Improved Eaton Bounds for Linear Combinations of Bounded Random Variables, With Statistical Applications

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The problem of evaluating tail probabilities for linear combinations of independent, possibly nonidentically distributed, bounded random variables arises in various statistical contexts, mainly connected with nonparametric inference. A remarkable inequality on such tail probabilities has been established by Eaton. The significance of Eaton's inequality is substantiated by a recent result of Pinelis showing that the minimum B_{EP} of Eaton's bound B_E and a traditional Chebyshev bound yields an inequality that is optimal within a fairly general class of bounds. Eaton's bound, however, is not directly operational, because it is not explicit; apparently, it never has been studied numerically, and its many potential statistical applications have not yet been considered. A simpler inequality recently proposed by Edelman for linear combinations of iid Bernoulli variables is also considered, but it appears considerably less tight than Eaton's original bound. This article has three main objectives. First, we put Eaton's exact bound B_E into numerically tractable form and tabulate it, along with B_{EP}, which makes them readily applicable; the resulting conservative critical values are provided for standard significance levels. Second, we show how further improvement can be obtained over the Eaton–Pinelis bound B_{EP} if the number n of independent variables in the linear combination under study is taken into account. The resulting improved Eaton bounds B^*_{EP} and the corresponding conservative critical values are also tabulated for standard significance levels and most empirically relevant values of n. Finally, various statistical applications are discussed: permutation t tests against location shifts, permutation t tests against regression or trend, permutation tests against serial correlation, and linear signed rank tests against various alternatives, all in the presence of possibly nonidentically distributed (e.g., heteroscedastic) data. For permutation t tests and linear signed rank tests, the improved Eaton bounds are compared numerically with other available bounds. The results indicate that the sharpened Eaton bounds often yield sizable improvements over all other bounds considered.

KEY WORDS: Bounded random variables; Conservative test; Eaton bounds; Heteroscedasticity; Nonnormality; Nonparametric test; Permutation test; Serial correlation; Signed rank tests; t test.

1. INTRODUCTION

Let $Y_1, \ldots, Y_n$ be independent random variables with mean 0 and $|Y_i| \leq 1$, $i = 1, \ldots, n$. We do not require that the Y_i's be identically or symmetrically distributed. Denote by $\mathbf{a} = (a_1, \ldots, a_n)'$ a n-tuple of fixed real numbers such that $\sum_{i=1}^n a_i^2 = 1$. The vector $\mathbf{a}$ need not be specified. Let also $\phi(z) = (2\pi)^{-1/2}\exp(-z^2/2)$ and $\Phi(x) = \int_{-\infty}^x \phi(z)\,dz$ denote the standard normal probability density and distribution functions. We study here the distribution of $\sum_{i=1}^n a_i Y_i$.

Many problems in nonparametric inference lead one to consider statistics of the form $\sum_{i=1}^n a_i Y_i$. Important examples include linear signed rank tests, permutation t tests against location shift, permutation t tests against regression or trend, and permutation tests against serial dependence. Except for very special cases, the distributions of such statistics are either unknown or quite difficult to compute. In most cases only large sample approximations are available (e.g., normal approximations). These require additional regularity assumptions (e.g., on the constants $a_1, \ldots, a_n$), however, and may be highly inaccurate.

In such contexts it is clear that finite sample bounds on the tail areas of $\sum_{i=1}^n a_i Y_i$ can be quite useful. On this issue, Eaton (1974, thm. 2) proved the following fundamental result: for all $y > 0$,

$$P\left[\left|\sum_{i=1}^n a_i Y_i\right| \geq y\right] \leq 2 \inf_{0 \leq c \leq y} \int_c^\infty \left(\frac{z-c}{y-c}\right)^3 \phi(z)\,dz \equiv 2B_E(y). \quad (1)$$

Eaton did not, however, provide any explicit solution to the problem of minimizing the integral expression in (1). Instead, he suggested several upper bounds for $B_E(y)$ (see his cor. 1 and 2) and conjectured that for $y > \sqrt{2}$,

$$B_E(y) \leq \bar{B}_E(y) \equiv (2e^3/9)\phi(y)y^{-1}. \quad (2)$$

No attempt apparently has been made to study Eaton's bound numerically or to implement it, and no table of the bound $B_E(y)$ seems to be available so far. In the particular case when $Y_1, \ldots, Y_n$ are independent Bernoulli variables such that $P[Y_i = 1] = P[Y_i = -1] = \frac{1}{2}$, $i = 1, \ldots, n$, Edelman (1990, lem. 2) has proposed the alternative simpler inequality

$$P\left[\left|\sum_{i=1}^n a_i Y_i\right| \geq y\right] \leq 2\{1 - \Phi[y - (1.5/y)]\} \equiv 2B_{Ed}(y), \quad \text{for } y > 0. \quad (3)$$

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As we discuss in Section 3, B_{Ed} is less tight than B_{E} for most practically relevant values of y ($y \geq 1.1$).

Eaton's result has also been revisited recently by Pinelis (1991). Observing that Eatons' bound can be improved for small y ($0 < y < 2\sqrt{2/\pi} = 1.596$) by a second-order Chebyshev bound y^{-2} (or simply by 1), Pinelis proposed the alternative bound $2B_{\mathrm{EP}}(y) = \min\{2B_{\mathrm{E}}(y), y^{-2}, 1\}$ and showed that it is optimal within the context of Eaton's approach, in the sense that it is tightest among all bounds based on expectations of convex functions of a standard normal variable; see Pinelis (1991, prop. 4.7, with $r = 1$). Pinelis also provided a proof of (2). For most values of y of practical relevance ($y > 2\sqrt{2/\pi} = 1.596$), the bound B_{EP} coincides with Eaton's bound B_{E}, so that B_{E} enjoys the optimality of B_{EP} for $y > 1.596$.

