 0.05$) under $\mathcal{M}_1$ with weak instruments, nominal level: 5%

					$T = 200$									$T = 300$								
					$l = 1$			$l = 3$			$l = 5$			$l = 1$			$l = 3$			$l = 5$		
ρ	ϕ	σ_v	λ	CP	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	4.5	4.3	0.00	4.3	4.3	0.00	4.4	4.3	0.00	4.5	4.4	0.00	4.4	4.3	0.00	4.3	4.0
		2.21	1.010	0.5	0.11	10.3	9.8	0.06	7.2	6.9	0.05	6.7	6.4	0.09	10.1	9.9	0.05	7.3	7.0	0.04	6.1	6.2
		2.21	1.010	5.0	0.35	61.5	60.3	0.20	40.9	39.8	0.16	31.1	30.4	0.29	62.1	60.5	0.17	40.5	39.5	0.13	32.0	30.8
	0.2	3.82	0.338	0.0	0.00	4.7	4.6	0.00	4.2	4.4	0.00	4.5	4.5	0.00	4.6	4.5	0.00	4.3	4.5	0.00	4.4	4.1
		3.82	0.338	0.5	0.11	12.7	12.1	0.06	8.3	8.2	0.05	7.6	7.1	0.09	12.2	11.9	0.05	8.0	7.8	0.04	7.2	6.8
		3.82	0.338	5.0	0.35	75.3	73.8	0.20	53.6	51.9	0.16	42.5	41.1	0.29	76.0	73.9	0.17	54.7	53.3	0.13	44.1	42.5
	0.3	4.92	0.204	0.0	0.00	4.9	4.6	0.00	4.5	4.5	0.00	4.6	4.5	0.00	4.4	4.4	0.00	4.4	4.5	0.00	4.5	4.3
		4.92	0.204	0.5	0.11	12.4	11.9	0.06	8.0	7.9	0.05	7.5	7.2	0.09	12.0	11.9	0.05	8.0	7.7	0.04	7.1	6.9
		4.92	0.204	5.0	0.35	74.6	72.9	0.20	52.5	50.8	0.16	41.6	40.5	0.29	75.1	73.2	0.17	53.8	52.1	0.13	43.6	42.0
	0.4	5.81	0.146	0.0	0.00	4.9	4.8	0.00	4.4	4.6	0.00	4.5	4.5	0.00	4.4	4.4	0.00	4.6	4.5	0.00	4.5	4.3
		5.81	0.146	0.5	0.11	11.5	11.0	0.06	7.8	7.6	0.05	7.3	6.9	0.09	11.4	11.3	0.05	7.6	7.4	0.04	7.0	6.7
		5.81	0.146	5.0	0.35	70.1	68.2	0.20	47.8	46.1	0.16	37.9	36.5	0.29	70.4	68.4	0.17	49.1	47.2	0.13	39.3	37.8
	0.5	6.57	0.114	0.0	0.00	4.9	4.8	0.00	4.4	4.6	0.00	4.5	4.4	0.00	4.4	4.5	0.00	4.6	4.6	0.00	4.6	4.5
		6.57	0.114	0.5	0.11	10.5	10.1	0.06	7.4	7.3	0.05	6.7	6.5	0.09	10.4	10.4	0.05	7.4	7.1	0.04	6.5	6.3
		6.57	0.114	5.0	0.35	63.5	61.7	0.20	41.5	39.8	0.16	32.3	31.5	0.29	63.8	61.6	0.17	42.4	40.8	0.13	33.6	32.2
	0.6	7.25	0.094	0.0	0.00	5.0	5.0	0.00	4.4	4.6	0.00	4.5	4.5	0.00	4.4	4.3	0.00	4.5	4.6	0.00	4.6	4.5
		7.25	0.094	0.5	0.11	9.7	9.3	0.06	7.1	6.8	0.05	6.4	6.3	0.09	9.2	9.1	0.05	6.7	6.6	0.04	6.2	5.9
		7.25	0.094	5.0	0.35	54.8	52.7	0.20	34.3	32.9	0.16	26.3	25.5	0.29	54.9	53.2	0.17	35.0	33.6	0.13	27.1	25.8
	0.7	7.86	0.080	0.0	0.00	5.2	5.0	0.00	4.6	4.8	0.00	4.6	4.5	0.00	4.3	4.2	0.00	4.8	4.6	0.00	4.6	4.5
		7.86	0.080	0.5	0.11	8.5	8.3	0.06	6.4	6.3	0.05	6.0	5.9	0.09	8.1	8.1	0.05	6.2	6.1	0.04	5.8	5.8
		7.86	0.080	5.0	0.35	44.1	42.9	0.20	27.0	25.8	0.16	20.4	19.8	0.29	44.4	43.0	0.17	27.3	26.0	0.13	20.9	20.0
	0.8	8.42	0.070	0.0	0.00	5.1	4.8	0.00	4.6	4.8	0.00	4.8	4.8	0.00	4.3	4.3	0.00	4.8	4.8	0.00	4.9	4.7
		8.42	0.070	0.5	0.11	7.5	7.4	0.06	5.8	5.7	0.05	5.7	5.6	0.09	7.0	6.8	0.05	5.7	5.7	0.04	5.7	5.5
		8.42	0.070	5.0	0.35	32.2	30.5	0.20	19.4	18.3	0.16	14.7	14.3	0.29	32.1	30.4	0.17	18.7	18.3	0.13	15.0	14.4
	0.9	8.94	0.062	0.0	0.00	5.0	4.9	0.00	4.9	5.1	0.00	5.0	4.7	0.00	4.4	4.4	0.00	4.9	4.9	0.00	4.9	4.7
		8.94	0.062	0.5	0.11	6.3	6.1	0.06	5.4	5.4	0.05	5.4	5.1	0.09	5.7	5.6	0.05	5.5	5.4	0.04	5.4	5.2
		8.94	0.062	5.0	0.35	18.9	18.1	0.20	11.9	11.3	0.16	9.7	9.4	0.29	19.0	18.0	0.17	11.6	11.2	0.13	9.8	9.3
	1.0	9.42	0.056	0.0	0.00	5.3	5.0	0.00	4.9	5.1	0.00	5.1	4.9	0.00	5.1	5.0	0.00	5.2	5.0	0.00	4.8	4.7
		9.42	0.056	0.5	0.11	5.3	5.1	0.06	5.0	4.9	0.05	4.9	4.9	0.09	5.2	5.1	0.05	5.2	5.2	0.04	4.7	4.8
		9.42	0.056	5.0	0.35	5.5	5.1	0.20	5.1	4.9	0.16	5.2	4.9	0.29	5.1	5.1	0.17	5.1	5.0	0.13	4.9	4.7
0.10	0.2	2.18	1.042	0.0	0.00	4.5	100.0	0.00	4.4	95.7	0.00	4.5	75.2	0.00	4.6	100.0	0.00	4.4	100.0	0.00	4.3	99.7
		2.18	1.042	0.5	0.11	9.6	100.0	0.06	6.8	96.2	0.05	6.4	78.1	0.09	9.4	100.0	0.05	7.0	100.0	0.04	5.9	99.7
		2.18	1.042	5.0	0.35	54.1	100.0	0.20	34.3	99.0	0.16	26.0	92.2	0.29	54.5	100.0	0.17	34.4	100.0	0.13	26.9	99.9
	0.3	3.07	0.524	0.0	0.00	4.7	99.9	0.00	4.4	88.9	0.00	4.6	63.7	0.00	4.6	100.0	0.00	4.4	100.0	0.00	4.4	98.1
		3.07	0.524	0.5	0.11	10.4	99.9	0.06	7.2	90.5	0.05	6.9	68.1	0.09	10.3	100.0	0.05	7.3	100.0	0.04	6.3	98.4
		3.07	0.524	5.0	0.35	61.7	100.0	0.20	40.5	97.6	0.16	31.2	89.6	0.29	61.7	100.0	0.17	40.6	100.0	0.13	31.6	99.6
	0.4	3.74	0.352	0.0	0.00	4.8	99.3	0.00	4.5	77.8	0.00	4.6	52.0	0.00	4.3	100.0	0.00	4.4	99.4	0.00	4.4	92.9
		3.74	0.352	0.5	0.11	10.3	99.5	0.06	7.1	80.4	0.05	6.7	56.8	0.09	10.2	100.0	0.05	7.1	99.5	0.04	6.3	93.8
		3.74	0.352	5.0	0.35	60.5	99.9	0.20	39.1	94.6	0.16	30.1	83.5	0.29	60.3	100.0	0.17	39.7	99.8	0.13	30.8	98.4
	0.5	4.30	0.267	0.0	0.00	4.9	96.9	0.00	4.5	62.5	0.00	4.6	41.4	0.00	4.4	100.0	0.00	4.5	96.3	0.00	4.4	81.8
		4.30	0.267	0.5	0.11	9.6	97.1	0.06	7.0	66.4	0.05	6.4	45.9	0.09	9.5	100.0	0.05	6.8	96.7	0.04	6.1	83.8
		4.30	0.267	5.0	0.35	55.4	99.3	0.20	35.2	88.3	0.16	26.6	74.1	0.29	55.6	100.0	0.17	35.7	99.0	0.13	27.6	94.6
	0.6	4.79	0.216	0.0	0.00	4.9	89.1	0.00	4.5	46.3	0.00	4.5	32.2	0.00	4.5	99.4	0.00	4.6	86.6	0.00	4.5	65.2
		4.79	0.216	0.5	0.11	8.9	89.9	0.06	6.7	51.9	0.05	6.1	35.8	0.09	8.5	99.5	0.05	6.3	87.9	0.04	5.9	68.7
		4.79	0.216	5.0	0.35	48.2	96.6	0.20	29.7	77.4	0.16	22.4	61.8	0.29	48.4	99.9	0.17	30.2	95.5	0.13	22.9	85.6
	0.7	5.21	0.181	0.0	0.00	4.9	72.3	0.00	4.6	34.0	0.00	4.6	24.2	0.00	4.4	95.5	0.00	4.7	68.5	0.00	4.4	47.4
		5.21	0.181	0.5	0.11	8.1	75.0	0.06	6.2	37.8	0.05	5.8	26.9	0.09	7.5	95.8	0.05	6.0	70.9	0.04	5.7	50.6
		5.21	0.181	5.0	0.35	39.1	88.6	0.20	23.3	61.4	0.16	17.8	47.8	0.29	39.8	98.3	0.17	23.5	85.0	0.13	18.1	70.9
	0.8	5.60	0.157	0.0	0.00	4.9	51.1	0.00	4.6	24.0	0.00	4.6	18.0	0.00	4.4	80.8	0.00	4.7	46.9	0.00	4.6	32.4
		5.60	0.157	0.5	0.11	7.2	53.5	0.06	5.6	26.0	0.05	5.5	19.5	0.09	6.7	82.1	0.05	5.6	49.1	0.04	5.4	34.5
		5.60	0.157	5.0	0.35	28.6	71.0	0.20	17.0	43.5	0.16	13.1	32.8	0.29	28.5	89.8	0.17	16.8	65.3	0.13	13.4	50.2
	0.9	5.96	0.139	0.0	0.00	4.9	30.3	0.00	4.9	15.9	0.00	5.0	13.2	0.00	4.4	54.2	0.00	4.7	28.2	0.00	4.7	21.1
		5.96	0.139	0.5	0.11	6.1	31.8	0.06	5.5	17.3	0.05	5.3	13.9	0.09	5.8	55.3	0.05	5.1	29.2	0.04	5.2	21.7
		5.96	0.139	5.0	0.35	17.4	45.2	0.20	10.9	26.0	0.16	9.0	20.2	0.29	16.9	65.5	0.17	10.5	39.0	0.13	8.7	29.0
	1.0	6.28	0.125	0.0	0.00	5.0	17.3	0.00	4.9	11.0	0.00	4.7	9.6	0.00	4.9	29.8	0.00	5.1	16.6	0.00	4.6	13.5
		6.28	0.125	0.5	0.11	5.1	17.2	0.06	5.0	11.1	0.05	4.7	9.8	0.09	4.9	30.0	0.05	5.1	16.7	0.04	4.6	13.4
		6.28	0.125	5.0	0.35	5.0	17.1	0.20	4.9	11.1	0.16	4.8	9.6	0.29	4.9	29.4	0.17	5.0	16.6	0.13	4.5	13.6

