Coefficients of determination [*]

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Contents

1. Coefficient of determination: R^2

Let $y = X\beta + \varepsilon$ be a model that satisfies the assumptions of the classical linear model, where y and ε are $T \times 1$ vectors, X is a $T \times k$ matrix and β is $k \times 1$ coefficient vector. We wish to characterize to which extent the variables included in X (excluding the constant, if there is one) explain y.

A first method consists in computing R^2, the "coefficient of determination", or $R = \sqrt{R^2}$, the "coefficient of multiple correlation". Let

$$\hat{y} = X\hat{\beta}\ ,\ \hat{\varepsilon} = y - \hat{y}\ ,\ \bar{y} = \sum_{t=1}^{T} y_t/T = i'y/T\ , \tag{1.1}$$

$$i = (1, 1, \ldots, 1)' \text{ the unit vector of dimension } T, \tag{1.2}$$

$$SST = \sum_{t=1}^{T} (y_t - \bar{y})^2 = (y - i\bar{y})'(y - i\bar{y})\ ,\ \text{(total sum of squares)} \tag{1.3}$$

$$SSR = \sum_{t=1}^{T} (\hat{y}_t - \bar{y})^2 = (\hat{y} - i\bar{y})'(\hat{y} - i\bar{y})\ ,\ \text{(regression sum of squares)} \tag{1.4}$$

$$SSE = \sum_{t=1}^{T} (y_t - \hat{y}_t)^2 = (y - \hat{y})'(y - \hat{y}) = \hat{\varepsilon}'\hat{\varepsilon}\ ,\ \text{(error sum of squares).} \tag{1.5}$$

We can then define "variance estimators" as follows:

$$\hat{V}(y) = SST/T\ , \tag{1.6}$$

$$\hat{V}(\hat{y}) = SSR/T\ , \tag{1.7}$$

$$\hat{V}(\varepsilon) = SSE/T\ . \tag{1.8}$$

1.1 Definition $R^2 = 1 - \left(\hat{V}(\varepsilon)/\hat{V}(y)\right) = 1 - (SSE/SST)\ .$

1.2 Proposition $R^2 \leq 1\ .$

PROOF This result is immediate on observing that $SSE/SST \geq 0$. □

1.3 Lemma $y'y = \hat{y}'\hat{y} + \hat{\varepsilon}'\hat{\varepsilon}$.

PROOF We have

$$y = \hat{y} + \hat{\varepsilon} \text{ and } \hat{y}'\hat{\varepsilon} = \hat{\varepsilon}'\hat{y} = 0, \tag{1.9}$$

hence

$$y'y = (\hat{y} + \hat{\varepsilon})'(\hat{y} + \hat{\varepsilon}) = \hat{y}'\hat{y} + \hat{y}'\hat{\varepsilon} + \hat{\varepsilon}'\hat{y} + \hat{\varepsilon}'\hat{\varepsilon} = \hat{y}'\hat{y} + \hat{\varepsilon}'\hat{\varepsilon} .$$

□

1.4 Proposition *If one of the regressors is a constant, then*

$$\begin{aligned} SST &= SSR + SSE, \\ \hat{V}(y) &= \hat{V}(\hat{y}) + \hat{V}(\varepsilon) . \end{aligned}$$

PROOF Let $A = I_T - i(i'i)^{-1} i' = I_T - \frac{1}{T} ii'$. Then, $A'A = A$ and

$$Ay = \left[I_T - \frac{1}{T} ii' \right] y = y - i\bar{y}.$$

If one of the regressors is a constant, we have

$$i'\hat{\varepsilon} = \sum_{t=1}^{T} \hat{\varepsilon}_t = 0$$

hence

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T} \hat{y}_t &= \frac{1}{T} i'\hat{y} = \frac{1}{T} i'(y - \hat{\varepsilon}) = \frac{1}{T} i'y = \bar{y} , \\ A\hat{\varepsilon} &= \hat{\varepsilon} - \frac{1}{T} ii'\hat{\varepsilon} = \hat{\varepsilon} , \\ A\hat{y} &= \hat{y} - \frac{1}{T} ii'\hat{y} = \hat{y} - i\bar{y} , \end{aligned}$$

