RANK TESTS FOR SERIAL DEPENDENCE

BY JEAN-MARIE DUFOUR

Université de Montréal

Abstract. A family of linear rank statistics is proposed in order to test the independence of a time series, under the assumption that the random variables involved have symmetric distributions with zero medians, without the standard assumptions of normality or identical distributions. The family considered includes analogues of the sign, Wilcoxon signed-rank and van der Waerden tests for symmetry about zero and tables constructed for these tests remain applicable in the present context. The tests proposed are exact and may be applied to assess serial dependence at lag one or greater. The procedures developed are illustrated by a test of the efficiency of forward exhange rates as predictors of future spot rates during the German hyperinflation.

Keywords: Serial dependence; Rank tests; Nonparametric tests: Runs; Wilcoxon signed-rank test; Serial correlation.

1. INTRODUCTION

The purpose of this paper is to describe a family of linear rank tests aimed at testing the independence of a sequence of random variables under a simple symmetry assumption, without the standard assumptions of normality or identical distributions.

Namely, let $X_1, \ldots, X_n$ be a time series of random variables having symmetric (marginal) probability density functions (pdf's) with zero medians; it is not required that these pdf's be identical in other respects. We want to test the null hypothesis (H_0) that $X_1, \ldots, X_n$ are mutually independent against the alternative that these variables are 'positively (or negatively) serially dependent'.

However, there is a problem in defining precisely the latter notion in a nonparametric context, especially since we did not assume covariances are finite. By the symmetry assumption, we can see easily that, under H_0, med $(X_t X_{t+k}) = 0$, $t = 1, \ldots, n-k$, where k is a positive integer ($1 \leq k < n$) and med $(X_t X_{t+k})$ refers to the median of $X_t X_{t+k}$. Then, a simple way of defining 'positive serial dependence' (at lag k) consists in saying that the medians of the variables $X_t X_{t+k}$ are positive:

$$\text{med}\,(X_t X_{t+k}) > 0, \qquad t = 1, \ldots, n-k; \tag{1.1}$$

similarly, 'negative serial dependence' can be defined via negative medians:

$$\text{med}\,(X_t X_{t+k}) < 0, \qquad t = 1, \ldots, n-k. \tag{1.2}$$

Another concept of dependence one could also use is the concept of 'positive (or negative) quadrant dependence' introduced by Lehmann (1966), to which the reader is referred for further details. It will be sufficient here to indicate that,

0143-9782/81/03 0117-12 $02.50/0
JOURNAL OF TIME SERIES ANALYSIS Vol. 2, No. 3

under the symmetry assumption, if X_t and X_{t+k} are 'positively (negatively) quadrant dependent' and the events $X_t \leq 0$ and $X_{t+k} \leq 0$ are not independent, then med $(X_t X_{t+k}) > 0(<0)$; further, if second moments exist, we also have in this case $E(X_t X_{t+k}) > 0(<0)$, so that the alternatives (1.1) and (1.2) could be restated in terms of positive and negative autocorrelations. Under wide conditions thus, the tests described in this paper may be viewed as tests against serial correlation. Furthermore, it is easy to see that alternatives of the type (1.1) or (1.2) include as well a large variety of trend alternatives (monotonic, cyclic, etc.).

A natural statistic to look at for such a test is the sequence of the products $X_t X_{t+k}$, $t = 1, \ldots, n-k$. We propose to test H_0 (against serial dependence at lag k) by applying linear rank tests for symmetry about zero to this sequence. The tests obtained in this way are both easy to run and applicable against a wide set of alternative hypotheses, such as the ones where (1.1) or (1.2) holds. It is not required that X_t, $t = 1, \ldots, n$, have identical distributions; in particular, variances may differ under H_0, a feature which may be useful even with Gaussian series. The linear rank statistics used integrate information about both the signs of the observations and their sizes (via the ranks of the absolute values $|X_t X_{t+k}|$, $t = 1, \ldots, n-k$). The exact null distributions of the test statistics can be obtained easily and, in some standard cases, are already well tabulated. Finally by changing k, one has the opportunity of assessing dependence at various lags.