Table A10. Size comparison of joint tests ($H_0 : \phi = \phi_0, \rho = \rho_0$) under $\mathcal{M}_2$ with weak instruments, nominal level: 5%

					$T = 200$									$T = 300$								
					$l = 1$			$l = 3$			$l = 5$			$l = 1$			$l = 3$			$l = 5$		
ρ	ϕ	σ_v	λ	CP	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*	$\bar{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	4.8	4.7	0.00	5.0	4.8	0.00	5.2	5.0	0.00	4.8	4.9	0.00	5.0	4.9	0.00	5.1	4.9
		2.21	1.010	0.5	0.11	4.7	4.8	0.06	4.8	4.8	0.05	5.2	5.0	0.09	4.8	4.7	0.05	4.9	5.1	0.04	5.2	4.9
		2.21	1.010	5.0	0.35	5.0	4.9	0.20	4.9	5.1	0.16	5.2	5.2	0.29	5.0	5.1	0.17	5.2	5.2	0.13	5.2	5.1
	0.2	3.82	0.338	0.0	0.00	4.8	5.0	0.00	5.0	4.8	0.00	5.2	5.0	0.00	4.8	4.6	0.00	4.9	4.9	0.00	5.3	5.1
		3.82	0.338	0.5	0.11	4.8	4.9	0.06	5.0	4.9	0.05	5.1	4.9	0.09	4.8	4.6	0.05	4.9	4.9	0.04	5.2	5.2
		3.82	0.338	5.0	0.35	5.0	5.0	0.20	5.1	4.9	0.16	5.1	5.0	0.29	4.9	4.9	0.17	5.1	5.0	0.13	5.5	5.4
	0.3	4.92	0.204	0.0	0.00	4.8	5.1	0.00	5.0	5.0	0.00	5.3	5.1	0.00	4.7	4.7	0.00	4.9	4.8	0.00	5.2	5.1
		4.92	0.204	0.5	0.11	5.0	5.0	0.06	5.0	4.9	0.05	5.2	5.0	0.09	4.7	4.7	0.05	4.9	4.9	0.04	5.3	5.2
		4.92	0.204	5.0	0.35	5.0	5.1	0.20	5.1	5.1	0.16	5.1	5.2	0.29	4.9	4.9	0.17	5.1	4.9	0.13	5.5	5.4
	0.4	5.81	0.146	0.0	0.00	5.0	5.2	0.00	5.2	5.1	0.00	5.3	5.1	0.00	4.7	4.6	0.00	5.0	4.9	0.00	5.1	5.1
		5.81	0.146	0.5	0.11	4.9	5.1	0.06	5.1	4.9	0.05	5.2	5.2	0.09	4.8	4.6	0.05	5.0	5.0	0.04	5.2	5.1
		5.81	0.146	5.0	0.35	4.9	5.1	0.20	5.1	5.0	0.16	5.1	5.3	0.29	4.8	4.9	0.17	5.0	4.9	0.13	5.3	5.3
	0.5	6.57	0.114	0.0	0.00	5.0	5.3	0.00	5.3	5.1	0.00	5.3	5.1	0.00	4.7	4.6	0.00	5.0	5.0	0.00	5.2	5.0
		6.57	0.114	0.5	0.11	5.0	5.2	0.06	5.2	5.0	0.05	5.3	5.1	0.09	4.9	4.6	0.05	5.0	5.0	0.04	5.1	5.1
		6.57	0.114	5.0	0.35	4.9	4.9	0.20	5.0	4.9	0.16	5.1	5.1	0.29	4.9	4.8	0.17	5.0	4.9	0.13	5.3	5.1
	0.6	7.25	0.094	0.0	0.00	5.0	5.3	0.00	5.3	5.1	0.00	5.4	5.1	0.00	4.8	4.7	0.00	5.1	5.0	0.00	5.2	5.0
		7.25	0.094	0.5	0.11	5.0	5.1	0.06	5.2	5.1	0.05	5.3	5.1	0.09	4.8	4.6	0.05	5.1	5.0	0.04	5.1	5.0
		7.25	0.094	5.0	0.35	4.8	4.7	0.20	5.0	4.8	0.16	5.1	5.1	0.29	4.8	4.6	0.17	5.1	4.9	0.13	5.4	5.2
	0.7	7.86	0.080	0.0	0.00	5.0	5.4	0.00	5.3	5.0	0.00	5.3	5.1	0.00	4.9	4.7	0.00	5.1	5.0	0.00	5.2	5.0
		7.86	0.080	0.5	0.11	5.0	5.2	0.06	5.2	5.0	0.05	5.3	5.1	0.09	4.8	4.6	0.05	5.1	5.0	0.04	5.2	5.0
		7.86	0.080	5.0	0.35	4.7	4.6	0.20	5.0	4.9	0.16	5.2	5.0	0.29	4.7	4.5	0.17	5.1	4.9	0.13	5.3	5.2
	0.8	8.42	0.070	0.0	0.00	5.1	5.3	0.00	5.4	5.1	0.00	5.3	5.0	0.00	4.9	4.7	0.00	5.1	5.0	0.00	5.1	5.0
		8.42	0.070	0.5	0.11	5.1	5.3	0.06	5.2	5.0	0.05	5.3	5.1	0.09	4.8	4.6	0.05	5.1	4.9	0.04	5.2	4.9
		8.42	0.070	5.0	0.35	4.6	4.7	0.20	5.0	4.8	0.16	5.1	5.0	0.29	4.7	4.3	0.17	5.0	4.8	0.13	5.2	5.0
	0.9	8.94	0.062	0.0	0.00	5.1	5.3	0.00	5.4	5.1	0.00	5.4	5.1	0.00	4.9	4.7	0.00	5.1	5.0	0.00	5.1	5.0
		8.94	0.062	0.5	0.11	5.2	5.2	0.06	5.3	5.1	0.05	5.4	5.1	0.09	4.9	4.7	0.05	5.1	5.0	0.04	5.0	5.0
		8.94	0.062	5.0	0.35	4.9	4.8	0.20	5.0	4.9	0.16	5.1	5.0	0.29	4.6	4.4	0.17	4.9	4.9	0.13	5.0	4.9
	1.0	9.42	0.056	0.0	0.00	5.1	5.2	0.00	5.2	5.0	0.00	5.3	5.0	0.00	5.3	5.3	0.00	5.2	5.1	0.00	4.9	4.8
		9.42	0.056	0.5	0.11	5.2	5.1	0.06	5.2	5.1	0.05	5.3	5.0	0.09	5.2	5.2	0.05	5.1	5.0	0.04	4.9	4.7
		9.42	0.056	5.0	0.35	5.5	5.4	0.20	5.4	5.2	0.16	5.3	5.0	0.29	5.2	5.1	0.17	5.2	5.0	0.13	4.9	4.8
0.10	0.2	2.18	1.042	0.0	0.00	4.7	4.6	0.00	4.9	4.6	0.00	5.1	4.8	0.00	4.8	4.6	0.00	5.0	4.8	0.00	5.2	4.6
		2.18	1.042	0.5	0.11	4.8	4.7	0.06	4.8	4.6	0.05	5.2	4.9	0.09	5.0	4.6	0.05	5.0	4.9	0.04	5.2	4.7
		2.18	1.042	5.0	0.35	6.0	5.6	0.20	5.4	5.0	0.16	5.5	5.1	0.29	5.9	5.7	0.17	5.7	5.4	0.13	5.7	5.3
	0.3	3.07	0.524	0.0	0.00	4.6	4.7	0.00	4.9	4.6	0.00	5.3	4.8	0.00	4.7	4.4	0.00	5.1	4.9	0.00	5.2	4.6
		3.07	0.524	0.5	0.11	4.8	4.8	0.06	5.1	4.7	0.05	5.3	4.9	0.09	4.9	4.4	0.05	5.2	4.9	0.04	5.2	4.7
		3.07	0.524	5.0	0.35	6.2	5.7	0.20	5.4	5.0	0.16	5.5	5.3	0.29	6.1	5.8	0.17	5.8	5.6	0.13	5.9	5.2
	0.4	3.74	0.352	0.0	0.00	4.7	4.8	0.00	5.1	4.7	0.00	5.2	4.8	0.00	4.8	4.4	0.00	5.0	4.7	0.00	5.2	4.7
		3.74	0.352	0.5	0.11	4.9	4.8	0.06	5.1	4.6	0.05	5.2	4.8	0.09	4.8	4.4	0.05	5.1	4.9	0.04	5.4	4.8
		3.74	0.352	5.0	0.35	6.1	5.6	0.20	5.5	5.0	0.16	5.4	5.2	0.29	6.0	5.6	0.17	5.8	5.5	0.13	5.9	5.3
	0.5	4.30	0.267	0.0	0.00	4.9	4.8	0.00	5.2	4.8	0.00	5.3	4.9	0.00	4.7	4.4	0.00	4.8	4.7	0.00	5.2	4.8
		4.30	0.267	0.5	0.11	5.0	4.8	0.06	5.1	4.7	0.05	5.2	4.9	0.09	4.7	4.4	0.05	5.0	4.8	0.04	5.2	4.9
		4.30	0.267	5.0	0.35	6.1	5.6	0.20	5.5	4.9	0.16	5.4	5.2	0.29	5.9	5.4	0.17	5.7	5.4	0.13	5.9	5.3
	0.6	4.79	0.216	0.0	0.00	4.9	4.9	0.00	5.3	4.8	0.00	5.4	5.0	0.00	4.6	4.5	0.00	5.0	4.8	0.00	5.1	4.8
		4.79	0.216	0.5	0.11	5.0	4.8	0.06	5.1	4.7	0.05	5.3	4.9	0.09	4.7	4.4	0.05	5.0	4.9	0.04	5.2	4.8
		4.79	0.216	5.0	0.35	5.9	5.3	0.20	5.3	4.8	0.16	5.4	5.1	0.29	5.7	5.3	0.17	5.4	5.2	0.13	5.7	5.1
	0.7	5.21	0.181	0.0	0.00	5.0	5.0	0.00	5.3	4.9	0.00	5.4	4.8	0.00	4.8	4.6	0.00	5.0	4.9	0.00	5.2	4.7
		5.21	0.181	0.5	0.11	5.0	4.9	0.06	5.1	4.8	0.05	5.4	4.8	0.09	4.7	4.4	0.05	5.1	5.0	0.04	5.2	4.9
		5.21	0.181	5.0	0.35	5.6	5.1	0.20	5.2	4.7	0.16	5.3	4.9	0.29	5.4	4.9	0.17	5.4	4.9	0.13	5.5	5.0
	0.8	5.60	0.157	0.0	0.00	4.9	5.1	0.00	5.3	4.9	0.00	5.5	4.8	0.00	4.8	4.6	0.00	5.1	4.8	0.00	5.1	4.7
		5.60	0.157	0.5	0.11	5.1	4.9	0.06	5.2	4.7	0.05	5.4	4.8	0.09	4.6	4.4	0.05	5.1	4.9	0.04	5.2	4.8
		5.60	0.157	5.0	0.35	5.1	4.8	0.20	5.2	4.7	0.16	5.2	4.9	0.29	5.0	4.6	0.17	5.3	4.8	0.13	5.4	4.8
	0.9	5.96	0.139	0.0	0.00	5.0	5.1	0.00	5.3	4.9	0.00	5.5	4.8	0.00	4.9	4.6	0.00	5.0	4.8	0.00	5.1	4.7
		5.96	0.139	0.5	0.11	5.1	5.0	0.06	5.2	4.8	0.05	5.4	4.8	0.09	4.7	4.4	0.05	5.1	4.8	0.04	5.0	4.6
		5.96	0.139	5.0	0.35	4.7	4.3	0.20	5.0	4.6	0.16	5.2	4.7	0.29	4.6	4.2	0.17	5.0	4.7	0.13	5.0	4.5
	1.0	6.28	0.125	0.0	0.00	5.1	5.1	0.00	5.2	4.8	0.00	5.4	4.9	0.00	5.2	4.9	0.00	5.2	5.0	0.00	5.0	4.7
		6.28	0.125	0.5	0.11	5.2	5.0	0.06	5.2	4.7	0.05	5.4	4.8	0.09	5.2	4.8	0.05	5.1	5.0	0.04	5.0	4.5
		6.28	0.125	5.0	0.35	5.2	4.7	0.20	5.1	4.9	0.16	5.3	4.8	0.29	5.1	4.9	0.17	5.1	4.8	0.13	4.9	4.5