and, using the fact that $A\hat{\varepsilon} = \hat{\varepsilon}$ and $\hat{y}'\hat{\varepsilon} = 0$,

$$\begin{aligned} SST &= (y - i\bar{y})'(y - i\bar{y}) = y'A'Ay = y'Ay \\ &= (\hat{y} + \hat{\varepsilon})' A (\hat{y} + \hat{\varepsilon}) \\ &= \hat{y}'A\hat{y} + \hat{y}'A\hat{\varepsilon} + \hat{y}'A\hat{\varepsilon} + \hat{\varepsilon}'A\hat{\varepsilon} \\ &= \hat{y}'A\hat{y} + \hat{\varepsilon}'\hat{\varepsilon} \\ &= (A\hat{y})'(A\hat{y}) + \hat{\varepsilon}'\hat{\varepsilon} = SSR + SSE\,. \end{aligned}$$

□

1.5 Proposition *If one of the regressors is a constant,*

$$R^2 = \frac{\hat{V}(\hat{y})}{\hat{V}(y)} = \frac{SSR}{SST} \quad \textit{and} \quad 0 \leq R^2 \leq 1\,.$$

PROOF By the definition of R^2, we have $R^2 \leq 1$ and

$$R^2 = 1 - \frac{\hat{V}(\varepsilon)}{\hat{V}(y)} = \frac{\hat{V}(y) - \hat{V}(\varepsilon)}{\hat{V}(y)} = \frac{\hat{V}(\hat{y})}{\hat{V}(y)} = \frac{SSR}{SST}$$

hence $R^2 \geq 0$. □

1.6 Proposition *If one of the regressors is a constant, the empirical correlation between y and $\hat{y}$ is non-negative and equal to $\sqrt{R^2}$.*

PROOF The empirical correlation between y and $\hat{y}$ is defined by

$$\hat{\rho}(y,\hat{y}) = \frac{\hat{C}(y,\hat{y})}{\left[\hat{V}(y)\hat{V}(\hat{y})\right]^{1/2}}$$

where

$$\hat{C}(y,\hat{y}) = \frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})(\hat{y}_t - \bar{y}) = \frac{1}{T}(Ay)'(A\hat{y})$$

and $A = I_T - \frac{1}{T} ii'$. Since one of the regressors is a constant,

$$A\hat{\varepsilon} = \hat{\varepsilon} \; , \; Ay = A\hat{y} + \hat{\varepsilon} \; , \; \hat{\varepsilon}' (A\hat{y}) = \hat{\varepsilon}'\hat{y} = 0$$

and

$$\begin{aligned} \hat{C}(y, \hat{y}) &= \frac{1}{T}(A\hat{y} + \hat{\varepsilon})'(A\hat{y}) = \frac{1}{T}(A\hat{y})'(A\hat{y}) = \hat{V}(\hat{y}) \; , \\ \hat{\rho}(y, \hat{y}) &= \frac{\hat{V}(\hat{y})}{\left[\hat{V}(y)\hat{V}(\hat{y})\right]^{1/2}} = \left[\frac{\hat{V}(\hat{y})}{\hat{V}(y)}\right]^{1/2} = \sqrt{R^2} \geq 0 \; . \end{aligned}$$

□

2. Significance tests and R^2

2.1. Relation of R^2 with a Fisher statistic

R^2 is descriptive statistic which measures the proportion of the “variance” of the dependent variable y explained by suggested explanatory variables (excluding the constant). However, R^2 can be related to a significance test (under the assumptions of the Gaussian classical linear model).