None of the available nonparametric tests against randomness seems applicable under the same conditions. In particular, the tests considered by Wallis and Moore (1941), Moore and Wallis (1943), David (1947), Stuart (1952), Cox and Stuart (1955), Goodman (1958) and Granger (1963) use only information about signs and depend on the assumption that the observations have identical distributions under the null hypothesis. Similarly, the rank tests (based on the ranks of the original observations X_t) studied by Wald and Wolfowitz (1943), Mann (1945), Daniels (1950), Stuart (1954), Knoke (1977) and Aiyar et al. (1979) depend crucially on the same assumption; furthermore, the tests of Mann, Daniels and Aiyar et al. are designed specifically against monotonic trend alternatives. It is also interesting to note that the widely used tests based on sample correlation coefficients are large sample tests (see Box and Jenkins, 1970, chapters 2 and 6), while the classical parametric tests of von Neumann (1941) and the modified von Neumann test (see Theil, 1971, pp. 218–219) deal with serial correlation at lag 1 only and assume homoskedasticity.

Potential applications of the tests described below are to be found especially in studies of the behaviour of speculative prices, such as stock prices or exchange rates, usually in view of testing market efficiency. One important problem, in this area, involves testing whether a given time series follows a simple random walk, i.e., testing whether a series of price (or log-price) changes $P_t - P_{t-1}$ are independent with zero mean (or median); for a general discussion, see Fama (1970). Evidence concerning the non-normality and symmetry of distributions of stock price or exchange rate changes has been presented by several authors, including Mandelbrot (1963, 1967), Fama (1965), Praetz (1972), Blattberg and Gonedes (1974), Giddy and Dufey (1975). Another related problem consists in testing whether a given predictor of a market variable is 'optimal', e.g., the forward exchange rate between two moneys as a predictor of the corresponding future spot rate: this usually involves testing whether the prediction errors $P_t - \hat{P}_{t-1}$, where $\hat{P}_{t-1}$ is a predictor of P_t, have mean zero and are uncorrelated; we give below an example of such a test. Furthermore, in many of these problems, the assumption of identical distributions is neither essential to the null hypothesis tested nor realistic; for example, with long historical series, heteroskedasticity is quite likely to be observed.

The family of tests considered is described in section 2 and the exact null distributions of the test statistics are obtained in section 3. In sections 4 and 5, special tests in the family are considered; in particular, we derive an exact run test for which critical points can be obtained from tables of the binomial distribution, and an analogue of the Wilcoxon signed-rank test for which tables built for this last test may be used. In section 6, we discuss some power comparisons with alternative tests and present the results of a small Monte-Carlo experiment showing that a number of well-known alternative tests of randomness may be completely unreliable under conditions of heteroskedasticity. In section 7, the procedures developed are illustrated by a test of the optimality of the forward exchange rate as a predictor of the future spot rate during the German hyperinflation. Finally, in section 8, a few concluding remarks are made.

2. DESCRIPTION OF THE TESTS

Let $Z_t = X_t X_{t+k}$, $t = 1, \ldots, n-k$, where $1 \leq k < n$. It is easy to see that, under H_0, each random variable Z_t, has a continuous distribution symmetric about zero; hence

$$P[Z_t \geq 0] = P[Z_t < 0] = \tfrac{1}{2}, \qquad t = 1, \ldots, n-k. \tag{2.1}$$

In a wide variety of cases, non-independence will break this pattern. In particular, under an alternative of positive (negative) serial dependence at lag k, the medians of the variables Z_t, $t = 1, \ldots, n-k$, will be shifted towards the right (left). This suggests to test whether these variables have median zero against such alternatives. In order to do this, we shall consider the family of simple linear rank tests for symmetry about zero, applied to the variables Z_t, $t = 1, 2, \ldots, n-k$. These are based on test statistics of the form:

$$S_k = \sum_{t=1}^{N} u(Z_t) a_N(R_t^+) \tag{2.2}$$

where $N = n - k$ and

$u(\cdot)$ is an indicator function such that

$$u(z) = 1, \quad \text{if } z \geq 0$$
$$= 0, \quad \text{if } z < 0;$$

$$R_t^+ = \sum_{i=1}^{N} u(|Z_t| - |Z_i|),$$

the rank of $|Z_t|$ when $|Z_1|, \ldots, |Z_N|$ are ranked in increasing order;

$$a_N(\cdot) \text{ is a score function transforming the ranks } R_t^+. \tag{2.3}$$

(Note that Z_t, R_t^+ and N are also functions of k; this could be made explicit by using instead Z_{tk}, R_{tk}^+ and N_k; but, to avoid heavy notations in the sequel, we will adopt the convention of letting k implicit).

