Distribution and quantile functions *

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1. Monotonic functions

In this section, we review some properties of monotonic functions, which are important to study distribution and quantile functions.

1.1. Definitions

1.1 Definition MONOTONIC FUNCTION. *Let D a non-empty subset of $\mathbb{R}$, $f : D \to E$, where E is a non-empty subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, and let I be a non-empty subset of D.*

(a) *f is nondecreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \quad \forall x_1, x_2 \in I .$$

(b) *f is nonincreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2), \quad \forall x_1, x_2 \in I .$$

(c) *f is strictly increasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \forall x_1, x_2 \in I .$$

(d) *f is strictly decreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \quad \forall x_1, x_2 \in I .$$

(e) *f is monotonic on I iff f is nondecreasing, nonincreasing, increasing or decreasing.*

(f) *f is strictly monotonic on I iff f is strictly increasing or decreasing.*

1.2 Definition MONOTONICITY AT A POINT. *Let D a non-empty subset of $\mathbb{R}$, $f : D \to E$, where E is a non-empty subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, and let $x \in D$.*

(a) *f is nondecreasing at x iff there is an open neighborhood I of x such that*

$$x_1 < x \Rightarrow f(x_1) \le f(x), \quad \forall x_1 \in I \cap D ,$$

$$\text{and } \; x < x_2 \Rightarrow f(x) \le f(x_2), \quad \forall x_2 \in I \cap D ;$$

(b) *f is nonincreasing at x iff there is an open neighborhood I of x such that*

$$x_1 < x \Rightarrow f(x_1) \ge f(x), \quad \forall x_1 \in I \cap D ,$$

$$\text{and } \; x < x_2 \Rightarrow f(x) \ge f(x_2), \quad \forall x_2 \in I \cap D ;$$

(c) *f is strictly increasing at x iff there is an open neighborhood I of x such that*

$$x_1 < x \Rightarrow f(x_1) < f(x), \quad \forall x_1 \in I \cap D,$$

$$\text{and } x < x_2 \Rightarrow f(x) < f(x_2), \quad \forall x_2 \in I \cap D;$$

(d) *f is strictly decreasing on I iff there is an open neighborhood I of x such that*

$$x_1 < x \Rightarrow f(x_1) > f(x), \quad \forall x_1 \in I \cap D,$$

$$\text{and } x < x_2 \Rightarrow f(x) > f(x_2), \quad \forall x_2 \in I \cap D.$$

(e) *f is monotonic at x iff f is nondecreasing, nonincreasing, increasing or decreasing at x.*

(f) *f is strictly monotonic at x iff f is strictly increasing or decreasing at x.*

1.3 Remark It is clear that:

(a) an increasing function is also nondecreasing;

(b) a decreasing function is also nonincreasing;

(c) if f is nondecreasing (alt., strictly increasing), the function

$$g(x) = -f(x)$$

is nonincreasing (alt., strictly decreasing) on I, and the function

$$h(x) = -f(-x)$$

is nondecreasing on $I_1 = \{x : -x \in I\}$..

1.2. Continuity properties of monotonic functions

1.4 Proposition LIMITS OF MONOTONIC FUNCTIONS. *Let $I = (a,b) \subseteq \mathbb{R}$, where $-\infty \leq a < b \leq \infty$, and $f : I \to \mathbb{R}$ be a nondecreasing function on I. Then the function f has the following properties.*

(a) *For each $x \in (a,b)$, set*

$$\begin{aligned} f(x_+) &= \lim_{\delta \downarrow 0} \left\{ \inf_{x<y<x+\delta} f(y) \right\},\ f\left(x^+\right) = \lim_{\delta \downarrow 0} \left\{ \sup_{x<y<x+\delta} f(y) \right\}, \\ f(x_-) &= \lim_{\delta \downarrow 0} \left\{ \inf_{x-\delta<y<x} f(y) \right\},\ f\left(x^-\right) = \lim_{\delta \downarrow 0} \left\{ \sup_{x-\delta<y<x} f(y) \right\}. \end{aligned}$$

Then, the four limits $f(x_+)$, $f(x^+)$, $f(x_-)$ *and* $f(x^-)$ *are finite and, for any* $\delta > 0$ *such that* $[x-\delta,\, x+\delta] \subseteq (a,b)$,

$$f(x-\delta) \le f(x_-) \le f\left(x^-\right) \le f(x) \le f(x_+) \le f\left(x^+\right) \le f(x+\delta).$$