Consider the model

$$y_t = \beta_1 + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} + \varepsilon_t \ , \ t = 1, \ldots, T.$$

We wish to test the hypothesis that none of these variables (excluding the constant) should appear in the equation:

$$H_0 : \beta_2 = \beta_3 = \cdots = \beta_k = 0 \ .$$

The Fisher statistic for H_0 is

$$F = \frac{(S_\omega - S_\Omega)/q}{S_\Omega/(T-k)} \sim F(q, T-k)$$

where $q = k-1$, S_Ω is the error sum of squares from the estimation of the unconstrained model

$$\Omega : y = X\beta + \varepsilon \ ,$$

where $X = [i, X_2, \ldots, X_k]$ and S_ω s the error sum of squares from the estimation of the constrained model

$$\omega : y = i\beta_1 + \varepsilon \ ,$$

where $i = (1, 1, \ldots, 1)'$. We see easily that

$$\begin{aligned} S_\Omega &= \left(y - X\hat{\beta}\right)' \left(y - X\hat{\beta}\right) = SSE \ , \\ \hat{\beta}_1 &= (i'i)^{-1} i'y = \frac{1}{T}\sum_{t=1}^{T} y_t = \bar{y} \ , \ \text{(under } \omega) \end{aligned}$$

$$S_\omega = (y - i\bar{y})'(y - i\bar{y}) = SST$$

and

$$\begin{aligned} F &= \frac{(SST - SSE)/(k-1)}{SSE/(T-k)} = \frac{\left[1 - \frac{SSE}{SST}\right]/(k-1)}{\frac{SSE}{SST}/(T-k)} \\ &= \frac{R^2/(k-1)}{(1-R^2)/(T-k)} \sim F(k-1, T-k) \ . \end{aligned}$$

As R^2 increases, F increases.

2.2. General relation between R^2 and Fisher tests

Consider the general linear hypothesis

$$H_0 : C\beta = r$$

where $C : q \times k$, $\beta : k \times 1$, $r : q \times 1$ and $\text{rank}(C) = q$. The values of R^2 for the constrained and unconstrained models are respectively:

$$R_0^2 = 1 - \frac{S_\omega}{SST}\ ,\ R_1^2 = 1 - \frac{S_\Omega}{SST},$$

hence

$$S_\omega = \left(1 - R_0^2\right) SST\ ,\ S_\Omega = \left(1 - R_1^2\right) SST\ .$$

The Fisher statistic for testing H_0 may thus be written

$$\begin{aligned} F &= \frac{(S_\omega - S_\Omega)/q}{S_\Omega/(T-k)} = \frac{\left(R_1^2 - R_0^2\right)/q}{\left(1 - R_1^2\right)/(T-k)} \\ &= \left(\frac{T-k}{q}\right)\frac{R_1^2 - R_0^2}{1 - R_1^2}. \end{aligned}$$

If $R_1^2 - R_0^2$ is large, we tend to reject H_0. If $H_0 : \beta_2 = \beta_3 = \cdots = \beta_k = 0$, then

$$q = k - 1\ ,\ S_\omega = SST\ ,\ R_0^2 = 0$$

and the formula for F above gets reduced of the one given in section 2.1.

3. Uncentered coefficient of determination: $\widetilde{R}^2$

Since R^2 can take negative values when the model does not contain a constant, R^2 has little meaning in this case. In such situations, we can instead use a coefficient where the values of y_t are not centered around the mean.

3.1 Definition $\widetilde{R}^2 = 1 - \left(\hat{\varepsilon}'\hat{\varepsilon}/y'y\right)$.

$\tilde{R}^2$ is called the "uncentered coefficient of determination" on "uncentered R^2" and $\tilde{R} = \sqrt{\tilde{R}^2}$ the "uncentered coefficient of multiple correlation".

3.2 Proposition $0 \leq \tilde{R}^2 \leq 1$.

PROOF This follows directly from Lemma **1.3**: $y'y = \hat{y}'\hat{y} + \hat{\varepsilon}'\hat{\varepsilon}$. □

4. Adjusted coefficient of determination: $\bar{R}^2$

4.1. Definition and basic properties

An unattractive property of the R^2 coefficient comes form the fact that R^2 cannot decrease when explanatory variables are added to the model, even if these have no relevance. Consequently, choosing to maximize R^2 can be misleading. It seems desirable to penalize models that contain too many variables.