Under the null hypothesis that $Z_1, \ldots, Z_N$ have pdf's symmetric about zero, the distribution of S_k is obtained in standard cases with the assumption that the observations $Z_1, \ldots, Z_N$ are independent (see Hájek, 1969, chapter V). The difficulty in the present case is that, even under H_0, the variables $Z_1, \ldots, Z_N$ are not independent (although they have marginal distributions symmetric about zero). We show, in this paper, that this difficulty is in fact of no consequence here, in the sense that standard null distributions of S_k can be used.

We shall furthermore assume that the score function in (2.2) is nonnegative:

$$a_N(r) \geq 0 \quad \text{for all } r. \tag{2.4}$$

Then, under the alternative (1.1) of positive dependence, we expect the number of positive Z_t's to be greater than under H_0, and hence S_k to take a relatively large value: against this one-sided alternative we use a critical region of the form $\{S_k \geq c\}$. Similarly, against the alternative (1.2) of negative dependence, we use a critical region of the form $\{S_k \geq c'\}$, while a two-sided test has a critical region of the form $\{S_k \geq c \text{ or } S_k \leq c'\}$.

Finally, let us note that, if ties are present among the $|Z_t|$'s, the formula used in (2.3) in order to define R_t^+ (given by Hájek, 1969, p. 103) has the property of assigning to the tied observations the highest rank associated with each tie. A somewhat more natural, although computationally less convenient, alternative approach would consist in using average ranks or average scores for the tied observations (see Hájek, 1969, chapter VII). In any case, under the continuity assumption adopted here, the method of treatment of ties can make a difference only in a set with probability zero.

3. NULL DISTRIBUTIONS OF THE TEST STATISTICS

First, let us introduce the following notations:

$$\boldsymbol{X} = (X_1, \ldots, X_n), \qquad \boldsymbol{Z} = (Z_1, \ldots, Z_N),$$

$$|\boldsymbol{X}| = (|X_1|, \ldots, |X_n|), \qquad |\boldsymbol{Z}| = (|Z_1|, \ldots, |Z_N|),$$

$$u(\boldsymbol{X}) = (u(X_1), \ldots, u(X_n)), \qquad u(\boldsymbol{Z}) = (u(Z_1), \ldots, u(Z_N))$$

$$\boldsymbol{R}^+ = (R_1^+, \ldots, R_N^+),$$

$$a_N(\boldsymbol{R}^+) = (a_N(R_1^+), \ldots, a_N(R_N^+)),$$

$$E = \{0, 1\}, \qquad E^n = \underbrace{E \times E \times \cdots \times E}_{n \text{ times}}. \tag{3.1}$$

Since $X_1, \ldots, X_n$ have continuous distributions, the variables $Z_1, \ldots, Z_N$ also have continuous (marginal) distributions. Consequently, the vector $\boldsymbol{R}^+$ is with probability 1 a permutation of $(1, 2, \ldots, N)$, although all permutations may not be equally likely, even under H_0. Similarly, the vector $a_N(\boldsymbol{R}^+)$ is a permutation of $(a_N(1), \ldots, a_N(N))$, and the rank statistic $S_k = \sum_{t=1}^{N} u(Z_t)a_N(R_t^+)$ can be viewed as a linear combination of the set of constants $a_N(1), \ldots, a_N(N)$. Furthermore, if the random variables $u(Z_1), \ldots, u(Z_N)$ and the random vector $\boldsymbol{R}^+$ are mutually independent with

$$P[u(Z_t)=0]=P[u(Z_t)=1]=\tfrac{1}{2}, \qquad t=1,\ldots,N,$$

the characteristic function of S_k is easily computed to be (with $i=\sqrt{-1}$):

$$E[\exp(i\tau S_k)]=(\tfrac{1}{2})^N \prod_{r=1}^{N} [1+e^{i\tau a_N(r)}], \quad \tau \text{ real.} \tag{3.2}$$

The distribution of S_k is thus completely determined for a given score function $a_N(\cdot)$ and is identical to the null distribution of S_k obtained under standard assumptions (where $Z_1, \ldots, Z_N$ are assumed independent).