This article has three main objectives. First, we put Eaton's exact bound B_{E} into a numerically tractable form and then evaluate it (hence also the Eaton–Pinelis bound B_{EP}) by numerical methods. We then compute the corresponding conservative critical values for standard significance levels. Because the resulting table does not depend on n or $\mathbf{a}$, these critical values are applicable even if the constants $a_1, \ldots, a_n$ or the sample size n are not specified. Numerical comparisons of B_{EP} with Edelman's bound B_{Ed} also show that B_{EP} is often substantially sharper. Second, building again on Eaton's (1974) results, we observe that the bounds B_{EP} and B_{E} can be improved in an operational way if the number n of independent random variables in $\sum a_t Y_t$ is taken into account. The resulting bound $B^*_{\mathrm{EP}}(y; n)$, which depends on y and n (but not on $a_1, \ldots, a_n$), is always tighter than the bound $B_{\mathrm{E}}(y)$ and never larger than $B_{\mathrm{EP}}(y)$. The bound $B^*_{\mathrm{EP}}(y; n)$ can improve the "optimal" Eaton–Pinelis bound $B_{\mathrm{EP}}(y)$ because it is based on the expectation of a function of a standardized binomial variable (instead of a standard normal one). Because the sample size n is typically known in applications, the improved bound B^*_{EP} can easily be used in practically all situations where the alternative bounds B_{E}, B_{EP}, and B_{Ed} apply. In Section 3 we also tabulate the critical values based on $B^*_{\mathrm{EP}}(y; n)$ for standard significance levels and several values of n. Third, we show how the bounds derived can be applied in statistical problems of practical interest, including permutation t tests against location shift, permutation t tests against regression or trend, permutation tests against autocorrelation, and linear signed rank tests.

The improved bound B^*_{EP} is presented in Section 2. The tabulation of the bounds B_{EP} and B^*_{EP} is given in Section 3; for the purpose of comparison, B_{E}, $\bar{B}_{\mathrm{E}}$, and B_{Ed} are also tabulated. The statistical applications to hypothesis testing problems are described in Section 4.

2. AN IMPROVED EATON INEQUALITY

In this section we establish an inequality that improves the bounds given by Eaton (1974) and Pinelis (1991). As in Eaton (1974), let $\mathcal{F}_1$ be the class of all functions $f: \mathbb{R} \to \mathbb{R}$ such that f is symmetric (about 0) and admits a derivative $\dot{f}$ such that $t^{-1}[\dot{f}(t + \Delta) - \dot{f}(-t + \Delta)]$ is nondecreasing in $t > 0$ for all $\Delta \geq 0$. A sufficient condition for a symmetric function f to be in $\mathcal{F}_1$ is given by Eaton (1974, lem. 1) and requires the existence of a third derivative $\dddot{f}$. (The notation $\dot{f}, \ddot{f}, \dddot{f}$ is used for derivatives.) A slightly more general version of this lemma, where $\dddot{f}$ may not exist at a finite number of points, is required here. Actually, this extended lemma is also necessary for the proof of Eaton's theorem 2, because the function $f_c(x) = [(|x| - c)_+]^3$ defined there (Eaton 1974, eq. 3.2) has no third derivative at $x = c$. This is because the left derivative is $(\dddot{f}_c)_l(c) = 0$, whereas the right derivative is $(\dddot{f}_c)_r(c) = 6$. The proof of this extended lemma is briefly sketched in the Appendix.

Lemma 1. Suppose that the function $f: \mathbb{R} \to \mathbb{R}$ is symmetric (about 0) and that $\ddot{f}$ exists and everywhere admits nondecreasing left and right derivatives, $\dddot{f}_l$ and $\dddot{f}_r$. Also suppose that $\dddot{f}_l = \dddot{f}_r$ everywhere, except possibly in a finite set of points. Then $f \in \mathcal{F}_1$.

Proposition 1. Let $Y_1, \ldots, Y_n$ be independent random variables, with $E(Y_i) = 0$ and $|Y_i| \leq 1$, $i = 1, \ldots, n$. Then for any fixed vector $\mathbf{a} = (a_1, \ldots, a_n)'$ such that $\sum_{i=1}^n a_i^2 = 1$,

$$\begin{aligned} P\left[\left|\sum_{i=1}^n a_i Y_i\right| \geq y\right] &\leq 2 \min\{B_{\mathrm{E}}(y; n), .5y^{-2}, .5\} \\ &\equiv 2B^*_{\mathrm{EP}}(y; n) \\ &\leq 2 \min\{B_{\mathrm{E}}(y), .5y^{-2}, .5\} \\ &\equiv 2B_{\mathrm{EP}}(y) \end{aligned} \tag{4}$$

for $y > 0$, where

$$B_{\mathrm{E}}(y; n) = (.5) \inf_{0 \leq c < y} \left\{(.5)^n \sum_{m=0}^n \binom{n}{m} f_c[(n/4)^{-1/2} \times (m - (n/2))]/(y - c)^3\right\}, \tag{5}$$

$f_c(x) = [(|x| - c)_+]^3$, $\binom{n}{m} = n!/[m!(n - m)!]$, and

$$\begin{aligned} B_{\mathrm{E}}(y) &= \inf_{0 \leq c < y} \int_c^\infty \left(\frac{z - c}{y - c}\right)^3 \phi(z)\, dz \\ &= \inf_{0 \leq c < y} \{[\phi(c)(2 + c^2) - (1 - \Phi(c))(c^3 + 3c)]/(y - c)^3\}. \end{aligned} \tag{6}$$

Furthermore, $B^*_{\mathrm{EP}}(y; n) \leq B^*_{\mathrm{EP}}(y; n + 1)$, for $y > 0$.

The proof of this proposition is given in the Appendix. Note that the last expression in (6) is much more convenient for minimization purposes than Eaton's integral form (1). Further, we can write

$$B_{\mathrm{E}}(y; n) = (.5) \inf_{0 \leq c < y} \{E[f_c(T_n)]/(y - c)^3\},$$

where $T_n = n^{-1/2} \sum_{i=1}^n U_i$ and $U_1, \ldots, U_n$ are independent Bernoulli variables such that $P[U_i = -1] = P[U_i = 1] = .5$. Consequently, there is no contradiction between the fact that $B^*_{\mathrm{EP}}(y; n)$ improves the Eaton–Pinelis bound $B_{\mathrm{EP}}(y)$ and the optimality property given by Pinelis (1991) for B_{EP}. According to the latter, B_{EP} is optimal in a class of bounds based on expectations of convex functions of standard normal variables, whereas $B^*_{\mathrm{EP}}(y; n)$ is based on the expected value of a function of a nonnormal variable (T_n). It is also useful to observe that the bound $B^*_{EP}(y; n)$ increases monotonically with n.