Since

$$R^2 = 1 - \frac{\hat{V}(\varepsilon)}{\hat{V}(y)} ,$$

where

$$\hat{V}(\varepsilon) = \frac{SSE}{T} = \frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_t^2 \, , \; \hat{V}(y) = \frac{SST}{T} = \frac{1}{T}\sum_{t=1}^{T} (y_t - \bar{y})^2 \, ,$$

Theil (1961, p. 213) suggested to replace $\hat{V}(\varepsilon)$ and $\hat{V}(y)$ by "unbiased estimators":

$$s^2 = \frac{SSE}{T-k} = \frac{1}{T-k}\sum_{t=1}^{T} \hat{\varepsilon}_t^2 \, ,$$
$$s_y^2 = \frac{SST}{T-1} = \frac{1}{T-1}\sum_{t=1}^{T} (y_t - \bar{y})^2 \, .$$

4.1 Definition *R^2 adjusted for degrees of freedom is defined by*

$$\bar{R}^2 = 1 - \frac{s^2}{s_y^2} = 1 - \frac{T-1}{T-k}\left(\frac{SSE}{SST}\right) .$$

4.2 Proposition $\bar{R}^2 = 1 - \frac{T-1}{T-k}\left(1 - R^2\right) = R^2 - \frac{k-1}{T-k}\left(1 - R^2\right)$.

PROOF

$$\bar{R}^2 = 1 - \frac{T-1}{T-k}\left(\frac{SSE}{SST}\right) = 1 - \frac{T-1}{T-k}\left(1 - R^2\right)$$

$$= 1 - \frac{T-k+k-1}{T-k}\left(1-R^2\right) = 1 - \left(1 + \frac{k-1}{T-k}\right)\left(1-R^2\right)$$
$$= 1 - \left(1-R^2\right) - \frac{k-1}{T-k}\left(1-R^2\right) = R^2 - \frac{k-1}{T-k}\left(1-R^2\right) . \quad Q.E.D.$$

□

4.3 Proposition $\overline{R}^2 \leq R^2 \leq 1$.

PROOF The result follows from the fact that $1-R^2 \geq 0$ and **(4.2)**. □

4.4 Proposition $\overline{R}^2 = R^2 \quad iff \quad (k=1 \quad or \quad R^2 = 1)$.

4.5 Proposition $\overline{R}^2 \leq 0 \quad iff\, R^2 \leq \frac{k-1}{T-1}$.

$\overline{R}^2$ can be negative even if $R^2 \geq 0$. If the number of explanatory variables is increased, R^2 and k both increase, so that $\overline{R}^2$ can increase or decrease.

4.6 Remark When several models are compared on the basis of R^2 or $\overline{R}^2$, it is important to have the same dependent variable. When the dependent variable (y) is the same, maximizing $\overline{R}^2$ is equivalent to minimizing the standard error of the regression

$$s = \left[\frac{1}{T-k}\sum_{t=1}^{T}\hat{\varepsilon}_t^2\right]^{1/2} .$$

4.2. Criterion for $\overline{R}^2$ increase through the omission of an explanatory variable

Consider the two models:

$$y_t = \beta_1 X_{t1} + \cdots + \beta_{k-1} X_{t(k-1)} + \varepsilon_t \quad , \; t = 1, \ldots, T, \tag{4.1}$$

$$y_t = \beta_1 X_{t1}, + \cdots + \beta_{k-1} X_{t(k-1)} + \beta_k X_{tk} + \varepsilon_t \quad , t = 1, \ldots, T. \tag{4.2}$$

We can then show that the value of $\overline{R}^2$ associated with the restricted model (4.1) is larger than the one of model (4.2) if the t statistic for testing $\beta_k = 0$ is smaller than 1 (in absolute value).