We will now show that the above condition does hold in our problem.

THEOREM 1. *Let $X_1, \ldots, X_n$ be a sequence of independent real random variables having pdf's symmetric about zero, k a positive integer $(1 \leq k < n-1)$ and $Z_t = X_tX_{t+k}$, $t=1,\ldots,n-k$. Then, using the notations in* (3.1) *and* (2.3), *with $N=n-k$, the random vector $u(\boldsymbol{Z})$ is independent of the random vectors $|\boldsymbol{Z}|$ and $\boldsymbol{R}^+$, and the elements of $u(\boldsymbol{Z})$ are mutually independent with*

$$P[u(Z_t)=0]=P[u(Z_t)=1]=\tfrac{1}{2}, \qquad t=1,\ldots,n-k. \tag{3.3}$$

Proof. By the symmetry assumption, the sign and the absolute value of X_t are independent for each t (see Lehmann 1975, pp. 169–170). Thus, since $X_1, \ldots, X_n$ are independent, the random vectors $u(\boldsymbol{X})$ and $|\boldsymbol{X}|$ are independent; furthermore $u(X_1), \ldots, u(X_n)$ are mutually independent with

$$P[u(X_t)=0]=P[u(X_t)=1]=\tfrac{1}{2}, \qquad t=1,\ldots,n,$$

so that the 2^n different values which the vector $u(\boldsymbol{X})$ may take in E^n have the same probability $(\frac{1}{2})^n$. Then, since

$$u(Z_t)=u(X_t)u(X_{t+k})+[1-u(X_t)][1-u(X_{t+k})], \qquad t=1,\ldots,n-k$$

with probability 1, we can see easily that exactly 2^k of these values give each possible value of $u(\boldsymbol{Z})$ in E^{n-k} and thus each of these has the same probability $(\frac{1}{2})^{n-k}$. Therefore, $u(Z_1), \ldots, u(Z_{n-k})$ are mutually independent with (3.3) holding, as well as independent of the vectors $|\boldsymbol{Z}|$ and $\boldsymbol{R}^+$ (for these are both functions of $|\boldsymbol{X}|$ only). Q.E.D.

From theorem 1, we conclude that the distribution of the test statistic $S_k = \sum_{t=1}^{N} u(Z_t)a_N(R_t^+)$ is completely determined under H_0. The mean and variance of

S_k under H_0 are easily computed:

$$E(S_k) = \tfrac{1}{2} \sum_{t=1}^{N} a_N(t), \tag{3.4}$$

$$\operatorname{var}(S_k) = \tfrac{1}{4} \sum_{t=1}^{N} a_N^2(t). \tag{3.5}$$

Furthermore, the distribution of S_k is symmetric about $E(S_k)$ and approximately normal for $\max_{1 \le r \le N} [a_N^2(r)/\sum_{t=1}^{N} a_N^2(t)]$ sufficiently small (see Hájek 1969, p. 106).

By specifying the score function $a_N(\cdot)$, we will now examine special cases in this family of test statistics.

4. AN EXACT RUNS TEST

If we consider the constant score function $a_N(r) = 1$, the statistic S_k takes the form:

$$S_k = \sum_{t=1}^{n-k} u(Z_t). \tag{4.1}$$

S_k is here the number of non-negative values in the sequence $Z_1, \ldots, Z_{n-k}$, i.e., the statistic of the sign test applied to $Z_1, \ldots, Z_{n-k}$. From theorem 1, it is easy to see that, under H_0, $S_k \sim Bi(n-k, \frac{1}{2})$, i.e., S_k follows a binomial distribution with number of trials $n-k$ and probability of 'success' $\frac{1}{2}$. Then, the fact that S_k takes a relatively big (small) value can be viewed as evidence of positive (negative) serial dependence.