Table A11. Power comparison of joint tests ($H_0 : \phi = 1, \rho = 0.05$) under $\mathcal{M}_2$ with weak instruments, nominal level: 5%

					$T = 200$									$T = 300$								
					$l = 1$			$l = 3$			$l = 5$			$l = 1$			$l = 3$			$l = 5$		
ρ	ϕ	σ_v	λ	CP	$\tilde{\pi}_1$	AR	AR^*	$\tilde{\pi}_1$	AR	AR^*	$\tilde{\pi}_1$	AR	AR^*	$\tilde{\pi}_1$	AR	AR^*	$\tilde{\pi}_1$	AR	AR^*	$\tilde{\pi}_1$	AR	AR^*
0.05	0.1	2.21	1.010	0.0	0.00	4.4	4.5	0.00	4.4	4.2	0.00	4.1	4.0	0.00	4.3	4.2	0.00	4.4	4.1	0.00	4.4	4.2
		2.21	1.010	0.5	0.11	10.0	10.0	0.06	7.0	7.1	0.05	6.4	6.1	0.09	10.3	9.7	0.05	7.0	6.9	0.04	6.1	6.0
		2.21	1.010	5.0	0.35	62.2	60.0	0.20	39.2	38.2	0.16	30.6	29.7	0.29	62.3	60.6	0.17	41.4	40.2	0.13	33.0	31.8
	0.2	3.82	0.338	0.0	0.00	4.7	4.6	0.00	4.5	4.5	0.00	4.4	4.4	0.00	4.2	4.1	0.00	4.4	4.0	0.00	4.3	4.3
		3.82	0.338	0.5	0.11	12.2	11.9	0.06	8.0	8.1	0.05	7.3	7.0	0.09	12.4	12.0	0.05	7.9	7.7	0.04	6.9	6.5
		3.82	0.338	5.0	0.35	75.6	73.7	0.20	52.9	51.2	0.16	42.6	40.6	0.29	76.1	74.1	0.17	54.3	53.0	0.13	44.2	42.5
	0.3	4.92	0.204	0.0	0.00	4.7	4.6	0.00	4.5	4.6	0.00	4.6	4.5	0.00	4.2	4.1	0.00	4.5	4.2	0.00	4.4	4.4
		4.92	0.204	0.5	0.11	12.5	11.7	0.06	8.0	8.1	0.05	7.3	7.0	0.09	12.1	11.7	0.05	7.7	7.5	0.04	7.0	6.7
		4.92	0.204	5.0	0.35	74.6	72.8	0.20	52.0	50.4	0.16	41.6	40.1	0.29	75.5	73.4	0.17	53.5	51.9	0.13	43.4	42.0
	0.4	5.81	0.146	0.0	0.00	4.7	4.5	0.00	4.4	4.7	0.00	4.6	4.6	0.00	4.3	4.2	0.00	4.5	4.3	0.00	4.6	4.4
		5.81	0.146	0.5	0.11	11.3	11.0	0.06	7.5	7.7	0.05	7.0	6.8	0.09	11.3	11.2	0.05	7.5	7.2	0.04	6.7	6.7
		5.81	0.146	5.0	0.35	70.3	68.1	0.20	47.3	46.0	0.16	37.5	36.1	0.29	71.0	68.9	0.17	49.2	47.6	0.13	39.5	37.9
	0.5	6.57	0.114	0.0	0.00	4.7	4.6	0.00	4.3	4.6	0.00	4.6	4.6	0.00	4.3	4.2	0.00	4.7	4.4	0.00	4.6	4.5
		6.57	0.114	0.5	0.11	10.6	10.2	0.06	7.1	7.3	0.05	6.7	6.4	0.09	10.3	10.4	0.05	7.0	6.9	0.04	6.6	6.2
		6.57	0.114	5.0	0.35	63.6	61.4	0.20	41.6	40.1	0.16	32.1	31.0	0.29	63.9	61.8	0.17	43.0	41.7	0.13	34.1	32.9
	0.6	7.25	0.094	0.0	0.00	4.8	4.7	0.00	4.5	4.7	0.00	4.6	4.6	0.00	4.3	4.2	0.00	4.7	4.5	0.00	4.6	4.4
		7.25	0.094	0.5	0.11	9.7	9.3	0.06	6.8	6.9	0.05	6.4	6.2	0.09	9.6	9.3	0.05	6.6	6.5	0.04	6.3	6.1
		7.25	0.094	5.0	0.35	54.7	52.7	0.20	34.2	33.1	0.16	26.6	25.4	0.29	55.3	52.9	0.17	35.5	34.4	0.13	27.9	26.5
	0.7	7.86	0.080	0.0	0.00	4.9	5.0	0.00	4.7	4.9	0.00	4.7	4.7	0.00	4.1	4.3	0.00	4.6	4.6	0.00	4.8	4.6
		7.86	0.080	0.5	0.11	8.5	8.1	0.06	6.5	6.4	0.05	6.1	5.9	0.09	8.5	8.0	0.05	6.4	6.2	0.04	6.0	6.0
		7.86	0.080	5.0	0.35	44.4	42.5	0.20	26.8	25.7	0.16	20.4	19.7	0.29	43.8	42.8	0.17	27.3	26.5	0.13	21.7	20.9
	0.8	8.42	0.070	0.0	0.00	5.0	5.0	0.00	4.9	4.9	0.00	4.9	4.8	0.00	4.3	4.3	0.00	4.7	4.7	0.00	4.8	4.8
		8.42	0.070	0.5	0.11	7.2	6.9	0.06	6.1	6.0	0.05	5.6	5.6	0.09	7.3	7.0	0.05	5.9	5.7	0.04	5.8	5.6
		8.42	0.070	5.0	0.35	31.8	30.0	0.20	18.8	17.9	0.16	14.7	14.2	0.29	31.9	30.4	0.17	19.6	18.7	0.13	15.3	14.8
	0.9	8.94	0.062	0.0	0.00	5.0	5.2	0.00	5.0	5.1	0.00	5.2	5.0	0.00	4.6	4.5	0.00	4.8	4.8	0.00	4.8	4.8
		8.94	0.062	0.5	0.11	6.1	6.1	0.06	5.7	5.6	0.05	5.6	5.4	0.09	6.1	5.8	0.05	5.5	5.3	0.04	5.4	5.4
		8.94	0.062	5.0	0.35	19.1	18.3	0.20	11.4	11.2	0.16	9.6	9.2	0.29	18.7	17.9	0.17	11.9	11.3	0.13	9.7	9.4
	1.0	9.42	0.056	0.0	0.00	5.1	5.2	0.00	5.2	5.0	0.00	5.3	5.0	0.00	5.3	5.3	0.00	5.2	5.1	0.00	4.9	4.8
		9.42	0.056	0.5	0.11	5.2	5.1	0.06	5.2	5.1	0.05	5.3	5.0	0.09	5.2	5.2	0.05	5.1	5.0	0.04	4.9	4.7
		9.42	0.056	5.0	0.35	5.5	5.4	0.20	5.4	5.2	0.16	5.3	5.0	0.29	5.2	5.1	0.17	5.2	5.0	0.13	4.9	4.8
0.10	0.2	2.18	1.042	0.0	0.00	4.4	100.0	0.00	4.5	95.6	0.00	4.2	75.6	0.00	4.2	100.0	0.00	4.3	100.0	0.00	4.4	99.7
		2.18	1.042	0.5	0.11	8.9	100.0	0.06	6.5	95.9	0.05	6.1	78.1	0.09	9.4	100.0	0.05	6.7	100.0	0.04	5.9	99.7
		2.18	1.042	5.0	0.35	54.4	100.0	0.20	33.0	99.0	0.16	25.5	91.9	0.29	54.7	100.0	0.17	35.1	100.0	0.13	27.5	99.9
	0.3	3.07	0.524	0.0	0.00	4.6	99.9	0.00	4.5	88.5	0.00	4.3	64.1	0.00	4.3	100.0	0.00	4.4	100.0	0.00	4.5	98.1
		3.07	0.524	0.5	0.11	9.9	99.9	0.06	6.7	90.1	0.05	6.4	68.0	0.09	10.2	100.0	0.05	7.0	100.0	0.04	6.1	98.3
		3.07	0.524	5.0	0.35	62.0	100.0	0.20	39.3	97.8	0.16	30.4	88.8	0.29	62.2	100.0	0.17	41.2	100.0	0.13	32.6	99.5
	0.4	3.74	0.352	0.0	0.00	4.8	99.3	0.00	4.6	77.7	0.00	4.4	52.1	0.00	4.3	100.0	0.00	4.4	99.5	0.00	4.5	92.8