(b) *For each* $x \in (a,b)$, *we have*

$$f(x_+) = f\left(x^+\right)\ ,\ f(x_-) = f\left(x^-\right)\ ,$$

and the function $f(x)$ *has finite unilateral limits:*

$$f(x+) \equiv \lim_{y \downarrow x} f(y) = f(x_+) = f\left(x^+\right)\ ,\ f(x-) \equiv \lim_{y \uparrow x} f(y) = f(x_-) = f\left(x^-\right)\ .$$

(c) *For each* $x \in (a,b)$,

$$\sup_{a<y<x} f(y) = f(x-) \le f(x) \le f(x+) = \inf_{x<y<b} f(y)\ .$$

(d) *If* $a < x < y < b$, *then*

$$f(x+) \le f(y-)\ .$$

(e) *If* $a = -\infty$, *the function* $f(x)$ *has a limit in the extended real numbers* $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ *as* $x \to -\infty$,

$$-\infty \le f(-\infty) \equiv \lim_{x \to -\infty} f(x) < \infty$$

and, if $b = \infty$, *the function* $f(x)$ *has a limit in* $\overline{\mathbb{R}}$ *as* $x \to \infty$ *:*

$$-\infty < f(+\infty) \equiv \lim_{x \to \infty} f(x) \le \infty.$$

1.5 Theorem CONTINUITY OF MONOTONIC FUNCTIONS. *Let* $I = (a,b) \subseteq \mathbb{R}$, *where* $-\infty \le a < b \le \infty$, *and* $f : I \to \mathbb{R}$ *be a nondecreasing function on* I. *Then the function* f *has the following properties.*

(a) *For each* $x \in (a,b)$, f *is continuous at* x *iff*

$$f(x-) = f(x+)\ .$$

(b) *The only possible kind of discontinuity of* f *on* (a,b) *is a jump.*

(c) *The set of points of* (a,b) *at which* f *is discontinuous is countable (possibly empty).*

(d) *The function*

$$f_R(x) = f(x+)\ , \quad x \in (a,b)$$

is right continuous at every point of (a,b), i.e.,

$$\lim_{y\downarrow x} f_R(y) = f_R(x) , \quad \forall x \in (a,b) .$$

(e) *The function*

$$f_L(x) = f(x-)$$

is left continuous at every point of (a,b), i.e.,

$$\lim_{y\uparrow x} f_L(y) = f_L(x) , \quad \forall x \in (a,b) .$$

1.6 Theorem CHARACTERIZATION OF THE CONTINUITY OF MONOTONIC FUNCTIONS. *Let* $f : D \to \mathbb{R}$ *a monotonic function, where* D *is a non-empty subset of* $\mathbb{R}$ *and* I *a non-empty subset of* D. *Then*

$$f \text{ is continuous on } I \text{ iff } f(I) \text{ is an interval.}$$

1.7 Definition HOMEOMORPHISM. *Let* I *and* J *be two subsets of* $\mathbb{R}$, *and* $f : I \to J$. *We say that* f *is an homeomorphism iff* $f : I \to J$ *is a bijection such that* f *and* f^{-1} *are continuous.*

1.8 Theorem MONOTONE INVERSE FUNCTION THEOREM. *Let* I *be an interval in* $\mathbb{R}$, *and* $f : I \to \mathbb{R}$. *If* f *is continuous and strictly monotonic, then* $J = f(I)$ *is an interval and the function* $f : I \to J$ *is an homeomorphism.*

1.9 Theorem STRICT MONOTONICITY AND HOMEOMORPHISMS BETWEEN INTERVALS. *Let* I *and* J *be intervals in* $\mathbb{R}$ *and* $f : I \to J$.

(a) *If* f *is an homeomorphism, then* f *is strictly monotonic.*

(b) f *is an homeomorphism* $\Leftrightarrow f$ *is continuous and strictly monotonic*
$\Leftrightarrow f^{-1} : J \to I$ *exists and is an homeomorphism*
$\Leftrightarrow f^{-1} : J \to I$ *exists, and* f^{-1} *is a continuous strictly monotonic.*

1.10 Lemma CHARACTERIZATION OF RIGHT (LEFT) CONTINUOUS FUNCTIONS BY DENSE SETS. *Let* f_1 *and* f_2 *be two real-valued functions defined on the interval* (a,b) *such that the functions* f_1 *and* f_2 *are either both right continuous or both left continuous at each point* $x \in (a,b)$, *and let* D *be a dense subset of* (a,b). *If*

$$f_1(x) = f_2(x) , \quad \forall x \in D ,$$

then

$$f_1(x) = f_2(x) , \quad \forall x \in (a,b) .$$

1.11 Theorem CHARACTERIZATION OF MONOTONIC FUNCTIONS BY DENSE SETS. *Let f_1 and f_2 be two monotonic nondecreasing functions on (a,b), let D be a dense subset of (a,b), and suppose*