In particular, taking $k = 1$ and assuming $X_1, \ldots, X_n$ are all different from zero (an event with probability 1), we can see that S_1 is the number of times consecutive X_t's have the same sign; thus $(n-1) - S_1$ is the number of times changes of sign occur in the sequence $X_1, \ldots, X_n$ and $n - S_1$ is (with probability 1) the total number of runs in the sequence $u(X_1), \ldots, u(X_n)$. Therefore $(n-1) - S_1 \sim Bi(n-1, \frac{1}{2})$ and the null distribution of the total number of runs $R = n - S_1$ can be characterized by

$$R - 1 \sim Bi(n-1, \tfrac{1}{2}); \tag{4.2}$$

hence the mean and the variance of R are

$$E(R) = 1 + \frac{n-1}{2} = \frac{n+1}{2}, \qquad \operatorname{var}(R) = \frac{n-1}{4}. \tag{4.3}$$

Too small a number of runs indicates positive serial dependence while too big a number indicates negative serial dependence.

This runs test or close variants of it were studied in the past by David (1947), Goodman (1958) and Granger (1963). Nevertheless, one should note here that David's test is conditional on the number of +'s and −'s in the sequence while the tests of Goodman and Granger are large sample tests.

5. THE SIGNED-RANK TEST AND OTHER TESTS

If we take $a_N(r) = r$, the statistic S_k becomes

$$S_k = \sum_{t=1}^{N} u(Z_t)R_t^+. \tag{5.1}$$

S_k is the sum of the ranks of the non-negative Z_t's, the test statistic associated with the Wilcoxon signed-rank test against symmetry about zero, when applied to $Z_1, \ldots, Z_N$. The distribution of S_k, under H_0, is exactly the same as the null distribution of the Wilcoxon test statistic (as can be seen easily by comparing with Lehmann, 1975, p. 165). It has been very extensively tabulated by Wilcoxon, Katt and Wilcox (1968).

Other tests can, of course, be generated by using other score functions. We will mention the analogues of two other well-known tests of symmetry. If we take the scores

$$a_N(r) = E|V|^{(r)}, \tag{5.2}$$

where $|V|^{(r)}$ is the rth order statistic from the absolute values of a $N(0, 1)$ random sample, we get an analogue of the Fraser test (also called normal scores test). If we take

$$a_N(r) = \phi^{-1}\left(\frac{1}{2}+\frac{1}{2}\,\frac{r}{N+1}\right), \tag{5.3}$$

where $\phi(\cdot)$ is the cumulative distribution function of a $N(0, 1)$ random variable and $\phi^{-1}(\cdot)$ is the inverse function of $\phi(\cdot)$, we get an analogue of the van der Waerden test. (For further details, see Hájek and Sidák 1967, pp. 108–111).

An interesting feature of these tests is that, in contrast with the runs test of section 4, they use not only information concerning the signs of the random variable $Z_1, \ldots, Z_N$, but also their sizes via the ranks $R_1^+, \ldots, R_N^+$.

6. COMPARISON WITH OTHER TESTS

In a recent simulation study by Lepage and Zeidan (1979) the powers of the sign, Wilcoxon and van der Waerden tests described above (with $k = 1$) were compared with those of a number of well-known alternative tests, including the tests proposed by Moore and Wallis (1943), Wald and Wolfowitz (1943), Mann (1945), Daniels (1950), Foster and Stuart (1955), von Neumann (1941) and Anderson (1942). The alternative hypothesis considered were linear and cyclical trends with normal dependent errors. In 48 different simulations (12 different models with sample sizes of 15, 25, 35 and 50) reported by Lepage and Zeidan (1979, tables 1 to 4), the Wilcoxon and van der Waerden tests showed the best power in 20 cases and a power at least as high as the best competitor in 8 other cases (nominal level considered: 0.05). In general, the powers of the Wilcoxon and van der Waerden tests were very close, with the van der Waerden test being slightly more powerful and the Wilcoxon test computationally easier. However, the power of the sign (or runs) test was appreciably lower than that of the two others; this is not surprising since this test uses only information about signs, a feature which nevertheless has the advantage of great simplicity. Thus, in terms of power, it appears that, under a wide range of circumstances, the tests suggested compare favorably with several well-known alternative tests.