		3.74	0.352	0.5	0.11	9.8	99.5	0.06	6.8	80.1	0.05	6.4	57.0	0.09	10.0	100.0	0.05	7.0	99.6	0.04	6.1	93.8
		3.74	0.352	5.0	0.35	60.8	99.9	0.20	38.2	94.4	0.16	29.5	82.9	0.29	61.0	100.0	0.17	40.1	99.9	0.13	31.8	98.2
	0.5	4.30	0.267	0.0	0.00	4.8	96.6	0.00	4.5	62.4	0.00	4.6	40.9	0.00	4.4	100.0	0.00	4.4	96.4	0.00	4.6	82.0
		4.30	0.267	0.5	0.11	9.4	97.3	0.06	6.6	66.9	0.05	6.2	45.3	0.09	9.3	100.0	0.05	6.7	96.9	0.04	5.9	84.2
		4.30	0.267	5.0	0.35	55.7	99.3	0.20	34.4	88.1	0.16	26.3	73.7	0.29	55.7	100.0	0.17	35.9	99.1	0.13	28.8	94.5
	0.6	4.79	0.216	0.0	0.00	4.8	88.7	0.00	4.5	47.3	0.00	4.6	31.6	0.00	4.4	99.5	0.00	4.6	86.8	0.00	4.7	65.6
		4.79	0.216	0.5	0.11	8.7	90.1	0.06	6.5	51.8	0.05	6.1	35.3	0.09	8.5	99.6	0.05	6.4	87.9	0.04	5.8	68.9
		4.79	0.216	5.0	0.35	48.4	96.8	0.20	29.2	76.9	0.16	22.2	61.8	0.29	48.6	99.9	0.17	30.3	95.7	0.13	24.0	85.8
	0.7	5.21	0.181	0.0	0.00	4.7	72.1	0.00	4.6	34.0	0.00	4.7	23.6	0.00	4.4	95.6	0.00	4.7	69.1	0.00	4.8	48.6
		5.21	0.181	0.5	0.11	7.9	74.9	0.06	6.2	37.2	0.05	5.9	26.3	0.09	7.9	95.9	0.05	6.0	71.4	0.04	5.8	51.5
		5.21	0.181	5.0	0.35	39.2	88.9	0.20	23.2	61.5	0.16	17.7	47.4	0.29	39.2	98.4	0.17	24.0	84.8	0.13	18.8	70.3
	0.8	5.60	0.157	0.0	0.00	4.9	50.3	0.00	4.8	23.6	0.00	5.0	17.5	0.00	4.4	80.8	0.00	4.6	47.4	0.00	4.8	33.0
		5.60	0.157	0.5	0.11	6.9	53.3	0.06	5.8	25.6	0.05	5.6	19.0	0.09	6.9	82.1	0.05	5.7	49.6	0.04	5.5	35.2
		5.60	0.157	5.0	0.35	28.5	71.1	0.20	16.4	43.4	0.16	13.3	32.3	0.29	28.6	90.1	0.17	17.4	65.3	0.13	13.5	50.3
	0.9	5.96	0.139	0.0	0.00	5.0	30.2	0.00	4.9	15.9	0.00	5.1	12.8	0.00	4.5	54.0	0.00	4.9	28.8	0.00	4.8	21.4
		5.96	0.139	0.5	0.11	6.0	31.8	0.06	5.5	16.8	0.05	5.4	13.4	0.09	6.1	54.8	0.05	5.3	30.3	0.04	5.4	22.5
		5.96	0.139	5.0	0.35	16.9	45.1	0.20	10.5	25.1	0.16	8.9	19.5	0.29	17.2	65.3	0.17	11.1	39.6	0.13	9.1	30.2
	1.0	6.28	0.125	0.0	0.00	5.1	17.7	0.00	5.2	11.3	0.00	5.1	9.7	0.00	5.2	29.8	0.00	5.2	17.1	0.00	4.9	14.2
		6.28	0.125	0.5	0.11	5.0	17.5	0.06	5.0	11.4	0.05	5.1	9.7	0.09	5.0	29.9	0.05	5.2	17.1	0.04	4.8	14.0
		6.28	0.125	5.0	0.35	5.0	16.9	0.20	5.0	11.1	0.16	5.1	9.5	0.29	4.9	29.2	0.17	5.1	16.8	0.13	4.6	13.9

J Different classes of high-frequency instruments

J.1 Classes of realized measures not robust to jumps

These classes of realized measures have been proposed to provide robustness to various types of market microstructure effects (bid-ask bounce, stale quotes, misreported prices) and improve the efficiency of volatility estimates. We consider five broad classes of realized measures, all of which are consistent estimators of the quadratic variation (QV) in the absence of jumps. It is important to note that when jumps are absent, the QV corresponds to the integrated volatility (IVol).

1. **Realized volatility (RV)**: RV is defined as the sum of squared intraday returns. By dividing an interval of time, *e.g.*, $[T_0, T_1]$, into n subintervals, $T_0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T_1$, we can define intraday returns, $r_{i,n} = p_{t_{i,n}} - p_{t_{i-1,n}}$, and then $RV_t = \sum_{i=1}^{n} r_{i,n}^2$. Andersen et al. (2001) showed that the RV is a consistent estimator for the QV:
$$RV_t \xrightarrow{p} IVol_t = \int_0^t \sigma_s^2 ds. \tag{J.8}$$

2. **RV with optimal sampling (RVbr)**: A standard RV estimator with optimal sampling is proposed by Bandi and Russell (2008), where the optimal sampling frequency is calculated using estimates of integrated quarticity and variance of the microstructure noise. This bias-corrected estimator removes the estimated impact of market microstructure noise. In the empirical applications below, we compute RVbr with an estimated optimal sampling frequency, which is the key feature of this estimator.

3. **Multi-scales RV (MSRV)**: The multi-scales RV by Zhang (2006) uses a combination of several high and lower frequencies to remove the noise and estimate the volatility. It is a generalization of two-scales RV [Zhang et al. (2005)] and can be defined as:
$$MSRV_t = [r,r]_t^{(K)} - \frac{\bar{n}_K}{\bar{n}_J}[r,r]_t^{(J)} \xrightarrow{p} IVol_t, \quad 1 \le J < K \le n, \tag{J.9}$$
where J and K are the time scales and $\bar{n}_i = (n-i+1)/i$ with $i = J, K$.