3. EXPLICIT BOUNDS AND CRITICAL VALUES

The numerical evaluation of $B^*_{EP}(y; n)$ and $B_{EP}(y)$, as defined in Proposition 1, is possible by means of standard optimization techniques. Table 1 provides numerical values for B^*_{EP} (with $n = 20$), B_{EP}, B_E, $\bar{B}_E$, and B_{Ed}.

Using obvious notation, let $S^*_{EP}(\alpha/2; n)$, $S_{EP}(\alpha/2)$, $S_E(\alpha/2)$, $S_{Ed}(\alpha/2)$, and $\bar{S}_E(\alpha/2)$ denote the two-sided critical values derived from the various bounds considered; that is, let $S.(\alpha/2)$ be the positive solution of $B.(y) = \alpha/2$. Because $P[|\sum_{i=1}^n a_i Y_i| \geq S.(\alpha/2)] \leq 2B.(S.(\alpha/2)) = \alpha$, $|\sum_{i=1}^n a_i Y_i| \geq S.(\alpha/2)$ is a conservative critical region for a two-sided test with level α based on $\sum_{i=1}^n a_i Y_i$. Note that $S.(\alpha)$ can be used as a (conservative) upper critical value for a one-sided test with level α only if $\sum_{i=1}^n a_i Y_i$ is symmetrically distributed, an assumption we have not made so far. For example, this will hold when Y_i, $i = 1, \ldots, n$ have symmetric distributions, as in Edelman (1990), where Y_i is symmetric Bernoulli with $P(Y_i = -1) = P(Y_i = 1) = \frac{1}{2}$. Table 2 provides critical values based on the four bounds B_{EP}, B_E, $\bar{B}_E$, and B_{Ed}, which do not depend on the number of variables n. The significance levels considered are α = .40, .25, .20, .10, .05, .025, .01, .005, .0025, .001, .0005. Table 3 contains critical values based on the improved bound B^*_{EP} for n = 5 (1) 15 (5) 100 (10) 150 and $\alpha \leq .20$; for $\alpha > .20$, the critical values based on B_{EP} can be used (see Table 2). A more detailed table, with critical values for n = 5 (1) 100 (10) 150, is available in a working paper (Dufour and Hallin 1992b, table 3). For very small values of n ($n < 10$) and small α, the equation $B_{EP}(y; n) = \alpha/2$ may not have a solution; in such cases, we do not report a critical value. Note that $\sqrt{n}$ is the maximum possible value of $|\sum_{i=1}^n a_i Y_i|$.

All calculations were performed using the GAUSS (1991) statistical package. Computation of the functions $B_{EP}(y)$ and $B^*_{EP}(y; n)$, which requires minimizing appropriate functions over the interval $0 \leq c < y$, were evaluated by a double grid search over $0 \leq c < y$. A grid with intervals of length 10^{-4} was first used to locate a first approximate value c_1 of c; then a second grid search over the set $[c_1 - 10^{-4}, c_1 + 10^{-4}]$ with intervals of length 10^{-6} was used to locate the optimal value of c. The critical values were obtained by solving for y the equations $B_{EP}(y) = \alpha$ and $B^*_{EP}(y; n) = \alpha$ through the quasi-Newton algorithm for solving nonlinear equations (NLSYS) available in GAUSS. The validity of each figure obtained was also checked by direct substitution. To keep the conservative character of the critical values reported, the values in Tables 2 and 3 have been rounded to the closest number with four decimals not smaller than the one actually computed (with higher precision).

Table 1. Numerical Values of the Improved Eaton Bounds B^*_{EP} (with n = 20) and B_{EP}, Eaton's Original Bound B_E, Edelman's Bound B_{Ed}, and Eaton's Conjectured Bound $\bar{B}_E$, for y = .2(.2)4.0

y	B^*_{EP} (n = 20)	B_{EP}	B_E	$\bar{B}_E$	B_{Ed}
.2	.5000	.5000	99.7356	8.7270	1.0000
.4	.5000	.5000	12.4670	4.1094	.9996
.6	.5000	.5000	3.6939	2.4789	.9713
.8	.5000	.5000	1.5584	1.6163	.8588
1.0	.5000	.5000	.7979	1.0800	.6915
1.2	.3472	.3472	.4617	.7223	.5199
1.4	.2551	.2551	.2908	.4774	.3712
1.6	.1922	.1948	.1948	.3094	.2538
1.8	.1278	.1311	.1311	.1958	.1669
2.0	.0811	.0848	.0848	.1205	.1057
2.2	.0492	.0528	.0528	.0720	.0645
2.4	.0284	.0317	.0317	.0416	.0379
2.6	.0155	.0183	.0183	.0233	.0215
2.8	.0080	.0101	.0101	.0126	.0118
3.0	.0039	.0054	.0054	.0066	.0062
3.2	.0018	.0028	.0028	.0033	.0032
3.4	.0008	.0014	.0014	.0016	.0015
3.6	.0003	.0007	.0007	.0008	.0007
3.8	.0001	.0003	.0003	.0003	.0003
4.0	.0000	.0001	.0001	.0001	.0001

Table 2. Conservative Critical Values Associated With B_{EP}, B_E, B_{Ed}, and $\bar{B}_E$, at Usual Nominal Probability Levels α

α	$S_{EP}(\alpha)$	$S_E(\alpha)$	$S_{Ed}(\alpha)$	$\bar{S}_E(\alpha)$
.40	1.1181	1.2589	1.3580	1.4828
.25	1.4143	1.4724	1.6076	1.6946
.20	1.5812	1.5861	1.7159	1.7910
.10	1.9264		2.0231	2.0738
.05	2.2222		2.2977	2.3345
.025	2.4875		2.5486	2.5766
.010	2.8042		2.8523	2.8730
.005	3.0242		3.0652	3.0822
.0025	3.2308		3.2663	3.2804
.001	3.4868		3.5168	3.5282
.0005	3.6697		3.6964	3.7062

NOTE: Two-sided tests reject at level α if $|\Sigma a_i Y_i| \geq S.(\alpha/2)$; one-sided tests reject if $\Sigma a_i Y_i \geq S.(\alpha)$ ($\leq -S.(\alpha)$) provided $\Sigma a_i Y_i$ is symmetrically distributed with respect to 0.