As indicated earlier, an important characteristic of the same tests is the allowance for non-identical distributions of the X_t's under the null hypothesis (provided symmetry about zero is preserved), a property which is not shared by the available alternatives. A frequent instance of such a situation is one where variances (more generally, dispersion parameters) differ. In order to show clearly that a number of well-known alternative tests may be very unreliable (in the sense that the actual level may differ dramatically from the nominal level) under conditions of heteroskedasticity, we performed a small Monte-Carlo simulation with series of the form: $X_t = e^t Y_t$, $t = 1, \ldots, T$, where the Y_t's are i.i.d. random variables with zero median. Two distributions for Y_t[$N(0, 1)$ and Cauchy] and two sample sizes ($T = 30, 60$) were examined. The following tests were applied to these series; (A) serial dependence sign test ($k = 1$); (B) Wilcoxon serial dependence test ($k = 1$); (C) Wallis and Moore (1941) turning point test; (D) Moore and Wallis (1943) sign test; (E) Mann (1945) rank test; (F) Daniels (1950) rank test; (G) rank serial correlation test (Knoke, 1977); (H) first-order serial correlation coefficient $r_1 = \sum_{t=2}^{T} X_t X_{t-1} / \sum_{t=1}^{T} X_t^2$, using $1/\sqrt{T}$ as standard error and the normal asymptotic distribution to determine levels; (I) modified von Neumann test (Theil, 1971, pp. 218–219); (J) Wald and Wolfowitz (1943) permutation test.[1] These tests were performed in two-sided form at a nominal level of 0.05. The actual probabilities of rejection, estimated from 500 replications, are reported in table I. From these results, we can see that the sign and Wilcoxon tests exhibit frequencies quite consistent with the nominal level of 0.05 (as expected) while the actual levels of all the other tests are generally greater than the stated level of the test, in some cases by a wide margin (e.g., H, I and J). Thus, the

TABLE I

LEVELS OF ALTERNATIVE TESTS UNDER HETEROSKEDASTICITY*

$X_t = e^t Y_t, \quad t = 1, \ldots, T$

Nominal significance level: 5%. Number of replications: 500

Distribution of Y_t	T	A	B	C	D	E	F	G	H	I	J
$N(0, 1)$	30	4.60	3.60	14.4	12.2	21.6	12.4	10.8	30.4	57.4	75.2
	60	3.00	5.00	23.0	19.0	24.4	16.4	14.6	45.2	68.4	81.0
Cauchy	30	6.80	5.80	11.0	8.20	22.6	13.2	16.0	21.2	50.6	62.2
	60	2.60	4.80	10.8	12.6	27.4	17.6	14.2	36.2	61.8	72.6

* Frequencies are given in percentages. Tests are two-sided except the modified von Neumann test (against positive serial dependence only).

latter tests may be very misleading under such circumstances and procedures of the type described above are then clearly more appropriate.

7. EXAMPLE

A standard problem in studies of foreign exchange market consists in testing whether the forward exchange rate F_t is an 'optimal' predictor of the corresponding future spot rate S_{t+1}, both usually in log form. (For a survey of the work on these questions, see Levich, 1979). This is usually interpreted as implying that the errors of prediction $S_{t+1}-F_t$ have mean zero and are uncorrelated. Furthermore, evidence of non-normality is quite frequent in this area so that nonparametric methods seem indicated. We will consider here the case where S_t is the logarithm the exchange rate between the German mark and the U.S. dollar (DM/$US) and F_t is the logarithm of the one-month forward exchange rate, during the interesting episode of the German hyperinflation. The series studied is monthly and covers the period January 1921–August 1923. It was analyzed previously by Frenkel (1977, 1979) who, using a parametric (normal) test, found some evidence of bias and serial correlation. However, as it is frequent for such data, this series exhibits signs of non-normality and heteroskedasticity, so that non-parametric tests of the type described above seem more appropriate here.[2]

The sign and Wilcoxon signed-rank serial dependence tests (described in sections 4 and 5), for lags of 1 to 6 months, were applied to this series. The test statistics, jointly with their marginal significance levels for testing against positive or negative serial dependence, are reported in table II. We see from there that the sign test is not significant (at a level of less than 0.05) for any of the lags considered (although, if we test against positive serial dependence only, it is significant at a marginal level of 0.049 for lag 1). The sign tests, however, since they use only information about signs, cannot be expected to be very powerful in most situations. On the other hand, the signed-rank test is significant for lag 1, although not for greater lags; if we test against positive dependence only, the marginal

TABLE II

SERIAL DEPENDENCE TESTS FOR LOG FORWARD-LOG SPOT PREDICTION ERRORS (February 1921–August 1923)*

k	1	2	3	4	5	6
S_k						
Sign tests	20	19	18	18	15	15
	(0.099)	(0.136)	(0.185)	(0.122)	(0.557)	(0.424)
Signed-rank tests	329	294	273	228	208	180
	(0.047)	(0.101)	(0.115)	(0.361)	(0.423)	(0.653)

* Marginal significance levels for a two-sided test are given in parentheses; for one-sided tests against positive serial dependence, these levels are to be divided by 2; e.g., for the signed-rank test, $2P[S_1 \geq 329] = 0.047$.

significance level is 0.024 for lag 1 (0.050 and 0.057 for lags 2 and 3). This indicates some positive dependence at least in the short run and thus confirms, under less restrictive assumptions, the previous finding of Frenkel (1977, 1979).