4. **Realized kernels (RK)**: The realized kernel by Barndorff-Nielsen et al. (2008) is a robust measure of volatility, which ensures consistency and positive semi-definiteness. Several generalizations to handle more lags and various shapes of autocorrelation function are derived in Barndorff-Nielsen et al. (2011). In this paper, we use the latter variant, which is given by
$$RK = \sum_{h=-H}^{H} k\left(\frac{h}{H+1}\right)\gamma_h \tag{J.10}$$
where $k(x)$ is the kernel function and $\gamma_h = \sum_{i=|h|+1}^{n} r_{i,n} r_{i-h,n}$. We consider four types of kernel functions: (1) Bartlett kernel [RKbart: $k(x) = 1-x$, flat-top, $n^{1/6}$ rate]; (2) Cubic kernel [RKcub: $k(x) = 1-3x^2+2x^3$, flat-top, $n^{1/4}$ rate]; (3) Parzen kernel [RKnfp: $k(x) = \{1-6x^2+6x^3 \text{ if } 0 \le x \le 1/2\ , 2(1-x)^3 \text{ if } 1/2 \le x \le 1\}$, non-flat-top, $n^{1/5}$ rate]; (4) Tukey-Hanning kernel with power 2 [RKth2: $k(x) = \sin^2\{\pi/2(1-x)^2\}$, flat-top, $n^{1/4}$ rate].

5. **Realized range RV (RRV)**: The realized range RV [Christensen and Podolskij (2007)] uses sums of normalized squared high-low ranges for intra-daily periods rather than sums of squared returns. As a result, it is based on extremes from the entire price path and provides more information than returns sampled at fixed time intervals. Decomposing the daily time interval into K *non-overlapping*

intervals of size m_K, the estimator is given by:

$$RRV^{(m_K,K)} = \frac{1}{\lambda_{2,m_K}} \sum_{i=1}^{K} s_i^{(m_K)^2} \xrightarrow{p} IVol \tag{J.11}$$

where the range of the price process over the ith interval is given by $s_i^{(m_K)} = \max_{0\le h,l\le m_K} \left(p_{\frac{i-1+h/mK}{K}} - p_{\frac{i-1+l/mK}{K}}\right)$, $i = 1,\dots,K$, and $\lambda_{2,m_K} = E[\max_{0\le h,l\le m_K}(W_{h/m_K} - W_{l/m_K})^2]$ is the second moment of the range of a standard Brownian motion over the unit interval with m_K observed increments.

J.2 Classes of realized measures robust to jumps

In the presence of jumps, the RV is a consistent estimator of the QV [see Andersen and Bollerslev (1998), Andersen et al. (2001), Barndorff-Nielsen et al. (2002)], which is a combination of IVol and jump variation (JV):

$$RV_t \xrightarrow{p} \underbrace{\int_0^t \sigma_s^2 ds}_{IVol_t} + \underbrace{\sum_{0<s\le t} \kappa_s^2}_{JV_t}. \tag{J.12}$$

We consider two classes of jump-robust realized measures:

1. **Bipower variation (BV)**: The most widely used estimator of IVol in the presence of jumps is the Bipower variation of Barndorff-Nielsen and Shephard (2004). It is the sum of adjacent absolute returns:

$$BV_t := \frac{\pi}{2} \sum_{i=2}^{n} |r_{i-1,n}||r_{i,n}| \xrightarrow{p} IVol_t = \int_0^t \sigma_s^2 ds. \tag{J.13}$$

2. **Nearest neighbor truncated RV**: Andersen et al. (2012) used nearest neighbor truncation approach to estimate the integrated volatility, where the median RV (MedRV) and minimum RV (MinRV) estimators use min or median of blocks of returns (MinRV with blocks of two returns and MedRV with blocks of three returns). The proposed estimators are:

$$MinRV_n = \frac{\pi}{\pi-2}\left(\frac{n}{n-1}\right)\sum_{i=1}^{n-1}[\min(|r_{i,n}|,\ |r_{i+1,n}|)]^2, \tag{J.14}$$

$$MedRV_n = \frac{\pi}{6-4\sqrt{3}+\pi}\left(\frac{n}{n-2}\right)\sum_{i=2}^{n-2}[\text{med}(|r_{i-1,n}|,\ |r_{i,n}|,\ |r_{i+1,n}|)]^2. \tag{J.15}$$

J.3 Additional HF measures and jump variations

We also consider realized semivariance (RSV), JV, and signed JV (SJV) and squared logarithms of the latter (JV, SJV):

1. **Jump variation** Combining the results in equations (J.12) and (J.13), the contribution of the JV in the QV can be consistently estimated by

$$JV_t := RV_t - BV_t \xrightarrow{p} \sum_{0<s\le t} \kappa_s^2; \tag{J.16}$$

   see Barndorff-Nielsen and Shephard (2006).

2. **Realized semivariance** Barndorff-Nielsen et al. (2010) proposed RSV estimators which can capture the variation only due to negative or positive returns. These estimators are defined as:

$$RSV_t^+ := \sum_{j=1}^{n} r_{t_j}^2 \mathbf{1}_{\{r_{t_j}>0\}} \xrightarrow{p} \frac{1}{2}\int_0^t \sigma_s^2 ds + \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s>0\}}, \tag{J.17}$$

$$RSV_t^- := \sum_{j=1}^{n} r_{t_j}^2 \mathbf{1}_{\{r_{t_j}<0\}} \xrightarrow{p} \frac{1}{2}\int_0^t \sigma_s^2 ds + \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s<0\}}, \tag{J.18}$$

where the first term in the limit of both RSV^+ and RSV^- is one-half of the integrated variance. These estimators provide a complete decomposition of RV, in the sense that $RV = RSV^+ + RSV^-$.

3. **Signed jump variation** The variation due to the continuous component can be removed by subtracting one RSV from the other without estimating it separately. The remaining part is defined as the signed jump variation:

$$SJV_t := \lim_{n\to\infty}(RSV_t^+ - RSV_t^-) = \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s>0\}} - \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s<0\}}. \tag{J.19}$$

K Description of instruments

Table A12. Description of instruments

No	Classes of instruments		Subclasses
	HF realized measures not robust to jumps		
1-13	RV	Realized volatility	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
14-24	RVbr	Realized volatility with optimal sampling	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
25-35	MSRV	Multi-scales realized volatility	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
36-40	Rkcub	Realized Kernel with fat-top cubic kernel	1t, 5t, 10t, 20t, 50t
41-45	Rkbart	Realized Kernel with fat-top Bartlett kernel	1t, 5t, 10t, 20t, 50t
46-50	RKth2	Realized Kernel with fat-top Tukey-Hanning kernel (power 2)	1t, 5t, 10t, 20t, 50t
51-55	RKnfp	Realized Kernel with non-fat-top Parzen kernel	1t, 5t, 10t, 20t, 50t
56-58	RRV	Realized range volatility	1m, 5m, 10m
	HF realized measures robust to jumps		
59-71	BV	Bipower variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
72-77	MedRV	Nearest neighbor truncated median RV	1s, 5s, 30s, 1m, 5m, 10m
78-83	MinRV	Nearest neighbor truncated minimum RV	1s, 5s, 30s, 1m, 5m, 10m
	Aditional HF measures and jump variations		
84-96	RSVN	Realized semivariance due to negative returns	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
97-109	RSVP	Realized semivariance due to positive returns	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
110-120	JV	Jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
121-131	SJV	Signed jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
132-142	LJV	Log squared jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
143-153	LSJV	Log squared signed jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
154-156	PCF	HF principal component factor	1, 2, 3
	Other instruments		
157-162	ImV-C	Implied volatility (call option)	mean, min, max, q1, q2, q3
163-168	ImV-P	Implied volatility (put option)	mean, min, max, q1, q2, q3
169-174	ImV-A	Implied volatility (both call and put option)	mean, min, max, q1, q2, q3
175	1-day	Daily realized volatility	

Notes:

1. Sampling frequencies are tick, second and minute, *e.g.*, 1t stands for 1-tick, 1s stands for 1-second and 1m stands for 1-minute.
2. The use of 1-minute subsamples in the calculation of realized measures is denoted by ss.
3. Three principal component factors are extracted from HF instruments (1-109). PCF-1 stands for the largest factor.
4. Implied volatilities (ImV) are calculated from American options. We consider three classes: (1) only call options, (2) only put options, and (3) both call and put options. For each class, we use all implied volatilities at a given date to construct six ImV subclasses, which are mean, min, max, and three quantiles (q1, q2, q3).

Table A13. Strength comparison with daily past lags as instruments
(F-statistics from first-stage regression)

January 2009 - December 2013, $T = 1258$							
	# of instruments						
Ticker	1	2	3	4	5	6	7
GE	23.64	25.00	21.10	20.73	18.46	16.85	14.81
IBM	9.22	10.08	10.08	9.72	8.63	7.73	6.87
JPM	41.08	38.42	34.99	28.34	24.71	23.22	20.79
KO	6.19	10.24	8.82	9.00	8.31	7.08	7.00
PFE	14.99	11.17	7.53	7.43	7.41	7.45	7.06
PG	3.57	4.28	5.38	4.88	5.76	5.14	6.56
T	5.36	13.65	9.62	7.04	6.76	6.07	5.37
WMT	15.24	11.01	7.71	6.10	5.45	5.36	5.63
XOM	9.48	7.80	7.87	6.97	5.86	6.08	5.69
$CV\_Size(0.10)$	16.38	19.93	22.30	24.58	26.87	29.18	31.50

Notes:

1. The critical value (CV) is a function of one endogenous regressor, the number of instrumental variables, and the desired 10% maximal size of a 5% Wald test of $\phi = \phi_0$, for further details, see Table 5.2 of Stock and Yogo (2005).

2. Instruments are deemed weak if the first-stage F-statistic is less than the CV associated with the corresponding column.