$$f_1(x) = f_2(x), \quad \forall x \in D.$$

(*a*) *Then f_1 and f_2 have the same points of discontinuity, they coincide everywhere in (a,b), except possibly at points of discontinuity, and*

$$f_1(x+) - f_1(x-) = f_2(x+) - f_2(x-), \quad \forall x \in (a,b).$$

(*b*) *If furthermore f_1 and f_2 are both left continuous (or right continuous) at every point $x \in (a,b)$, they coincide everywhere on (a,b)*, i.e.,

$$f_1(x) = f_2(x), \quad \forall x \in (a,b).$$

1.3. Total variation

1.12 Lemma *For any $x \in \mathbb{R}$,*

$$\max\{x,0\} = \frac{1}{2}(|x|+x) = I(x \geq 0)x = I(x \geq 0)\,|x|, \tag{1.1}$$

$$\max\{-x,0\} = \frac{1}{2}(|x|-x) = -I(x \leq 0)x = I(x \leq 0)\,|x|, \tag{1.2}$$

$$\min\{x,0\} = -\max\{-x,0\} = \frac{1}{2}(x-|x|) = I(x \leq 0)x = -I(x \leq 0)\,|x|, \tag{1.3}$$

$$\min\{-x,0\} = -\max\{x,0\} = -\frac{1}{2}(|x|+x) = -I(x \geq 0)x = -I(x \leq 0)\,|x|. \tag{1.4}$$

1.13 Lemma *For any $x_1, x_2 \in \mathbb{R}$,*

$$\begin{aligned} \min\{x_1,0\}+\min\{x_2,0\} &\leq \min\{x_1+x_2,0\} \\ &\leq \max\{x_1+x_2,0\} \leq \max\{x_1,0\}+\max\{x_2,0\}, \end{aligned} \tag{1.5}$$

$$\min\{x_1,0\}-\max\{x_2,0\} \leq \min\{x_1-x_2,0\} \tag{1.6}$$

$$\leq \max\{x_1-x_2,0\} \leq \max\{x_1,0\}-\min\{x_2,0\}. \tag{1.7}$$

1.14 Lemma *For any $x_1, x_2 \in \mathbb{R}$,*

$$\begin{array}{ll} \max\{x_1-x_2,0\} \leq x_1 \leq \max\{x_1,x_2\} & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0 \\ \max\{x_1-x_2,0\} \geq x_1 \geq \min\{x_1,x_2\} & \text{otherwise,} \end{array} \tag{1.8}$$

$$\begin{array}{ll} \min\{x_1-x_2,0\} \geq x_1 \geq \min\{x_1,x_2\} & \text{if } x_1 \leq 0 \text{ and } x_2 \leq 0 \\ \min\{x_1-x_2,0\} \leq x_1 \leq \max\{x_1,x_2\} & \text{otherwise.} \end{array} \tag{1.9}$$

Since

$$\min\{x_1 - x_2, 0\} \le \max\{x_1 - x_2, 0\}, \tag{1.10}$$

we can write:

$$\begin{array}{ll} x_1 \le \min\{x_1 - x_2, 0\} \le \max\{x_1 - x_2, 0\} & \text{if } x_1 \le 0 \text{ and } x_2 \le 0, \\ \min\{x_1 - x_2, 0\} \le x_1 \le \max\{x_1 - x_2, 0\} & \text{if } x_1 \le 0 \text{ and } x_2 \ge 0, \\ \min\{x_1 - x_2, 0\} \le x_1 \le \max\{x_1 - x_2, 0\} & \text{if } x_1 \ge 0 \text{ and } x_2 \le 0, \\ \min\{x_1 - x_2, 0\} \le \max\{x_1 - x_2, 0\} \le x_1 & \text{if } x_1 \ge 0 \text{ and } x_2 \ge 0. \end{array} \tag{1.11}$$

1.15 Definition *Let $f : [a, b] \to \mathbb{R}$. The **total variation** of f over $[a, b]$, denoted by $V_a^b(f)$, is*

$$V_a^b(f) = \sup_{\mathscr{P}[a,b]} \sum_{k=1}^{n} |f(x_k) - f(x_k)| \tag{1.12}$$

where $\mathscr{P}[a, b]$ is the set of all partitions of $[a, b]$ with $n \ge 1$ points of subdivision $x_0, x_1, \dots, x_n$ such that $n \ge 1$ and

$$a = x_0 < x_1 < \cdots < x_n = b. \tag{1.13}$$

1.16 Definition *Let $f : [a, b] \to \mathbb{R}$. The **positive variation** of f over $[a, b]$ is*

$$P_a^b(f) := \sup_{\mathscr{P}[a,b]} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]^+ \tag{1.14}$$

*and the **negative variation** of f over $[a, b]$ is*

$$N_a^b f := \sup_{\mathscr{P}[a,b]} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]^- \tag{1.15}$$

where $x^+ := I(x \ge 0)\,|x|$ and $x^- := I(x \le 0)\,|x|$.