Inspection of Tables 1, 2 and 3 reveals the substantial superiority of the bounds B^*_{EP} and B_{EP} over their competitors. One sees easily that

$$B^*_{EP}(y; n) = B_{EP}(y) < B_{Ed}(y) < \bar{B}_E(y) < B_E(y), \quad \text{for} \quad 0 < y \leq .7,$$

$$B^*_{EP}(y; n) = B_{EP}(y) < B_{Ed}(y) < B_E(y) < \bar{B}_E(y), \quad \text{for} \quad .8 \leq y < 1.0,$$

$$B^*_{EP}(y; n) = B_{EP}(y) < B_E(y) < B_{Ed}(y) < \bar{B}_E(y), \quad \text{for} \quad 1.1 \leq y \leq 1.5,$$

and

$$B^*_{EP}(y; n) < B_{EP}(y) = B_E(y) < B_{Ed}(y) < \bar{B}_E(y), \quad \text{for} \quad 1.6 \leq y \leq 5.0.$$

(See Table 2 and the more detailed tabulation in Dufour and Hallin 1992b, table 1). From a practical standpoint, the most relevant ranking is the one for $y \geq 1.6$, where B^*_{EP} is strictly smaller than all the other bounds, whereas Eaton's bound ranks second (with $B_{EP} = B_E$). Further, for $\alpha \leq .10$, the critical values based on B^*_{EP} are always strictly smaller (hence less conservative) than those based on B_{EP}; see Table 3. Eaton's conjecture about $\bar{B}_E$ is numerically confirmed (for $.8 \leq y \leq 5.0$) but loses its relevance in view of the much better performance of B^*_{EP} and B_{EP}.

Table 3. Conservative Critical Values $S^*_{EP}(\alpha; n)$ Associated With B^*_{EP} at Usual Nominal Probability Levels α

n	α = .20	.10	.05	.025	.01	.005	.0025	.001	.0005
5	1.556	1.855	2.087	2.237	—	—	—	—	—
6	1.564	1.870	2.114	2.315	2.450	—	—	—	—
7	1.566	1.878	2.138	2.343	2.582	2.646	—	—	—
8	1.570	1.886	2.144	2.374	2.603	2.769	2.829	—	—
9	1.571	1.889	2.158	2.380	2.638	2.790	2.944	—	—
10	1.573	1.895	2.162	2.395	2.658	2.825	2.965	3.158	3.163
11	1.574	1.897	2.169	2.405	2.668	2.851	3.000	3.173	3.312
12	1.575	1.901	2.174	2.410	2.686	2.860	3.032	3.204	3.327
13	1.576	1.902	2.177	2.420	2.694	2.879	3.039	3.242	3.358
14	1.577	1.904	2.182	2.423	2.702	2.891	3.058	3.258	3.397
15	1.577	1.905	2.184	2.428	2.713	2.897	3.075	3.271	3.416
20	1.580	1.911	2.194	2.444	2.736	2.932	3.116	3.334	3.486
25	1.581	1.914	2.200	2.453	2.751	2.952	3.139	3.369	3.524
30	1.582	1.916	2.204	2.459	2.760	2.965	3.155	3.389	3.551
35	1.582	1.918	2.207	2.463	2.766	2.974	3.167	3.403	3.570
40	1.582	1.919	2.209	2.466	2.771	2.980	3.175	3.414	3.583
45	1.582	1.920	2.210	2.469	2.775	2.985	3.182	3.422	3.593
50	1.582	1.921	2.212	2.471	2.778	2.989	3.187	3.429	3.601
55	1.582	1.921	2.213	2.472	2.780	2.993	3.191	3.435	3.608
60	1.582	1.922	2.214	2.474	2.783	2.996	3.194	3.439	3.613
65	1.582	1.922	2.214	2.475	2.784	2.998	3.197	3.443	3.617
70	1.582	1.922	2.215	2.476	2.786	3.000	3.200	3.446	3.621
75	1.582	1.923	2.215	2.477	2.787	3.001	3.202	3.449	3.625
80	1.582	1.923	2.216	2.477	2.788	3.003	3.204	3.451	3.627
85	1.582	1.923	2.216	2.478	2.789	3.004	3.205	3.454	3.630
90	1.582	1.923	2.217	2.478	2.790	3.005	3.207	3.455	3.632
95	1.582	1.924	2.217	2.479	2.791	3.006	3.208	3.457	3.634
100	1.582	1.924	2.217	2.479	2.791	3.007	3.209	3.459	3.636
110	1.582	1.924	2.218	2.480	2.793	3.009	3.211	3.461	3.639
120	1.582	1.924	2.218	2.481	2.794	3.010	3.213	3.463	3.642
130	1.582	1.925	2.218	2.481	2.794	3.011	3.214	3.465	3.644
140	1.582	1.925	2.219	2.482	2.795	3.012	3.216	3.467	3.646
150	1.582	1.925	2.219	2.482	2.796	3.013	3.217	3.468	3.648

NOTE: Two-sided tests reject at level α if $|\Sigma a_i Y_i| \geq S^*_{EP}(\alpha/2; n)$; one-sided tests reject at level α if $\Sigma a_i Y_i \geq SE^*_{EP}(\alpha; n)$ ($\leq -S^*_{EP}(\alpha; n)$) provided $\Sigma a_i Y_i$ is symmetrically distributed about 0.