L Strength of IVs using F-statistic

In this section, we examine the strength of IVs using F-statistic. We investigate the strength of daily IVs since a pressing concern with an IV approach is the possible use of weak IVs, which can produce biased estimators [bias towards OLS estimates] and hypothesis tests with large size distortion. The existing econometric literature defines weak IVs based on the strength of the first-stage equation [Bekker (1994), Dufour (2003), Staiger and Stock (1997), and Stock and Yogo (2005)]. Following Stock and Yogo (2005), we employ the first-stage F-statistic to detect whether IVs are weak or not.

F-statistics for testing whether daily IVs [past lags of the endogenous variable] all have zero coefficients are reported in Table A13 with corresponding critical values associated with the desired maximum level of size distortion. From the table, we can see that many F-statistics are less than the corresponding critical value associated with the maximum asymptotic size of a Wald test [these critical values are obtained using weak-IV asymptotic distributions]. These results suggest that IV estimates are biased towards OLS estimates, and we need to use weak instrument robust inference methods: see Dufour (1997) for more details about the Wald test.

Now, we wish to check if the HF and other IVs are weak or not. We consider IBM stock and different classes of IVs. Results with other stocks are qualitatively similar and omitted to conserve space. Table A14 reports the first-stage F-statistics of all IVs. From the results, we can draw several conclusions: (1) most of the HF IVs are strong for IBM, but exceptions are JV and SJV HF classes, ImV-mean subclass, and daily IVs; (2) if we consider multiple IVs, then Wald-type tests fail to control the level in many cases; (3) in most cases, the value of F-statistic (which measures the strength of IVs) is maximum, when we consider only one instrument irrespective of it is weak or strong.

Table A14. Strength comparison of all IVs
(F-statistics from first-stage regression)
Ticker: IBM, January 2009 - December 2013, $T = 1258$

No	Instruments	$l=1$	$l=3$	$l=5$	No	Instruments	$l=1$	$l=3$	$l=5$	No	Instruments	$l=1$	$l=3$	$l=5$	No	Instruments	$l=1$	$l=3$	$l=5$
1	RV-1s	70.4	29.2	17.5	45	RKbart-50t	139.0	46.3	27.7	89	RSVN-10m	79.3	31.7	19.3	133	LJV-5s	46.2	23.4	13.9
2	RV-5s	69.3	29.9	17.7	46	RKth2-1t	132.5	46.6	27.9	90	RSVN-1t	99.3	34.2	20.7	134	LJV-30s	9.4	8.3	6.0
3	RV-30s	95.7	34.1	20.5	47	RKth2-5t	139.7	46.3	27.9	91	RSVN-5t	103.6	34.9	21.4	135	LJV-1m	24.2	13.4	9.0
4	RV-1m	99.4	35.0	21.1	48	RKth2-10t	142.7	47.3	28.3	92	RSVN-10t	106.1	37.2	22.7	136	LJV-5m	16.2	12.5	8.3
5	RV-5m	96.8	34.5	21.6	49	RKth2-20t	143.5	47.7	28.6	93	RSVN-20t	122.0	42.4	26.3	137	LJV-10m	23.2	11.8	8.8
6	RV-10m	92.0	33.0	20.4	50	RKth2-50t	138.8	46.1	27.7	94	RSVN-50t	110.3	40.0	24.1	138	LJV-1t	92.5	32.1	19.2
7	RV-1t	99.9	34.5	20.8	51	RKnfp-1t	142.3	47.4	28.3	95	RSVN-5m-ss	93.6	35.2	21.2	139	LJV-5t	62.8	22.0	13.8
8	RV-5t	106.3	35.6	21.9	52	RKnfp-5t	139.7	47.0	28.0	96	RSVN-10m-ss	89.6	34.3	20.6	140	LJV-10t	71.8	27.5	17.7
9	RV-10t	110.7	38.5	23.5	53	RKnfp-10t	136.7	46.0	27.4	97	RSVP-1s	70.0	28.9	17.3	141	LJV-20t	44.1	19.6	14.4
10	RV-20t	128.3	43.6	27.3	54	RKnfp-20t	139.6	46.4	27.7	98	RSVP-5s	68.7	29.3	17.4	142	LJV-50t	90.1	30.6	19.9
11	RV-50t	117.5	40.9	24.7	55	RKnfp-50t	135.4	45.1	27.0	99	RSVP-30s	93.4	33.0	19.8	143	LSJV-1s	24.0	13.8	11.6
12	RV-5m-ss	104.8	36.4	22.0	56	RRV-1m	96.6	34.4	20.7	100	RSVP-1m	95.4	33.2	19.9	144	LSJV-5s	10.4	16.2	11.7
13	RV-10m-ss	101.4	35.4	21.3	57	RRV-5m	85.3	33.5	20.5	101	RSVP-5m	81.5	29.8	18.6	145	LSJV-30s	19.7	17.7	13.7
14	RVbr-1s	84.5	31.0	19.1	58	RRV-10m	80.5	32.6	20.1	102	RSVP-10m	69.1	26.4	16.6	146	LSJV-1m	16.5	13.7	9.4
15	RVbr-5s	81.0	29.8	18.5	59	BV-1s	80.2	30.2	18.3	103	RSVP-1t	99.7	34.6	20.8	147	LSJV-5m	22.6	14.3	10.1
16	RVbr-30s	71.5	27.4	17.5	60	BV-5s	71.6	29.2	17.8	104	RSVP-5t	106.1	35.6	22.0	148	LSJV-10m	13.4	12.3	9.7
17	RVbr-1m	76.5	29.5	18.4	61	BV-30s	97.6	34.5	20.8	105	RSVP-10t	109.6	38.2	23.4	149	LSJV-1t	40.1	17.0	11.8
18	RVbr-5m	87.7	35.1	21.9	62	BV-1m	100.5	35.1	21.1	106	RSVP-20t	125.3	42.5	26.9	150	LSJV-5t	35.4	14.6	10.1
19	RVbr-10m	61.8	27.7	17.3	63	BV-5m	95.5	34.5	21.2	107	RSVP-50t	111.9	38.6	23.5	151	LSJV-10t	38.1	20.6	13.6
20	RVbr-1t	99.4	36.4	21.7	64	BV-10m	87.6	31.3	19.3	108	RSVP-5m-ss	94.2	33.5	20.3	152	LSJV-20t	33.5	12.7	8.3
21	RVbr-5t	93.0	33.8	20.2	65	BV-1t	99.8	34.4	20.9	109	RSVP-10m-ss	82.5	30.6	18.5	153	LSJV-50t	37.3	16.5	10.9
22	RVbr-10t	93.8	34.2	21.5	66	BV-5t	106.8	35.8	22.1	110	JV-1s	0.6	0.5	0.8	154	PCF-1	102.7	35.3	21.4
23	RVbr-20t	95.0	34.1	20.6	67	BV-10t	105.3	36.9	22.4	111	JV-5s	0.7	0.5	0.7	155	PCF-2	98.5	34.1	20.6
24	RVbr-50t	92.3	33.2	20.7	68	BV-20t	129.0	43.8	27.4	112	JV-30s	0.0	2.3	2.2	156	PCF-3	67.4	24.8	15.8
25	MSRV-1s	99.6	34.6	21.2	69	BV-50t	120.6	42.1	25.6	113	JV-1m	2.9	5.7	4.2	157	ImV-C-mean	23.4	18.3	12.3
26	MSRV-5s	92.9	32.3	20.4	70	BV-5m-ss	95.5	34.5	21.2	114	JV-5m	9.1	9.0	7.2	158	ImV-C-min	84.8	29.3	17.4
27	MSRV-30s	94.1	34.2	21.7	71	BV-10m-ss	95.5	34.5	21.2	115	JV-10m	15.4	10.8	6.9	159	ImV-C-max	1.3	1.3	0.9
28	MSRV-1m	98.0	36.1	22.4	72	MedRV-1s	72.5	29.6	17.9	116	JV-1t	0.5	0.9	1.2	160	ImV-C-q1	87.5	29.2	17.8
29	MSRV-5m	83.2	33.2	20.9	73	MedRV-5s	62.9	28.4	16.9	117	JV-5t	0.6	1.2	1.3	161	ImV-C-q2	80.5	29.6	17.6
30	MSRV-10m	81.4	30.6	18.6	74	MedRV-30s	94.0	33.6	20.2	118	JV-10t	0.3	0.7	1.0	162	ImV-C-q3	25.1	18.1	12.1
31	MSRV-1t	123.9	43.2	25.9	75	MedRV-1m	97.6	34.3	20.8	119	JV-20t	0.1	3.6	2.5	163	ImV-P-mean	27.5	12.4	9.2
32	MSRV-5t	123.2	43.7	26.0	76	MedRV-5m	95.9	34.6	21.1	120	JV-50t	0.6	1.3	1.3	164	ImV-P-min	63.1	21.1	13.1
33	MSRV-10t	128.3	44.1	26.3	77	MedRV-10m	91.3	32.5	20.1	121	SJV-1s	0.9	1.3	0.8	165	ImV-P-max	0.2	1.0	1.0
34	MSRV-20t	126.0	42.8	26.3	78	MinRV-1s	74.3	29.1	17.8	122	SJV-5s	0.2	0.7	1.4	166	ImV-P-q1	72.4	27.4	16.6
35	MSRV-50t	142.3	47.3	28.9	79	MinRV-5s	62.1	26.8	16.3	123	SJV-30s	0.8	1.6	3.4	167	ImV-P-q2	71.4	25.4	15.4
36	RKcub-1t	102.8	40.2	24.4	80	MinRV-30s	93.9	33.6	20.2	124	SJV-1m	0.5	2.0	2.5	168	ImV-P-q3	44.0	15.9	10.6
37	RKcub-5t	127.7	42.7	25.6	81	MinRV-1m	97.2	34.1	20.6	125	SJV-5m	0.2	1.9	2.8	169	ImV-A-mean	35.1	17.9	12.0
38	RKcub-10t	145.2	48.2	28.9	82	MinRV-5m	92.1	34.2	20.9	126	SJV-10m	0.4	1.8	2.0	170	ImV-A-min	68.8	22.7	13.7
39	RKcub-20t	136.4	45.5	27.2	83	MinRV-10m	79.7	29.2	18.0	127	SJV-1t	0.7	11.6	7.1	171	ImV-A-max	1.1	1.6	1.1
40	RKcub-50t	134.3	44.8	26.8	84	RSVN-1s	70.5	29.5	17.6	128	SJV-5t	0.0	1.4	1.3	172	ImV-A-q1	83.8	31.3	19.1
41	RKbart-1t	133.9	45.2	27.0	85	RSVN-5s	69.3	30.3	18.0	129	SJV-10t	0.4	0.7	0.9	173	ImV-A-q2	82.3	28.0	17.0
42	RKbart-5t	139.9	46.3	27.9	86	RSVN-30s	92.9	34.2	20.5	130	SJV-20t	0.0	0.7	0.7	174	ImV-A-q3	51.7	21.8	13.5
43	RKbart-10t	141.9	47.0	28.2	87	RSVN-1m	95.0	35.2	21.2	131	SJV-50t	0.0	0.5	0.6	175	1-day	9.2	10.1	8.6
44	RKbart-20t	143.8	47.8	28.6	88	RSVN-5m	88.1	35.1	21.6	132	LJV-1s	56.7	26.6	15.9		$CV_{Size,0.10}$	16.4	22.3	26.9