1.17 Definition *Let $f : I \to \mathbb{R}$ and $[a, b] \subseteq I$. We say that f is of **bounded variation** on $[a, b]$ iff $V_a^b(f) < \infty$.*

1.18 Proposition *Let $f : [a, b] \to \mathbb{R}$, and $\alpha \in \mathbb{R}$. Then*

$$V_a^b(\alpha) = P_a^b(\alpha) = N_a^b(\alpha) = 0, \tag{1.16}$$

$$V_a^b(f + \alpha) = V_a^b(f), \quad P_a^b(f + \alpha) = P_a^b(f), \quad N_a^b(f + \alpha) = N_a^b(f), \tag{1.17}$$

$$V_a^b(f) = 0 \Leftrightarrow f \text{ is constant over } [a, b]. \tag{1.18}$$

1.19 Proposition BOUNDED VARIATION OF MONOTONIC FUNCTIONS. *Let $f : [a, b] \to \mathbb{R}$, $\alpha \in \mathbb{R}$. If f is nondecreasing on $[a, b]$, then*

$$V_a^b(f) = P_a^b(f) = f(b) - f(a), \tag{1.19}$$

$$N_a^b f = 0\,, \tag{1.20}$$

$$V_a^b(\alpha f) = \alpha V_a^b(f)\,,\ \text{for } \alpha \geq 0\,, \tag{1.21}$$

and f is of bounded variation on $[a, b]$. If f is nonincreasing on $[a, b]$, then

$$V_a^b(f) = N_a^b f = f(a) - f(b)\,, \tag{1.22}$$

$$P_a^b(f) = 0\,, \tag{1.23}$$

$$V_a^b(\alpha f) = \alpha V_a^b(f)\,,\ \text{for } \alpha \geq 0\,, \tag{1.24}$$

and f is of bounded variation on $[a, b]$.

1.20 Proposition *Let $f : [a, b] \to \mathbb{R}$, and $g : [a, b] \to \mathbb{R}$. If f and g are both nondecreasing or nonincreasing on $[a, b]$, then*

$$V_a^b(f+g) = V_a^b(f) + V_a^b(g)\,, \tag{1.25}$$

$$V_a^b(f+g) = V_a^b(f) \Leftrightarrow g \text{ is constant over } [a, b]\,. \tag{1.26}$$

1.21 Proposition CANONICAL DECOMPOSITION OF TOTAL VARIATION. *Let $f : [a, b] \to \mathbb{R}$. If f is of bounded variation on $[a, b]$, then*

$$V_a^b(f) = P_a^b(f) + N_a^b f \tag{1.27}$$

and

$$f(b) - f(a) = P_a^b(f) - N_a^b(f). \tag{1.28}$$

1.22 Theorem *Let $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$. If $a \leq b \leq c$, then*

$$V_a^b(f) = V_a^c f + V_c^b f\,. \tag{1.29}$$

1.23 Theorem *Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$, and $\alpha \in \mathbb{R}$. Then*

$$V_a^b(\alpha f) = |\alpha| V_a^b(f)\,, \tag{1.30}$$

and

$$V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)\,, \tag{1.31}$$

where we set $|\alpha| V_a^b(f) = 0$ if $\alpha = 0$ and $V_a^b(f) = +\infty$.

1.24 Definition *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then the function*

$$V_f(x) := V_a^x f\,,\ x \in [a, b]\,, \tag{1.32}$$

*is called the **total variation function** of f,*

$$P_f(x) := P_a^x f\,,\ x \in [a, b]\,, \tag{1.33}$$

is called the ***positive variation function*** *of f, and*

$$N_f(x) := N_a^x f\,,\ x \in [a,\,b]\,, \tag{1.34}$$

is called the ***negative variation function*** *of f.*

1.25 Theorem MONOTONICITY OF VARIATION FUNCTIONS. *Let $f : [a,\,b] \to \mathbb{R}$ be a function of bounded variation on $[a,\,b]$.*