4. STATISTICAL APPLICATIONS

4.1 One-sample Permutation *t* Tests

Let $X_1, \ldots, X_n$ be independent random variables with (possibly nonidentical) unspecified distributions symmetric about a common median μ. It is well known that the classical one-sample Student statistic

$$T_n = n^{-1/2} \sum_{i=1}^{n} (X_i - \mu_0) \Big/ \left[(n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right]^{1/2}$$

can be used to test $H_o: \mu = \mu_0$ against $\mu > \mu_0$ if one considers its *permutational* null distribution; that is, the conditional distribution of T_n given $|X_1 - \mu_0|, \ldots, |X_n - \mu_0|$. This conditional test follows from classical unbiasedness and Neyman structure arguments; see Hoeffding (1952), Lehmann (1986, chaps. 5 and 6), Lehmann and Stein (1949), Pratt and Gibbons (1981, pp. 218, 233–234). The problem, of course, is that this permutational distribution cannot be tabulated explicitly. Several authors, therefore, have proposed bounding permutational tail areas to obtain conservative critical values; see Dufour and Hallin (1991) and Edelman (1986, 1990). The improved Eaton bound derived here also yields such conservative permutational critical values. Define (with the convention $0/0 = 0$)

$$S_n = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} (X_j - \mu_0)^2 \right]^{-1/2} |X_i - \mu_0| U_i,$$

where $U_i = \text{sgn}(X_i - \mu_0)$ denotes the sign of $X_i - \mu_0$ and $\text{sgn}(x) = x/|x|$. It is easy to verify that $S_n = n^{1/2} T_n / [n - 1 + T_n^2]^{1/2}$ is a monotonically increasing transformation of T_n. Further, the conditional distribution of S_n (given $|X_1 - \mu_0|, \ldots, |X_n - \mu_0|$) is symmetric about 0. Because S_n has the form $\sum_{i=1}^{n} a_i Y_i$ considered in Proposition 1, it follows that

$$P[T_n \geq z \,|\, |X_1 - \mu_0|, \ldots, |X_n - \mu_0|] \leq B^*_{EP}[n^{1/2} z/(n - 1 + z^2)^{1/2}; n] \quad (7)$$

for $z > 0$, so that

$$t_n(\alpha) = (n-1)^{1/2} S^*_{EP}(\alpha; n)/[n - (S^*_{EP}(\alpha; n))^2]^{1/2}, \quad (8)$$

where $S^*_{EP}(\alpha; n)$ is given by Table 3, can be used as a critical value for one-sided permutation t tests (provided $S^*_{EP}(\alpha; n)^2 < n$). Similarly, the one-sided critical region $T_n \leq -t_n(\alpha/2)$ against $\mu < \mu_0$ and the two-sided critical region $|T_n| \geq t_n(\alpha/2)$ are conservative at level α. Of course, in (7) and (8), the bound B^*_{EP} could be replaced by B_{EP} and $S^*_{EP}(\alpha; n)$ could be replaced by $S_{EP}(\alpha)$ from Table 2, yielding more conservative critical values (provided that $S_{EP}(\alpha)^2 < n$).

In Table 4, the bounds B^*_{EP} and B_{EP} from (7) are compared with B_{Ed} as well as with Edelman's (1986) exponential bound

$$P[T_n \geq z \,|\, |X_1 - \mu_0|, \ldots, |X_n - \mu_0|] \leq \exp(-nz^2/[2(n - 1 + z^2)]). \quad (9)$$

The figures in Table 4 demonstrate the substantial superiority

Table 4. Permutational One-Sample t Test (n = 20, 40). Comparison Between the Improved Eaton Bounds B^*_{EP} and B_{EP}, Edelman's (1990) Bound B_{Ed}, and Edelman's (1976) Exponential Bound

Tail area bounds	z =	1.0	1.5	2.0	2.5	3.0	3.5
$B^*_{EP}(n^{1/2}z/(n-1+z^2)^{1/2})$	$n = 20$	.5000	.2361	.1109	.0460	.0189	.0080
	$n = 40$	.5000	.2292	.0977	.0344	.0111	.0034
$B_{EP}(n^{1/2}z/(n-1+z^2)^{1/2})$	$n = 20$	.5000	.2361	.1143	.0497	.0219	.0101
	$n = 40$	.5000	.2292	.0994	.0361	.0122	.0040
$B_{Ed}(n^{1/2}z/(n-1+z^2)^{1/2})$	$n = 20$	.6915	.3356	.1444	.0605	.0260	.0118
	$n = 40$	.6915	.3222	.1248	.0434	.0142	.0046
$\exp(-nz^2/[2(n-1+z^2)])$	$n = 20$	.6065	.3469	.1757	.0841	.0402	.0198
	$n = 40$	.6065	.3359	.1556	.0631	.0235	.0084

of the improved Eaton bounds B^*_{EP} and B_{EP} over those suggested by Edelman (1986, 1990). Note that the latter, in turn, improve earlier bounds given by Bernstein (1924, 1927)—see also Godwin (1955, p. 936) and Uspensky (1937, pp. 204-206)—Bennett (1962, (7a) and (8a)), Craig (1933), and Hoeffding (1963, (2.2) and (2.3)). All the bounds considered here are uniform, in the sense that they do not take the $|X_i - \mu_0|$ values into account. Further improvements might be obtained (in certain cases) from nonuniform bounds; see Dufour and Hallin (1992).

To assess better the validity and tightness of the Eaton bounds in the context of permutational t tests, we present in Table 5 permutational critical values for S_n (given $|X_1|, \ldots, |X_n|$), at level .05 (two-sided test), associated with 100 independent samples of n iid $N(0, 1)$ random variables ($n = 20, 40$). The 100 samples were generated by a Monte Carlo simulation. For each sample of size n, we give the quantiles of order 0 (minimum), .25, .50, .75, and 1 (maximum) of the permutational critical values associated with the 100 samples considered. For each of these samples, the permutational critical value was estimated by a Monte Carlo simulation with 1,000 replications (keeping $|X_1|, \ldots, |X_n|$ fixed). The critical values in Table 5 are directly comparable with the conservative critical values in Tables 2 and 3. As expected, the maximum permutational critical value is smaller than the corresponding bound for each sample size; for example, for $n = 40$, the maximum critical value is 2.107, whereas Table 3 yields the bound $S^*_{EP}(.025; 20) = 2.444$. Of course, distributions other than the normal may yield permutational critical values closer to the bound S^*_{EP}. Finding which distribution of $(X_1, \ldots, X_n)'$ yields permutational critical values that tend to be closest to S^*_{EP} is beyond the scope of this article.