4.7 Proposition *If $\overline{R}^2_{k-1}$ and $\overline{R}^2_k$ are the values of $\overline{R}^2$ for models (4.1) and (4.2), then*

$$\overline{R}^2_k - \overline{R}^2_{k-1} = \frac{\left(1 - \overline{R}^2_k\right)}{(T - k + 1)} \left(t_k^2 - 1\right) \tag{4.3}$$

where t_k is the Student t statistic for testing $\beta_k = 0$ in model (4.2), and

$$\overline{R}^2_k \leq \overline{R}^2_{k-1} \quad \text{iff} \quad t_k^2 \leq 1 \quad \text{iff} \quad |t_k| \leq 1\ .$$

If furthermore $\overline{R}^2_k < 1$, then

$$\overline{R}^2_k \lesseqqgtr \overline{R}^2_{k-1} \quad \text{iff} \quad |t_k| \lesseqqgtr 1\ .$$

PROOF By definition,

$$\overline{R}^2_k = 1 - \frac{s_k^2}{s_y^2} \quad \text{and} \quad \overline{R}^2_{k-1} = 1 - \frac{s_{k-1}^2}{s_y^2}$$

where $s_k^2 = SS_k / (T - k)$ and $s_{k-1}^2 = SS_{k-1} / (T - k + 1)$. SS_k and SS_{k-1} are the sums of squared errors for the models with k and $k - 1$ explanatory variables. Since t_k^2 is the Fisher statistic for testing $\beta_k = 0$, we have

$$\begin{aligned} t_k^2 &= \frac{(SS_{k-1} - SS_k)}{SS_k / (T - k)} \\ &= \frac{\left[(T - k + 1)\, s_{k-1}^2 - (T - k)\, s_k^2\right]}{s_k^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(T-k+1)\left(1-\overline{R}^2_{k-1}\right)-(T-k)\left(1-\overline{R}^2_k\right)}{1-\overline{R}^2_k} \\
&= (T-k+1)\left(\frac{1-\overline{R}^2_{k-1}}{1-\overline{R}^2_k}\right)-(T-k)
\end{aligned}
$$

for $s^2_{k-1}=s^2_y\left(1-\overline{R}^2_{k-1}\right)$ and $s^2_k=s^2_y\left(1-\overline{R}^2_k\right)$. Consequently,

$$
1-\overline{R}^2_{k-1}=\left(1-\overline{R}^2_k\right)\frac{\left[t^2_k+(T-k)\right]}{T-k+1}
$$

and

$$
\begin{aligned}
\overline{R}^2_k-\overline{R}^2_{k-1} &= \left(1-\overline{R}^2_{k-1}\right)-\left(1-\overline{R}^2_k\right) \\
&= \left(1-\overline{R}^2_k\right)\left[\frac{t^2_k+(T-k)}{T-k+1}-1\right] \\
&= \left(1-\overline{R}^2_k\right)\left[\frac{t^2_k-1}{T-k+1}\right].
\end{aligned}
$$

□

4.3. Generalized criterion for $\overline{R}^2$ increase through the imposition of linear constraints

We will now study when the imposition of q linearly independent constraints

$$
H_0 : C\beta = r
$$

will raise or decrease $\overline{R}^2$, where $C : q\times k$, $r : q\times 1$ and $\text{rank}(C)=q$. Let $\overline{R}^2_{H_0}$ and $\overline{R}^2$ be the values of $\overline{R}^2$ for the constrained (by H_0) and unconstrained models, similarly, s^2_0 and s^2 are the values of the corresponding unbiased estimators of the error variance.