(a) *If $x_1, x_2 \in [a,b]$ and $x_1 \le x_2$, then*

$$|f(x_2) - f(x_1)| \le V_{x_1}^{x_2}(f)\,, \tag{1.35}$$

$$\max\{f(x_2) - f(x_1),\,0\} \le P_{x_1}^{x_2}(f)\,, \tag{1.36}$$

$$\max\{f(x_1) - f(x_2),\,0\} \le N_{x_1}^{x_2}(f)\,. \tag{1.37}$$

(b) *The functions $V_f(x)$, $P_f(x)$ and $N_f(x)$ are nondecreasing on $[a,\,b]$.*

1.26 Theorem *Let $f : [a,\,b] \to \mathbb{R}$ be a function of bounded variation on $[a,\,b]$. If $f(x)$ is continuous from the left at x_0, then $V_f(x)$ is continuous from the left at x_0.*

1.27 Proposition LIMITS OF VARIATION FUNCTIONS. *Let $f : [a,\,b] \to \mathbb{R}$ be a function of bounded variation on $[a,\,b]$. Then,*

$$P_f(x+) - P_f(x) = \frac{1}{2}\{|f(x+) - f(x)| + [f(x+) - f(x)]\} = \max\{f(x+) - f(x),\,0\}\,, \tag{1.38}$$

$$N_f(x+) - N_f(x) = \frac{1}{2}\{|f(x+) - f(x)| - [f(x+) - f(x)]\} = \max\{f(x) - f(x+),\,0\}\,, \tag{1.39}$$

$$V_f(x+) - V_f(x) = |f(x+) - f(x)|\,, \tag{1.40}$$

$$P_f(x) - P_f(x-) = \frac{1}{2}\{|f(x) - f(x-)| + [f(x) - f(x-)]\} = \max\{f(x) - f(x-),\,0\}\,, \tag{1.41}$$

$$N_f(x) - N_f(x-) = \frac{1}{2}\{|f(x) - f(x-)| - [f(x) - f(x-)]\} = \max\{f(x-) - f(x),\,0\}\,, \tag{1.42}$$

$$V_f(x) - V_f(x-) = |f(x) - f(x-)|\,. \tag{1.43}$$

1.28 Theorem *Let $f : [a,\,b] \to \mathbb{R}$ be a function of bounded variation on $[a,\,b]$ and $x_0 \in [a,\,b]$.*

(a) *If $f(x)$ is right-continuous at x_0, then $P_f(x)$, $N_f(x)$ and $V_f(x)$ are right-continuous at x_0.*

(b) *If $f(x)$ is left-continuous at x_0, then $P_f(x)$, $N_f(x)$ and $V_f(x)$ are left-continuous at x_0.*

(c) *$f(x)$ is continuous at x_0 $\Leftrightarrow$ $V_f(x)$ is continuous at x_0*
$\Leftrightarrow$ $P_f(x)$ and $N_f(x)$ are continuous at x_0.

1.29 Theorem *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then, for any $x \in [a, b]$,*

$$V_f(x) = P_f(x) + N_f(x)\,, \tag{1.44}$$

and

$$f(x) - f(a) = P_f(x) - N_f(x)\,. \tag{1.45}$$

1.30 Theorem MONOTONE REPRESENTATION OF FUNCTIONS OF BOUNDED VARIATION. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then f can be represented as the difference between two nondecreasing functions on $[a, b]$. In particular, we have:*

$$\begin{aligned} f(x) &= [f(a) + P_f(x)] - N_f(x) \\ &= [f(a) + V_f(x)] - U_f(x) \end{aligned} \tag{1.46}$$

where $U_f(x) := 2N_f(x)$, and the functions $f(a) + P_f(x)$, $f(a) + V_f(x)$, $N_f(x)$ and $U_f(x)$ are all nondecreasing on $[a, b]$.

1.31 Corollary MONOTONE CHARACTERIZATION OF FUNCTIONS OF BOUNDED VARIATION. *Let $f : [a, b] \to \mathbb{R}$. Then f is of bounded variation on $[a, b]$ if and only if it is the difference between two nondecreasing functions on $[a, b]$.*

1.32 Remark The decomposition of a function of bounded variation as the difference of two nondecreasing functions is not unique. For example, if

$$f(x) = f_1(x) - f_2(x) \tag{1.47}$$

where $f_1(x)$ and $f_2(x)$ are nondecreasing, then for any nondecreasing function $g(x)$,

$$f(x) = [f_1(x) + g(x)] - [f_2(x) + g(x)] \tag{1.48}$$

where $f_1(x) + g(x)$ and $f_2(x) + g(x)$ are nondecreasing.