4.2 Permutation t Tests Against Regression

As another application, consider the simple regression model $y_i = \beta x_i + e_i$, $i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are fixed regression constants, not all equal, and $e_1, \ldots, e_n$ are independent errors with possibly nonidentical distributions symmetric about 0. The classical Gaussian procedure for testing $H_o: \beta = \beta_0$ in this model relies on Student's statistic

$$T_n = (\hat\beta - \beta_0) \Big/ \left[(n-1)^{-1} \sum_{i=1}^n (y_i - \hat\beta x_i)^2 \Big/ \sum_{i=1}^n x_i^2 \right]^{1/2},$$

where $\hat\beta = (\sum_{i=1}^n x_i^2)^{-1} \sum_{i=1}^n x_i y_i$. Here again, the classical t distribution of T_n does not generally hold under H_0, and unbiasedness as well as Neyman structure arguments lead to conditioning on $|y_1 - \beta_0 x_1|, \ldots, |y_n - \beta_0 x_n|$. Setting

$$S_n = \sum_{i=1}^n |y_i - \beta_0 x_i| x_i U_i,$$

where $U_i = \text{sgn}(y_i - \beta_0 x_i)$, it can be shown that

$$S_n = \left[\frac{\sum_{i=1}^n x_i^2 \sum_{i=1}^n (y_i - \beta_0 x_i)^2}{\sum_{i=1}^n (y_i - \beta_0 x_i)^2 x_i^2} \right]^{1/2} \frac{T_n}{(n-1+T_n^2)^{1/2}},$$

which is again a monotonically increasing transformation; see Dufour and Hallin (1991). Conditional on $|y_1 - \beta_0 x_1|, \ldots, |y_n - \beta_0 x_n|$, S_n has a distribution symmetric about 0 and satisfies the conditions of Proposition 1, so that

$$P[T_n \geq z \,|\, |y_1 - \beta_0 x_1|, \ldots, |y_n - \beta_0 x_n|] \leq B^*_{EP}\left(\left[\frac{\sum_{i=1}^n x_i^2 \sum_{i=1}^n (y_i - \beta_0 x_i)^2}{\sum_{i=1}^n (y_i - \beta_0 x_i)^2 x_i^2} \right]^{1/2} \times z/(n-1+z^2)^{1/2}; n \right) \quad (10)$$

for $z > 0$. Critical values $t_n(\alpha)$ can be obtained from Table 3 and

$$t_n(\alpha) = (n-1)^{1/2} S^*_{EP}(\alpha; n) \div \left\{ \frac{\sum_{i=1}^n x_i^2 \sum_{i=1}^n (y_i - \beta_0 x_i)^2}{\sum_{i=1}^n (y_i - \beta_0 x_i)^2 x_i^2} - (S^*_{EP}(\alpha; n))^2 \right\}^{1/2} \quad (11)$$

(provided the denominator of $t_n(\alpha)$ is real and positive). Note that the bound in (10) is nonuniform here because it involves both the observations y_i and the regression constants x_i. Similarly, it is straightforward to see that the two-sided critical region $|T_n| \geq t_n(\alpha/2)$ has size not greater than α.

Table 5. Permutational One-Sample t Test (n = 20, 40). Distribution of Permutational Critical Values for Two-Sided Tests Based on S_n: 100 Normal Samples, $\alpha = .05$

Sample size	Minimum	Q(.25)	Q(.50)	Q(.75)	Maximum
$n = 20$	1.783	1.889	1.932	1.964	2.060
$n = 40$	1.817	1.898	1.944	1.988	2.107

NOTE: $Q(p)$ is the pth quantile of the empirical distribution of critical values. The permutational critical values were evaluated by a Monte Carlo simulation with 1,000 replications.

4.3 Permutation Tests Against First-Order Autocorrelation

Consider the first-order autoregressive model $X_t - \rho X_{t-1} = e_t$, $t = 0, 1, \ldots, n$, where $e_0, e_1, \ldots, e_n$ are independent disturbances with possibly nonidentical distributions symmetric about 0. Suppose that we wish to test $H_0 : \rho = 0$. Usual testing procedures are based on some properly normalized version of the first-order autocorrelation

$$r_1^{(n)} = \sum_{t=1}^{n} X_t X_{t-1} \Big/ \sum_{t=0}^{n} X_t^2.$$

Here again, unbiasedness and Neyman structure arguments suggest conditioning on $|X_0|, |X_1|, \ldots, |X_n|$. Define

$$S_n = \sum_{t=1}^{n} \left[\sum_{s=1}^{n} |X_s X_{s-1}|^2 \right]^{-1/2} |X_t| \, |X_{t-1}| U_t,$$

where $U_t = \text{sgn}(X_t X_{t-1})$ denotes the sign of $X_t X_{t-1}$. It can be shown that S_n is symmetrically distributed about 0 and satisfies the conditions of Proposition 1 (conditional on $|X_1|, \ldots, |X_n|$); see Dufour and Hallin (1990). Obviously,

$$r_1^{(n)} \geq z \quad \text{iff } S_n \geq \left[\sum_{t=1}^{n} |X_t X_{t-1}|^2 \right]^{-1/2} \left[\sum_{t=0}^{n} X_t^2 \right] z.$$

Accordingly, for all positive z,

$$P[r_1^{(n)} \geq z \,|\, |X_0|, \ldots, |X_n|] \leq B_{\text{EP}}^{*}\left(\left[\sum_{t=1}^{n} |X_t X_{t-1}|^2 \right]^{-1/2} \left[\sum_{t=0}^{n} X_t^2 \right] z; n \right), \quad (12)$$

and conservative critical values for $r_1^{(n)}$ are

$$r_1^{(n)}(\alpha) = \left[\sum_{t=1}^{n} |X_t X_{t-1}|^2 \right]^{-1/2} S_{\text{EP}}^{*}(\alpha; n) \Big/ \sum_{t=0}^{n} X_t^2, \quad (13)$$

irrespective of the sample size. The corresponding two-sided critical region is $|r_1^{(n)}| \geq r_1^{(n)}(\alpha/2)$. Here again, the innovations e_t need only satisfy a mild symmetry assumption (with respect to 0).