Notes:

1. The critical value (CV) is a function of one endogenous regressor, the number of instrumental variables, and the desired 10% maximal size of a 5% Wald test of $\phi = \phi_0$, for further details, see Table 5.2 of Stock and Yogo (2005).
2. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12.
3. Instruments are deemed weak if the first-stage F-statistic is less than the CV associated with the corresponding column.

M Complementary empirical results

Table A15. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Strong instruments
Ticker: IBM, January 2009 - December 2013, $T = 1258$

No	Instruments	$\bar{d}_{i,s}$	AR	AR^*
Panel A				
1	RSVN-5m-ss	0.8860	[0.950, 1.000]	[0.866, 1.000]
2	RSVN-5m	0.8855	[0.948, 1.000]	[0.864, 1.000]
3	RSVN-1m	0.8848	[0.947, 1.000]	[0.856, 1.000]
4	ImV-C-mean	0.8830	[0.964, 1.000]	[0.852, 1.000]
5	MinRV-5m	0.8828	[0.945, 1.000]	[0.867, 1.000]
6	RV-5m-ss	0.8825	[0.946, 1.000]	[0.863, 1.000]
7	BV-5m	0.8823	[0.945, 1.000]	[0.865, 1.000]
8	BV-5m-ss	0.8823	[0.945, 1.000]	[0.865, 1.000]
9	BV-10m-ss	0.8823	[0.945, 1.000]	[0.865, 1.000]
10	MedRV-5m	0.8823	[0.945, 1.000]	[0.866, 1.000]
Panel B				
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*
11	ImV-C-q3	0.8805	[0.964, 1.000]	[0.843, 1.000]
12	RV-1m	0.8800	[0.944, 1.000]	[0.857, 1.000]
13	ImV-C-q2	0.8795	[0.958, 1.000]	[0.860, 1.000]
14	RRV-1m	0.8790	[0.945, 1.000]	[0.858, 1.000]
15	MedRV-1m	0.8785	[0.944, 1.000]	[0.857, 1.000]
16	RV-5m	0.8783	[0.943, 1.000]	[0.858, 1.000]
17	BV-1m	0.8775	[0.944, 1.000]	[0.857, 1.000]
18	RSVN-10m-ss	0.8775	[0.949, 1.000]	[0.858, 1.000]
19	RSVN-10m	0.8760	[0.946, 1.000]	[0.861, 1.000]
20	RV-10m-ss	0.8758	[0.944, 1.000]	[0.857, 1.000]
Panel C				
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*
21	RSVN-30s	0.8753	[0.944, 1.000]	[0.848, 1.000]
22	RRV-5m	0.8750	[0.946, 1.000]	[0.855, 1.000]
23	MinRV-1m	0.8745	[0.943, 1.000]	[0.855, 1.000]
24	ImV-C-min	0.8743	[0.952, 1.000]	[0.834, 1.000]
25	MSRV-1m	0.8723	[0.939, 1.000]	[0.862, 1.000]
26	RSVP-1m	0.8715	[0.942, 1.000]	[0.852, 1.000]
27	RV-30s	0.8713	[0.942, 1.000]	[0.847, 1.000]
28	BV-30s	0.8713	[0.943, 1.000]	[0.847, 1.000]
29	ImV-C-q1	0.8710	[0.952, 1.000]	[0.840, 1.000]
30	MSRV-30s	0.8698	[0.935, 1.000]	[0.858, 1.000]

Notes: The instrument set consists of a constant and a lag of an instrument, $l = 1$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The inference procedures $[AR, AR^*]$ are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). The confidence intervals are constructed by projection technique described in Section 3.3. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and SE$(\hat{\lambda}) = 6.935$. We use 99 Monte Carlo replications for point-optimal type procedures. The average precision of an instrument set i over the proposed inference methods is measured by $\bar{d}_{i,s} := S^{-1}\sum_{i=1}^{S} d_i$, where $s \in S$ and S is the set of identification-robust inference methods.

Table A16. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Weak instruments
Ticker: IBM, January 2009 - December 2013, $T = 1258$

No	Instruments	$\bar{d}_{i,s}$	AR	AR^*
Panel A				
1	JV-1s	0.0000	[0.000, 1.000]	[0.000, 1.000]
2	JV-5s	0.0000	[0.000, 1.000]	[0.000, 1.000]
3	JV-30s	0.0000	[0.000, 1.000]	[0.000, 1.000]
4	JV-1t	0.0000	[0.000, 1.000]	[0.000, 1.000]
5	SJV-1s	0.0000	[0.000, 1.000]	[0.000, 1.000]
6	SJV-5s	0.0000	[0.000, 1.000]	[0.000, 1.000]
7	SJV-10t	0.0000	[0.000, 1.000]	[0.000, 1.000]
8	SJV-20t	0.0000	[0.000, 1.000]	[0.000, 1.000]
9	SJV-50t	0.0000	[0.000, 1.000]	[0.000, 1.000]
10	ImV-C-max	0.0000	[0.000, 1.000]	[0.000, 1.000]
11	ImV-P-max	0.0000	[0.000, 1.000]	[0.000, 1.000]
12	ImV-A-max	0.0000	[0.000, 1.000]	[0.000, 1.000]
Panel B				
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*
13	JV-20t	0.0038	[0.000, 1.000]	[0.000, 1.000]
14	SJV-5t	0.1875	[0.500, 1.000]	[0.250, 1.000]
Panel C				
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*
15	JV-10t	0.3130	[0.000, 1.000]	[0.000, 1.000]
16	JV-50t	0.3283	[0.000, 1.000]	[0.000, 1.000]
17	JV-5t	0.3308	[0.000, 1.000]	[0.000, 1.000]
18	SJV-30s	0.3400	[0.000, 1.000]	[0.000, 1.000]
19	SJV-1m	0.3845	[0.000, 1.000]	[0.000, 1.000]
20	SJV-10m	0.3975	[0.000, 1.000]	[0.000, 1.000]
21	SJV-5m	0.3998	[0.000, 1.000]	[0.000, 1.000]
22	JV-1m	0.4028	[0.911, 1.000]	[0.700, 1.000]
23	JV-10m	0.4075	[0.890, 1.000]	[0.740, 1.000]
24	LSJV-20t	0.4108	[0.883, 0.992]	[0.750, 1.000]
25	LSJV-5t	0.4208	[0.913, 1.000]	[0.770, 1.000]
26	1-day	0.4255	[0.870, 0.965]	[0.750, 1.000]
27	LJV-30s	0.4268	[0.927, 1.000]	[0.780, 1.000]
28	LJV-1m	0.4373	[0.933, 1.000]	[0.816, 1.000]
29	SJV-1t	0.6795	[0.810, 0.992]	[0.700, 1.000]
30	LJV-10m	0.7465	[0.905, 0.998]	[0.750, 1.000]

Notes: The instrument set consists of a constant and a lag of an instrument, $l = 1$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The inference procedures $[AR, AR^*]$ are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). The confidence intervals are constructed by projection technique described in Section 3.3. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and $\mathrm{SE}(\hat{\lambda}) = 6.935$. We use 99 Monte Carlo replications for point-optimal type procedures. The average precision of an instrument set i over the proposed inference methods is measured by $\bar{d}_{i,s} := S^{-1}\sum_{i=1}^{S} d_i$, where $s \in S$ and S is the set of identification-robust inference methods.