1.33 Theorem MINIMAL PROPERTY OF POSITIVE-NEGATIVE DECOMPOSITION OF FUNCTIONS OF BOUNDED VARIATION. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. If $g^+ : [a, b] \to \mathbb{R}$ and $g^- : [a, b] \to \mathbb{R}$ are nondecreasing functions on $[a, b]$ such that*

$$f(x) = f(a) + g^+(x) - g^-(x) \quad \forall x \in [a, b]\,, \tag{1.49}$$

then

$$P_f(x) \le g^+(x) - g^+(a) \quad \forall x \in [a, b]\,, \tag{1.50}$$

$$N_f(x) \le g^-(x) - g^-(a) \quad \forall x \in [a, b]\,. \tag{1.51}$$

If we note that

$$P_f(a) = N_f(a) = V_f(a) = 0\,, \tag{1.52}$$

it is natural to impose the same restriction $g^+(a) = g^-(a) = 0$. This yields the following result.

1.34 Theorem OPTIMALITY OF CANONICAL MONOTONE REPRESENTATIONS OF FUNCTIONS OF BOUNDED VARIATION. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. If $g^+ : [a, b] \to \mathbb{R}$ and $g^- : [a, b] \to \mathbb{R}$ are nondecreasing functions on $[a, b]$ such that*

$$f(x) = f(a) + g^+(x) - g^-(x) \quad \forall x \in [a, b], \tag{1.53}$$

and

$$g^+(a) = g^-(a) = 0 \tag{1.54}$$

then

$$P_f(x) \le g^+(x) \le V_f(x) \quad \forall x \in [a, b], \tag{1.55}$$

$$N_f(x) \le g^-(x) \le 2N_f(x) \quad \forall x \in [a, b]. \tag{1.56}$$

1.35 Lemma *Let $\mathscr{F}$ be a family of functions $f : I \to \mathbb{R}$ where I is some set, and $f_1, f_2 \in \mathscr{F}$. If*

$$f_1(x) \le f(x), \quad \forall x \in I, \forall f \in \mathscr{F}, \tag{1.57}$$

and

$$f_2(x) \le f(x), \quad \forall x \in I, \forall f \in \mathscr{F}, \tag{1.58}$$

then

$$f_1(x) = f_2(x), \quad \forall x \in I. \tag{1.59}$$

The above lemma is a *unicity* property: it means that only one element f_1 of $\mathscr{F}$ can satisfy the inequality (1.57).

1.36 Theorem CANONICAL MONOTONE REPRESENTATIONS OF FUNCTIONS OF BOUNDED VARIATION. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$, and $\mathscr{M}_I$ the set of the nondecreasing functions $g : [a, b] \to \mathbb{R}$ such that $g(a) = 0$. Then,*

(a) *there is a unique pair of nondecreasing functions $f^+, f^- \in \mathscr{M}_I$ such that*

$$f(x) = f(a) + f^+(x) - f^-(x) \quad \forall x \in [a, b], \tag{1.60}$$

and

$$\begin{aligned} &\{f(x) = f(a) + g_1(x) - g_2(x) \quad \forall x \in [a, b]\} \\ \Rightarrow &\{[f^+(x) \le g_1(x) \quad \text{and} \quad f^-(x) \le g_1(x)] \quad \forall x \in [a, b]\} \end{aligned} \tag{1.61}$$

for all $g_1, g_2 \in \mathscr{M}_I$; further,

$$f^+(x) = P_f(x) \quad \text{and} \quad f^-(x) = N_f(x) \quad \forall x \in [a, b]; \tag{1.62}$$

(b) *there is a unique pair of nondecreasing functions $v_f, u_f \in \mathscr{M}_I$ such that*

$$f(x) = f(a) + v_f(x) - u_f(x) \quad \forall x \in [a, b], \tag{1.63}$$

and

$$\{f(x) = f(a) + g_1(x) - g_2(x) \quad \forall x \in [a,\, b]\}$$
$$\Rightarrow \{[g_1(x) \le v_f(x) \quad \text{and} \quad g_2(x) \le u_f(x)] \quad \forall x \in [a,\, b]\} \tag{1.64}$$

for all $g_1,\, g_2 \in \mathscr{M}_I$ *; further,*

$$v_f(x) = V_f(x) = P_f(x) + N_f(x) \quad \text{and} \quad u_f(x) = 2N_f(x) \quad \forall x \in [a,\, b]\,. \tag{1.65}$$

1.4. Absolute continuity

1.37 Definition ABSOLUTE CONTINUITY. *Let* $f : [a,\, b] \to \mathbb{R}$. f *is said to be* absolutely continuous *on* $[a, b]$ *if, for any* $\varepsilon > 0$*, there is* $\delta > 0$ *such that*