4.4 Linear Signed Rank Statistics

Various testing procedures in nonparametric inference based on ranks rely on linear signed rank statistics of the form

$$T_n = \sum_{i=1}^{n} a_n(R_i^+, i) \text{sgn}(X_i),$$

where $X_1, \ldots, X_n$ denotes a series of observation, R_i^+ is the rank of $|X_i|$ among $|X_1|, \ldots, |X_n|$, and $a_n(r, i)$ is a score function. For example, with $a_n(r, i) = a_n(r)$, T_n can be used to test whether independent symmetrically distributed observations $X_1, \ldots, X_n$ have median 0; see Hájek and Sidák (1967), Hollander and Wolfe (1974), or Husková (1984). This includes, in particular, paired sample comparisons; see Hájek (1969, pp. 109–112) and Pratt and Gibbons (1981, chap. 3). Other useful applications include tests against regression or trend alternatives, where $a_n(r, i) = c_i a_n(r)$ and $c_1, \ldots, c_n$ are known constants (Puri and Sen 1985, chap. 3), and tests against serial dependence (Dufour 1981; Hallin, Laforet, and Mélard 1989; Hallin and Puri 1991). Except for a few cases, such as the one-sample sign and Wilcoxon tests, for which simple formulas and tables are available, the null distribution of this type of statistic must be obtained either by approximations or by computationally expensive algorithms. In particular, asymptotic normal approximations and their rates of convergence have been extensively studied; see Albers, Bickel, and van Zwet (1976), Hájek and Sidák (1967), Husková (1970, 1984), Koul and Staudte (1972), Puri and Sen (1985, chap. 3), Puri and Ralescu (1982, 1984), Puri and Seoh (1984a, b, c, 1985), Puri and Wu (1986), Ralescu and Puri (1985), Seoh (1990), Seoh and Puri (1985), Seoh, Ralescu, and Puri (1985), Thompson, Govindarajulu, and Doksum (1967), and Wu (1986, 1987). Convergence to normality, however, requires restrictive regularity assumptions on the scores and regression constants. For general score functions and given sample size, the normal approximation may be highly inaccurate, and there is no general guarantee that tests based on such approximations will not reject too often. Similar remarks also apply to approximations based on asymptotic expansions, like Edgeworth expansions (Albers, et al. 1976; Fellingham and Stoker 1969; Field and Ronchetti 1990; Puri and Seoh 1984a,b; Thompson, Govindarajulu, and Doksum 1967). For several examples showing that Edgeworth expansions may underestimate the actual tail probabilities in the case of linear signed rank statistics, see Dufour and Hallin (1992a).

When $X_1, \ldots, X_n$, are likely to have nonhomogeneous distributions (as in the case of heteroscedastic observations), it appears again safer to condition on $|X_1|, \ldots, |X_n|$ or an appropriate function of the latter, such as the rank vector $(R_1^+, \ldots, R_n^+)$. Then, provided that $X_1, \ldots, X_n$ are independent with distributions symmetric about 0, we straightforwardly obtain

$$P[T_n \geq z \,|\, R_1^+, \ldots, R_n^+] \leq B_{\text{EP}}^{*}(z/\sigma_n; n) \leq B_{\text{EP}}^{*}(z/\sigma_n; n) \quad (14)$$

for $z > 0$, where $\sigma_n = [\sum_{i=1}^{n} (a_n(R_i^+, i))^2]^{1/2}$, hence conservative (one-sided) critical values of the form $T_n(\alpha) = S_{\text{EP}}^{*}(\alpha; n)\sigma_n$.

Other explicit bounds (of the exponential, Chebyshev, and Berry–Esséen types) have been derived for this situation by Dufour and Hallin (1992a). Table 6 provides a comparison for the following two statistics:

$$T_n^{(1)} = \sum_{i=1}^{n} \text{sgn}(X_i) \cos\left[\pi\left(1 + \frac{R_i^+}{n+1} \right) \right] \div \left[\sum_{i=1}^{n} \cos^2\left(1 + \frac{i}{n+1} \right) \right]^{1/2},$$

$$T_n^{(2)} = \sum_{i=1}^{n} \text{sgn}(X_i) i^2 / [n(n+1)(2n+1)/6]^{1/2}.$$

$T_n^{(1)}$ is the optimal one-sample linear rank statistic against location shift when the observations are independent with a Cauchy distribution (Philippou 1984), whereas $T_n^{(2)}$ is optimal against quadratic trend under double-exponential densities. Both $T_n^{(1)}$ and $T_n^{(2)}$ are exactly standardized under the null hypothesis of independent observations with symmetric (possibly nonidentical) distributions. The best of the exponential, Chebyshev, and Berry–Esséen bounds proposed in Dufour and Hallin (1992a) are provided, for $n = 25$ and $n = 50$, along with the improved Eaton bounds (B_{EP}^{*} and B_{EP})

Table 6. Signed Rank Statistics $T_n^{(1)}$ and $T_n^{(2)}$

	n = 25					n = 50				
z/σ_n =	1.0	1.5	2.0	2.5	3.0	1.0	1.5	2.0	2.5	3.0
B_{Ed}	.6915	.3085	.1057	.0287	.0062	.6915	.3085	.1057	.0287	.0062
B^*_{EP}	.5000	.2222	.0819	.0218	.0042	.5000	.2222	.0834	.0230	.0048
B_{EP}	.5000	.2222	.0848	.0242	.0054	.5000	.2222	.0848	.0242	.0054
$T_n^{(1)}$ Best nonuniform	.3512	.2222	.0900	.0268	.0052	.2944	.2026	.0919	.0289	.0065
(Type)	(BE)	(C2)	(C4)	(C8)	(C12)	(BE)	(BE)	(C4)	(C6)	(C10)
Winner	BE	B^*_{EP} = C2	B^*_{EP}	B^*_{E}	C12	BE	BE	B^*_{EP}	B^*_{EP}	B^*_{EP}
Tail area	.1604	.0658	.0220	.0051	.0008	.1579	.0655	.0214	.0058	.0011
$T_n^{(2)}$ Best nonuniform	.4107	.2222	.0870	.0219	.0032	.3379	.2222	.0903	.0275	.0055
(Type)	(BE)	(C2)	(C4)	(C8)	(C12)	(BE)	(C2)	(C4)	(C6)	(C10)
Winner	BE	B^*_{EP} = C2	B^*_{EP}	C8	C12	BE	C2	B^*_{EP}	B^*_{EP}	B^*_{EP}
Tail area	.1625	.0691	.0219	.0045	.0005	.1595	.0655	.0216	.0050	.0009