8. CONCLUDING REMARKS

The tests described in this paper have a number of interesting features: besides being nonparametric, they are exact, can be used to assess dependence at various lags and do not require the standard assumption of identical distributions under the null hypothesis. The last property is especially distinctive since, under conditions of heteroskedasticity for example, alternative tests of randomness may be very misleading. Furthermore, we may note that the alternatives considered are very wide, including covariance stationary schemes as well as a number of nonstationary ones.

The main limitations are the assumptions of symmetry and zero (or known) median. In practice, what we test is the joint hypothesis that the X_t's are independent and have pdf's symmetric about zero (hence, if second moments exist, $E(X_t X_{t+k}) = 0$, $t = 1, \ldots, n-k$, for any k). The symmetry assumption may reasonably be maintained in a large number of situations. On the other hand, although there are many important problems for which the assumption of zero (or known) median is part of the null hypothesis, it would certainly be desirable to be able to relax it. This remains a topic for further research.

ACKNOWLEDGEMENT

The author is grateful to Arnold Zellner, Craig Ansley, Leonrd Dudley, Nicholas Kiefer, Yves Lepage, Robert E. Lucas, Roch Roy, Houston Stokes, Constance Van Eeden, William Wecker, an anonymous referee, and the members of the University of Chicago Econometrics and Statistics Colloquium for many helpful comments; special thanks go to Jacob Frenkel for supplying the data. Sophie Mahseredjian executed the computations. This work was supported by Grant No. 410-80-0581 of the Social Sciences and Humanities Research Council of Canada and was completed while the author was visiting the Center for Computational Research in Economics and Management Science at M.I.T.

Jean-Marie Dufour, Départment de Science Economique, Université de Montréal, C.P. 6128, Succursale A, Montréal, P.Q., H3C 3J7, Canada.

NOTES

[1] Most of these tests are described in detail by Lepage and Zeidan (1979). Computations were made on a CDC Cyber 173 computer using IMSL (June 1980) random number generators.

[2] The Kolmogorov-Smirnov test statistic against non-normality is $D = 0.196$ [5% critical value = 0.16; see Lielliefors (1968)]. Evidence of heteroskedasticity is reported by Frenkel (1979).

REFERENCES

AIYAR, R. J., C. L. GUILLIER and W. ALBERS (1979) Asymptotic Relative Efficiencies of Rank Tests for Trend Alternatives. *Journal of the American Statistical Association* 74, 226–231.

ANDERSON, R. L. (1942) Distribution of the Serial Correlation Coefficient. *Annals of Mathematical Statistics* 13, 1–13.

BLATTBERG, ROBERT C. and NICHOLAS GONEDES (1974) A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices. *Journal of Business* 47, 244–280.

BOX, GEORGE E. P. and GWILYM M. JENKINS (1970) *Time Series Analysis Forecasting and Control* Holden-Day: San Francisco.

COX, D. R. and A. STUART (1955) Some Quick Sign tests for Trend in Location and Dispersion. *Biometrika* 42, 80–95.

DANIELS, H. E. (1950) Rank Correlation and Population Models. *Journal of the Royal Statistical Society, Series B* 12, 171–181.

DAVID, F. N. (1947) A Power Function For Tests of Randomness in a Sequence of Alternatives. *Biometrika* 34, 335–339.

FAMA, EUGENE F. (1965) The Behaviour of Stock Market Prices. *Journal of Business* 38, 34–105.

FAMA, EUGENE F. (1970) Efficient Capital Markets: A Review of Theory and Empirical Work. *Journal of Finance* 25, 383–417.