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon \tag{1.66}$$

for every finite set of pairwise disjoint intervals (a_k, b_k) *such that*

$$(a_k, b_k) \subseteq [a,\, b]\,, \quad k = 1, \ldots, n, \tag{1.67}$$

and

$$\sum_{k=1}^{n} |b_k - a_k| < \delta\,. \tag{1.68}$$

1.38 Theorem MONOTONE REPRESENTATION OF ABSOLUTELY CONTINUOUS FUNCTIONS. *Let* $f : [a,\, b] \to \mathbb{R}$*. If* f *is absolutely continuous on* $[a,\, b]$*, then:*

(a) f *is of bounded variation on* $[a,\, b]$*;*

(b) f *can be represented as the difference between two absolutely continuous nondecreasing functions on* $[a,\, b]$*.*

1.5. Differentiation and integration of monotonic functions

In this subsection, $[a,\, b]$ represents a closed interval of the real numbers: $[a,\, b] \subseteq \mathbb{R}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

1.39 Theorem BOUNDEDNESS AND INTEGRABILITY OF MONOTONIC FUNCTIONS. *Let* $f : [a,\, b] \to \mathbb{R}$*. If* f *is nondecreasing on* $[a,\, b]$*, then* f *is measurable, bounded, and integrable on* $[a,\, b]$*.*

1.40 Theorem CONTINUOUS-JUMP DECOMPOSITION OF LEFT-CONTINOUS NONDECREASING FUNCTION. *Let* $f : [a,\, b] \to \mathbb{R}$*. If* f *is nondecreasing and continuous from the left on* $[a,\, b]$*, then* f *is the sum of a continuous function and a left-continuous jump function.*

1.41 Theorem DIFFERENTIABILITY OF MONOTONIC FUNCTIONS. *Let $f : [a, b] \to \mathbb{R}$ be a nondecreasing function on $[a, b]$. Then f is differentiable almost everywhere on $[a, b]$.*

1.42 Corollary DIFFERENTIABILITY OF FUNCTIONS OF BOUNDED VARIATION. *Let be $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then f is differentiable almost everywhere on $[a, b]$.*

1.43 Theorem DIFFERENTIABILITY AND ABSOLUTE CONTINUITY OF DEFINITE INTEGRALS. *Let be $f : [a, b] \to \mathbb{R}$. Suppose f is integrable on $[a, b]$ and let*

$$F(x) = \int_a^x f(x)\,dx\,. \tag{1.69}$$

Then:

(a) *$F(x)$ is differentiable and*

$$F'(x) = f(x) \tag{1.70}$$

for almost all $x \in [a, b]$;

(b) *$F(x)$ is absolutely continuous on $[a, b]$;*

(c) *if $f(x)$ is continuous at $x_0 \in (a, b)$, then $F(x)$ is differentiable at x_0 and*

$$F'(x_0) = f(x_0)\,. \tag{1.71}$$

1.44 Theorem INTEGRABILITY OF MONOTONIC FUNCTIONS. *Let $F : [a, b] \to \mathbb{R}$ be a nondecreasing function on $[a, b]$. Then the derivative $F'(x)$ is integrable on $[a, b]$ and*

$$\int_a^b F'(x)\,dx \leq F(b) - F(a)\,. \tag{1.72}$$

1.45 Theorem FUNDAMENTAL THEOREM OF CALCULUS FOR ABSOLUTELY CONTINUOUS FUNCTIONS (LEBESGUE). *Let $F : [a, b] \to \mathbb{R}$ be a nondecreasing function on $[a, b]$. If $F(x)$ is absolutely continuous on $[a, b]$, then the derivative $F'(x)$ exists for almost all $x \in [a, b]$, and*

$$\int_a^x F'(x)\,dx = F(x) - F(a)\,. \tag{1.73}$$

1.46 Corollary CHARACTERIZATION OF ABSOLUTELY CONTINUOUS FUNCTIONS. *Let $F : [a, b] \to \mathbb{R}$ be a nondecreasing function on $[a, b]$. The formula*

$$\int_a^x F'(x)\,dx = F(x) - F(a) \tag{1.74}$$

holds for all $x \in [a,b]$ if and only if $F(x)$ is absolutely continuous on $[a,b]$.

2. Generalized inverse of a monotonic function

2.1 Definition GENERALIZED INVERSE OF A NONDECREASING RIGHT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then the generalized inverse of f is defined by*

$$f^*(y) = \inf\{x \in (a,b) : f(x) \geq y\} \tag{2.1}$$

for $-\infty < y < \infty$ (with the convention $\inf(\emptyset) = b$). Further, we define f^{-1} as the restriction of f^* to the interval $(\inf(f), \sup(f)) \equiv (\inf\{f(x) : x \in (a,b)\}, \sup\{f(x) : x \in (a,b)\})$:

$$f^{-1}(y) = f^*(y) \quad \text{for } \inf(f) < y < \sup(f)\ . \tag{2.2}$$

2.2 Definition GENERALIZED INVERSE OF A NONDECREASING LEFT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, left continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then the generalized inverse of f is defined by*

$$f^{**}(y) = \sup\{x \in (a,b) : f(x) \leq y\} \tag{2.3}$$

for $-\infty < y < \infty$ (with the convention $\sup(\emptyset) = a$).