NOTE: Comparisons between Edelman's bound B_{Ed}, improved Eaton bounds B_{EP} and B^*_{EP}, and best nonuniform bound among exponential (Exp), Chebyshev (Ci, where *i* indicates the order of the optimal Chebyshev inequality used) and Berry–Esséen (BE) bounds; n = 25 and 50. Actual tail areas were evaluated by a Monte Carlo simulation with 10,000 replications.

and Edelman's B_{Ed}. We see that the improved Eaton bounds provide the tightest bounds in about one-half of the cases considered.

APPENDIX: PROOFS

Proof of Lemma 1. Setting $D = \{t \mid \ddot{f}_l(t) \neq \ddot{f}_r(t)\}$ and $T_\Delta = \{t > 0 \mid t + \Delta \in D \text{ or } -t + \Delta \in D\}$, for $\Delta \geq 0$, the proof is analogous to the one of Eaton's (1974) lemma 1, starting with $0 \leq t \notin T_\Delta$.

Proof of Proposition 1. From inequalities (3.4) and (3.5) of Eaton (1974), we have

$$P\left[\left|\sum_{i=1}^{n} a_i Y_i\right| \geq y\right] \leq E[f_c(T_n)]/(y-c)^3$$

for $0 \leq c < y$, where $T_n = n^{-1/2} \sum_{i=1}^{n} U_i$ and $U_1, \ldots, U_n$ are independent random variables with $P[U_i = 1] = P[U_i = -1] = .5$, $i = 1, \ldots, n$. Let

$$B(y; c, n) = (.5)E[f_c(T_n)]/(y-c)^3, \qquad 0 \leq c < y,$$

and $V_i = (U_i + 1)/2$, $i = 1, \ldots, n$. It is clear that $V_1, \ldots, V_n$ are independent random variables with $P[V_i = 0] = P[V_i = 1] = .5$, $i = 1, \ldots, n$, so that the variable $B_n = \sum_{i=1}^{n} V_i$ has a binomial distribution $B(n, .5)$. Because $T_n = (n/4)^{-1/2}(B_n - (n/2))$, we have

$$E[f_c(T_n)] = (.5)^n \sum_{m=0}^{n} \binom{n}{m} f_c[(n/4)^{-1/2}(m - (n/2))] \Big/ (y-c)^3$$

for $0 \leq c < y$; hence

$$P\left[\left|\sum_{i=1}^{n} a_i Y_i\right| \geq y\right] \leq 2 \inf_{0 \leq c < y} B(y; c, n) \equiv 2B_E(y; n).$$

Further, by Chebyshev's inequality,

$$P\left[\left|\sum_{i=1}^{n} a_i Y_i\right| \geq y\right] \leq E\left[\left(\sum_{i=1}^{n} a_i Y_i\right)^2\right] \Big/ y^2$$

$$= \sum_{i=1}^{n} a_i^2 E(Y_i^2) \Big/ y^2 \leq y^{-2}, \qquad y > 0, \quad \text{(A.1)}$$

because $E(Y_i^2) \leq 1$ and $\sum_{i=1}^{n} a_i^2 = 1$. The first inequality in (4) follows.

To get the second inequality, we observe that the function $f_c(x)$ is symmetric (about 0) and that $\ddot{f}_c$ exists everywhere and admits nondecreasing left and right derivatives, $(\dddot{f}_c)_l$ and $(\dddot{f}_c)_r$, such that $(\dddot{f}_c)_l = (\dddot{f}_c)_r$ everywhere except at $x = \pm c$. By Lemma 1, $f \in \mathcal{F}_1$. Eaton's (1974) propositions 1 and 2 consequently hold, and $E[f_c(T_n)] \leq E[f_c(T_{n+1})] \leq E[f_c(Z)]$, where Z stands for a standard normal variable. Consequently, $B^*_{\text{EP}}(y; n) \leq B^*_{\text{EP}}(y; n+1)$ for $y > 0$. Further, for $c \geq 0$,

$$E[f_c(Z)] = 2\int_c^\infty (z-c)^3 \phi(z)\,dz$$

$$= 2\left\{\int_c^\infty z^3\phi(z)\,dz - 3c\int_c^\infty y^2\phi(z)\,dz + 3c^2\int_c^\infty z\phi(z)\,dz - c^3\int_c^\infty \phi(z)\,dz\right\}$$

$$= 2\{-c^3H_1 + 3c^2H_2 - 3cH_3 + H_4\},$$

with

$$H_1 = \int_c^\infty \phi(z)\,dz = 1 - \Phi(c), \quad H_2 = \int_c^\infty z\phi(z)\,dz = -\int_c^\infty \dot{\phi}(z)\,dz = \phi(c),$$

$$H_3 = \int_c^\infty z^2\phi(z)\,dz = \int_c^\infty [\ddot{\phi}(z) + \phi(z)]\,dz = -\dot{\phi}(c) + 1 - \Phi(c) = c\phi(c) + 1 - \Phi(c),$$

$$H_4 = \int_c^\infty z^3\phi(z)\,dz = \int_c^\infty [2z\phi(z) - \dot{\phi}(z) - \dddot{\phi}(z)]\,dz = 2\phi(c) + \phi(c) + \phi(c)(c^2 - 1) = (c^2 + 2)\phi(c);$$

hence $E[f_c(Z)] = 2\{\phi(c)(2 + c^2) - (1 - \Phi(c))(c^3 + 3c)\}$. Thus

$$2B_E(y; n) \leq \inf_{0 \leq c < y} \{E[f_c(Z)]/(y-c)^3\} \equiv 2B_E(y). \quad \text{(A.2)}$$

The second inequality in (4) follows on combining (A.2) and (A.1).

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