FOSTER, F. G. and A. STUART (1954) Distribution-free Tests in Time Series Based on the Breaking of Records. *Journal of the Royal Statistical Society, Series B* 16, 1–22.

FRENKEL, JACOB A. (1977) The Forward Exchange Rate, Expectations and the Demand for Money: The German Hyperinflation. *American Economic Review* 67, 653–670.

FRENKEL, JACOB A. (1979). Further Evidence on Expectations and the Demand for Money During the German Hyperinflation. *Journal of Monetary Economics* 5, 81–96.

GIDDY, DAN H. and GUNTER DUFEY (1975) The Random Behavior of Flexible Exchange Rates. *Journal of International Business Studies* 6, 1–32.

GOODMAN, LEO A. (1958) Simplified Runs Tests and Likelihood Ratio Tests for Markov Chains. *Biometrika* 45, 181–197.

GRANGER, C. W. J. (1963) A Quick Test for Serial Correlation Suitable for Use with Non-stationary Time Series. *Journal of the American Statistical Association* 58, 728–736.

HÁJEK, JAROSLAV (1969) *A Course in Nonparametric Statistics* Holden-Day: San Francisco.

HÁJEK, JAROSLAV and ZBYNEK SIDÁK (1967) *Theory of Rank Tests* Academic Press: New York and London.

KNOKE, J. D. (1977) Testing for Randomness against Autocorrelation: Alternative Tests. *Biometrika* 64, 523–529.

LEHMANN, E. L. (1966) Some Concepts of Dependence. *Annals of Mathematical Statistics* 37, 1137–1153.

LEHMANN, E. L. (1975) *Nonparametrics. Statistical Methods Based on Ranks* Holden-Day: San Francisco.

LEPAGE, YVES and HANNA ZEIDAN (1979) Sur l'indépendance d'une série d'observations, Technical Report 79-8, Département de mathématiques et de statistique, Université de Montréal [forthcoming in *Annales des Sciences Mathématiques du Québec* 5 (1981)].

LEVICH, RICHARD M. (1979) On the Efficiency of Markets for Foreign Exchange. In R. Dornbusch and J. A. Frenkel, eds, *International Economic Policy, Theory and Evidence* The Johns Hopkins Press: Baltimore.

LIELLIEFORS, HUBERT W. (1967) On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown. *Journal of the American Statistical Association* 64, 399–402.

MALDELBROT, BENOÎT (1963) The Variation of Certain Speculative Prices. *Journal of Business* 36, 394–419.

MALDELBROT, BENOÎT (1967) The Variation of Some Other Speculative Prices. *Journal of Business* 40, 393–413.

MANN, HENRY B. (1945) Nonparametric Test Against Trend. *Econometrica* 13, 245–259.

MOORE, GEOFFREY H. and W. ALLEN WALLIS (1943) Time Series Significance Tests Based on Signs of Differences. *Journal of the American Statistical Association* 38, 153–164.

NEUMANN, JOHN VON (1941) Distribution of the Ratio of the Mean Square Successive Difference to the Variance. *Annals of Mathematical Statistics* 12, 367–395.

PRAETZ, P. D. (1972) The Distribution of Share Price Changes. *Journal of Business* 45, 49–55.

STUART, ALAN (1952) The Power of Two Difference-sign Tests. *Journal of the American Statistical Association* 47, 416–424.

STUART, ALAN (1954) Asymptotic Relative Efficiencies of Distribution-Free Tests of Randomness against Normal Alternatives. *Jounal of the American Statistical Association* 49, 147–157.

THEIL, HENRI (1971) *Principles of Econometrics* John Wiley and Sons: New York.

WALD, A. and J. WOLFOWITZ (1943) An Exact Test for Randomness in the Non-parametric Case Based on Serial Correlation. *Annals of Mathematical Statistics* 14, 378–388.

WALLIS, W. ALLEN and GEOFFREY H. MOORE (1941) A Significance Test for Time Series Analysis. *Journal of the American Statistical Association* 36, 401–409.

WILCOXON, FRANK, S. K. KATTI and ROBERTA A. WILCOX (1973) Critical Values and Probability Levels for the Wilcoxon Rank Sum Test and the Wilcoxon Signed-rank Test, In *Selected Tables in Mathematical Statistics*, Volume I, Edited by the Institute of Mathematical Statistics, Providence: American Mathematical Society, 171–259.