Table A17. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Realized volatility vs. Bipower variation
Ticker: IBM, January 2009 - December 2013, $T = 1258$

Panel A: Realized volatility			
Instruments	$\bar{d}_{i,s}$	AR	AR^*
RV-1s	0.8590	[0.938, 1.000]	[0.836, 1.000]
RV-5s	0.8615	[0.938, 1.000]	[0.834, 1.000]
RV-30s	0.8713	[0.942, 1.000]	[0.847, 1.000]
RV-1m	0.8800	[0.944, 1.000]	[0.857, 1.000]
RV-5m	0.8783	[0.943, 1.000]	[0.858, 1.000]
RV-10m	0.8640	[0.939, 1.000]	[0.855, 1.000]
RV-1t	0.8415	[0.919, 0.997]	[0.831, 1.000]
RV-5t	0.8460	[0.922, 0.999]	[0.837, 1.000]
RV-10t	0.8495	[0.927, 1.000]	[0.850, 1.000]
RV-20t	0.8450	[0.928, 0.999]	[0.848, 1.000]
RV-50t	0.8498	[0.929, 1.000]	[0.844, 1.000]
RV-5m-ss	0.8825	[0.946, 1.000]	[0.863, 1.000]
RV-10m-ss	0.8758	[0.944, 1.000]	[0.857, 1.000]
Panel B: Bipower variation			
BV-1s	0.8670	[0.942, 1.000]	[0.858, 1.000]
BV-5s	0.8630	[0.939, 1.000]	[0.841, 1.000]
BV-30s	0.8713	[0.943, 1.000]	[0.847, 1.000]
BV-1m	0.8775	[0.944, 1.000]	[0.857, 1.000]
BV-5m	0.8823	[0.945, 1.000]	[0.865, 1.000]
BV-10m	0.8653	[0.941, 1.000]	[0.856, 1.000]
BV-1t	0.8460	[0.920, 0.996]	[0.832, 1.000]
BV-5t	0.8523	[0.924, 1.000]	[0.838, 1.000]
BV-10t	0.8520	[0.927, 1.000]	[0.850, 1.000]
BV-20t	0.8493	[0.929, 1.000]	[0.851, 1.000]
BV-50t	0.8560	[0.932, 1.000]	[0.848, 1.000]
BV-5m-ss	0.8823	[0.945, 1.000]	[0.865, 1.000]
BV-10m-ss	0.8823	[0.945, 1.000]	[0.865, 1.000]

Notes: The instrument set consists of a constant and a lag of an instrument, $l = 1$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The inference procedures $[AR, AR^*]$ are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). The confidence intervals are constructed by projection technique described in Section 3.3. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and $\text{SE}(\hat{\lambda}) = 6.935$. We use 99 Monte Carlo replications for point-optimal type procedures. The average precision of an instrument set i over the proposed inference methods is measured by $\bar{d}_{i,s} := S^{-1}\sum_{i=1}^{S} d_i$, where $s \in S$ and S is the set of identification-robust inference methods.

Table A18. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Strong instruments (Several lags)
Ticker: IBM, January 2009 - December 2013, $T = 1258$

	$l = 1$			$l = 3$			$l = 5$		
Instruments	$\bar{d}_{i,s}$	AR	AR^*	$\bar{d}_{i,s}$	AR	AR^*	$\bar{d}_{i,s}$	AR	AR^*
RSVN-5m-ss	0.8860	[0.950, 1.0]	[0.866, 1.0]	0.8618	[0.949, 1.0]	[0.838, 1.0]	0.8603	[0.941, 1.0]	[0.845, 1.0]
RSVN-5m	0.8855	[0.948, 1.0]	[0.864, 1.0]	0.8668	[0.955, 1.0]	[0.845, 1.0]	0.8695	[0.960, 1.0]	[0.856, 1.0]
RSVN-1m	0.8848	[0.947, 1.0]	[0.856, 1.0]	0.8570	[0.951, 1.0]	[0.851, 1.0]	0.8555	[0.946, 1.0]	[0.845, 1.0]
ImV-C-mean	0.8830	[0.964, 1.0]	[0.852, 1.0]	0.8218	[0.970, 1.0]	[0.828, 1.0]	0.8078	[0.950, 1.0]	[0.813, 1.0]
MinRV-5m	0.8828	[0.945, 1.0]	[0.867, 1.0]	0.8493	[0.935, 1.0]	[0.837, 1.0]	0.8465	[0.928, 1.0]	[0.825, 1.0]
RV-5m-ss	0.8825	[0.946, 1.0]	[0.863, 1.0]	0.8560	[0.939, 1.0]	[0.837, 1.0]	0.8523	[0.932, 1.0]	[0.840, 1.0]
BV-5m	0.8823	[0.945, 1.0]	[0.865, 1.0]	0.8508	[0.935, 1.0]	[0.835, 1.0]	0.8500	[0.932, 1.0]	[0.827, 1.0]
BV-5m-ss	0.8823	[0.945, 1.0]	[0.865, 1.0]	0.8508	[0.935, 1.0]	[0.835, 1.0]	0.8500	[0.932, 1.0]	[0.827, 1.0]
BV-10m-ss	0.8823	[0.945, 1.0]	[0.865, 1.0]	0.8508	[0.935, 1.0]	[0.835, 1.0]	0.8500	[0.932, 1.0]	[0.827, 1.0]
MedRV-5m	0.8823	[0.945, 1.0]	[0.866, 1.0]	0.8493	[0.936, 1.0]	[0.833, 1.0]	0.8515	[0.930, 1.0]	[0.826, 1.0]
1-day	0.4255	[0.870, 0.965]	[0.750, 1.0]	0.7490	[0.855, 0.974]	[0.760, 1.0]	0.7485	[0.838, 0.979]	[0.770, 1.0]

Notes: The instrument set consists of a constant and different lags of an instrument: $l = 1, 3, 5$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The inference procedures $[AR,\ AR^*]$ are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). The confidence intervals are constructed by projection technique described in Section 3.3. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and $SE(\hat{\lambda}) = 6.935$. We use 99 Monte Carlo replications for point-optimal type procedures. The average precision of an instrument set i over the proposed inference methods is measured by $\bar{d}_{i,s} := S^{-1} \sum_{i=1}^{S} d_i$, where $s \in S$ and S is the set of identification-robust inference methods.

Table A19. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Different combinations of strong instruments
Ticker: IBM, January 2009 - December 2013, $T = 1258$

Instrument set	$\bar{d}_{i,s}$	# of Instruments	AR	AR^*
RV-5m-ss, ImV-C-q3	0.8748	2	[0.954, 1.000]	[0.848, 1.000]
BV-5m-ss, ImV-C-q3	0.8775	2	[0.954, 1.000]	[0.850, 1.000]
RSVN-5m, ImV-C-q3	0.8820	2	[0.953, 1.000]	[0.857, 1.000]
RKcub-10t, ImV-C-q3	0.8583	2	[0.958, 0.995]	[0.854, 1.000]
BV-5m-ss, LJV-5s	0.8650	2	[0.936, 1.000]	[0.841, 1.000]
RKcub-10t, PCF-1, ImV-C-q3	0.8555	3	[0.967, 0.991]	[0.842, 1.000]
BV-5m-ss, LJV-5s, ImV-C-q3	0.8645	3	[0.946, 1.000]	[0.838, 1.000]
BV-5m-ss, LJV-5s, PCF-1	0.8568	3	[0.960, 0.999]	[0.844, 1.000]
BV-5m-ss, LJV-5s, PCF-1, ImV-C-q3	0.8553	4	[0.966, 0.996]	[0.829, 1.000]
BV-5m-ss, LJV-5s, LSJV-10t, PCF-1, ImV-C-q3	0.8493	5	[0.959, 0.997]	[0.820, 1.000]

Notes: The instrument set consists of a constant and various combinations of strong instruments. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The inference procedures $[AR,\ AR^*]$ are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). The confidence intervals are constructed by projection technique described in Section 3.3. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and SE($\hat{\lambda}$) = 6.935. We use 99 Monte Carlo replications for point-optimal type procedures. The average precision of an instrument set i over the proposed inference methods is measured by $\bar{d}_{i,s} := S^{-1}\sum_{i=1}^{S} d_i$, where $s \in S$ and S is the set of identification-robust inference methods.

Table A20. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Jump variation vs. log squared jump variation
Ticker: IBM, January 2009 - December 2013, $T = 1258$

Panel A: Jump variation			
Instruments	$\bar{d}_{i,s}$	AR	AR^*
JV-1s	0.0000	[0.000, 1.000]	[0.000, 1.000]
JV-5s	0.0000	[0.000, 1.000]	[0.000, 1.000]
JV-30s	0.0000	[0.000, 1.000]	[0.000, 1.000]
JV-1m	0.4028	[0.911, 1.000]	[0.700, 1.000]
JV-5m	0.7823	[0.925, 1.000]	[0.770, 1.000]
JV-10m	0.4075	[0.890, 1.000]	[0.740, 1.000]
JV-1t	0.0000	[0.000, 1.000]	[0.000, 1.000]
JV-5t	0.3308	[0.000, 1.000]	[0.000, 1.000]
JV-10t	0.3130	[0.000, 1.000]	[0.000, 1.000]
JV-20t	0.0038	[0.000, 1.000]	[0.000, 1.000]
JV-50t	0.3283	[0.000, 1.000]	[0.000, 1.000]
Panel B: Log squared jump variation			
LJV-1s	0.8573	[0.937, 1.000]	[0.830, 1.000]
LJV-5s	0.8635	[0.937, 1.000]	[0.815, 1.000]
LJV-30s	0.4268	[0.927, 1.000]	[0.780, 1.000]
LJV-1m	0.4373	[0.933, 1.000]	[0.816, 1.000]
LJV-5m	0.8175	[0.932, 1.000]	[0.800, 1.000]
LJV-10m	0.7465	[0.905, 0.998]	[0.750, 1.000]
LJV-1t	0.8275	[0.914, 0.997]	[0.827, 1.000]
LJV-5t	0.7940	[0.905, 0.996]	[0.806, 1.000]
LJV-10t	0.8193	[0.920, 1.000]	[0.838, 1.000]
LJV-20t	0.8080	[0.917, 0.997]	[0.811, 1.000]
LJV-50t	0.8393	[0.918, 0.999]	[0.833, 1.000]

Notes: The instrument set consists of a constant and a lag of an instrument, $l = 1$. We use logarithms of RV-RSVP and PCF classes of instruments given in Table A12. The inference procedures $[AR,\ AR^*]$ are proposed in Sections 3.1-3.2 and corresponding test statistics are given in equations (3.12) and (3.15). The confidence intervals are constructed by projection technique described in Section 3.3. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and SE($\hat{\lambda}$) = 6.935. We use 99 Monte Carlo replications for point-optimal type procedures. The average precision of an instrument set i over the proposed inference methods is measured by $\bar{d}_{i,s} := S^{-1}\sum_{i=1}^{S} d_i$, where $s \in S$ and S is the set of identification-robust inference methods.

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