4.8 Proposition *Let F be the Fisher statistic for testing H_0. Then*

$$s_0^2 - s^2 = \frac{qs^2}{T-k+q}(F-1)$$

and

$$s_0^2 \lesseqqgtr s^2 \quad \textit{iff} \quad F \lesseqqgtr 1\,.$$

PROOF If SS_0 and SS are the sum of squared errors for the constrained and unconstrained models, we have:

$$s_0^2 = \frac{SS_0}{T-k+q} \quad \text{and} \quad s^2 = \frac{SS}{T-k}\,.$$

The F statistic may then be written

$$\begin{aligned} F &= \frac{(SS_0 - SS)/q}{SS/(T-k)} \\ &= \frac{\left[(T-k+q)\,s_0^2 - (T-k)\,s^2\right]}{qs^2} = \frac{T-k+q}{q}\left(\frac{s_0^2}{s^2}\right) - \frac{T-k}{q} \end{aligned}$$

hence

$$\begin{aligned} s_0^2 &= s^2\frac{[qF + (T-k)]}{(T-k)+q}\,, \\ s_0^2 - s^2 &= s^2\frac{q\,(F-1)}{(T-k)+q}\,, \end{aligned}$$

and

$$s_0^2 \lesseqqgtr s^2 \quad \text{iff} \quad F \lesseqqgtr 1\,.$$

□

4.9 Proposition *Let F be the Fisher statistic for testing H_0. Then*

$$\overline{R}^2 - \overline{R}^2_{H_0} = \frac{q\left(1-\overline{R}^2\right)}{T-k+q}(F-1)$$

and

$$\overline{R}^2_{H_0} \gtreqqless \overline{R}^2 \quad \textit{iff} \quad F \lesseqqgtr 1 .$$

PROOF By definition,

$$\overline{R}^2_{H_0} = 1 - \frac{s_0^2}{s_y^2} \, , \, \overline{R}^2 = 1 - \frac{s^2}{s_y^2} \, .$$

Thus,

$$\begin{aligned} \overline{R}^2 - \overline{R}^2_{H_0} &= \frac{s^2 - s_0^2}{s_y^2} \\ &= \frac{q}{T-k+q}\left(\frac{s^2}{s_y^2}\right)(F-1) \\ &= \frac{q\left(1-\overline{R}^2\right)}{T-k+q}(F-1) \end{aligned}$$

hence

$$\overline{R}^2_{H_0} \gtreqqless \overline{R}^2 \quad \text{iff} \quad F \lesseqqgtr 1 .$$

□

On taking $q = 1$, we get property (4.3). If we test an hypothesis of the type

$$H_0 : \beta_k = \beta_{k+1} = \cdots = \beta_{k+l} = 0 \, ,$$

it is possible that $F > 1$, while all the statistics $|t_i|$, $i = k, \ldots, k+l$ are smaller than 1. This means that $\overline{R}^2$ increases when we omit one explanatory variable at a time, but decreases when they are all excluded from the regression. Further, it is also possible that $F < 1$, but $|t_i| > 1$ for all i: $\overline{R}^2$ increases when all the explanatory variables are simultaneously excluded, but decreases when only one is excluded.

5. Notes on bibliography

The notion of $\bar{R}^2$ was proposed by Theil (1961, p. 213). Several authors have presented detailed discussions of the different concepts of multiple correlation: for example, Theil (1971, Chap. 4), Schmidt (1976) and Maddala (1977, Sections 8.1, 8.2, 8.3, 8.9). The $\bar{R}^2$ concept is criticized by Pesaran (1974). The mean and bias of R^2 were studied by Cramer (1987) in the Gaussian case, and by Srivastava, Srivastava and Ullah (1995) in some non-Gaussian cases.

6. Chronological list of references

1. Theil (1961, p. 213) _ The $\overline{R}^2$ nation was proposed in this book.
2. Theil (1971, Chap. 4) _ Detailed discussion of R^2, $\overline{R}^2$ and partial correlation.
3. Pesaran (1974) _ Critique of $\overline{R}^2$.
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5. Maddala (1977, Sections 8.1, 8.2, 8.3, 8.9) _ Discussion of $R^2$and $\overline{R}^2$along with their relation with hypothesis tests.
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