2.3 Proposition GENERALIZED INVERSE BASIC EQUIVALENCE (RIGHT-CONTINUOUS FUNCTION). *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then, for $x \in (a,b)$ and for every real y,*

$$y \leq f(x) \Leftrightarrow f^*(y) \leq x\,, \tag{2.4}$$

$$y > f(x) \Leftrightarrow f^*(y) > x\,, \tag{2.5}$$

$$f[f^*(y)] \geq y\,. \tag{2.6}$$

2.4 Proposition GENERALIZED INVERSE BASIC EQUIVALENCE (LEFT-CONTINUOUS FUNCTION). *Let f be a real-valued, nondecreasing, left continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$. Then, for $x \in (a,b)$ and for every real y,*

$$y \leq f(x) \Leftrightarrow f^{**}(y) \leq x\,. \tag{2.7}$$

2.5 Proposition CONTINUITY OF THE INVERSE OF A NONDECREASING RIGHT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval (a,b) where $-\infty \leq a < b \leq \infty$, and set*

$$a(f) = \inf\{x \in (a,b) : f(x) > \inf(f)\}\ , \quad b(f) = \sup\{x \in (a,b) : f(x) < \sup(f)\}\ . \tag{2.8}$$

Then, f^* is nondecreasing and left continuous. Moreover

$$\lim_{y\to-\infty} f^*(y) = a\,, \quad \lim_{y\to\infty} f^*(y) = b \tag{2.9}$$

and

$$\lim_{y\to\inf(f)} f^{-1}(y) = a(f)\,, \quad \lim_{y\to\sup(f)} f^{-1}(y) = b(f)\,. \tag{2.10}$$

3. Distribution functions

3.1 Definition DISTRIBUTION AND SURVIVAL FUNCTIONS OF A RANDOM VARIABLE. *Let X be a real-valued random variable. The distribution function of X is the function $F(x)$ defined by*

$$F(x) = \mathbb{P}[X \le x]\,,\; x \in \mathbb{R}, \tag{3.1}$$

and its survival function is the function $G(x)$ defined by

$$G(x) = \mathbb{P}[X \ge x]\,,\; x \in \mathbb{R}. \tag{3.2}$$

3.2 Proposition PROPERTIES OF DISTRIBUTION FUNCTIONS. *Let X be a real-valued random variable with distribution function $F(x) = \mathbb{P}[X \le x]$. Then*

(a) *$F(x)$ is nondecreasing;*

(b) *$F(x)$ is right-continuous;*

(c) *$F(x) \to 0$ as $x \to -\infty$;*

(d) *$F(x) \to 1$ as $x \to \infty$;*

(e) *$\mathbb{P}[X = x] = F(x) - F(x-)$;*

(f) *for any $x \in \mathbb{R}$ and $q \in (0,1)$,*

$$\{\,\mathbb{P}[X \le x] \ge q \text{ and } \mathbb{P}[X \ge x] \ge 1-q\,\} \Longleftrightarrow \{\,\mathbb{P}[X < x] \le q \text{ and } \mathbb{P}[X > x] \le 1-q\,\}\,.$$

3.3 Remark In view of Proposition 3.2, the domain of a distribution function $F(x)$ can be extended to $\mathbb{R}$ $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, the extended real numbers, by setting

$$F(-\infty) = 0 \text{ and } F(\infty) = 1\,. \tag{3.3}$$

3.4 Proposition PROPERTIES OF SURVIVAL FUNCTIONS. *Let X be a real-valued random variable with survival function $G(x) = \mathbb{P}[X \ge x]$. Then*

(a) *$G(x)$ is nonincreasing;*

(b) *$G(x)$ is left-continuous;*

(*c*) $G(x) \to 1$ as $x \to -\infty$;

(*d*) $G(x) \to 0$ as $x \to \infty$;

(*e*) $\mathbb{P}[X = x] = G(x) - G(x+)$;

(*f*) $G(x) = 1 - F(x) + \mathbb{P}[X = x]$.

4. Indicator representations of distribution and survival functions

4.1 Definition INDICATOR FUNCTION OF A PROPOSITION. *Let p be a proposition which may be true or false. The indicator function associated with p is the function defined*

$$\mathbf{1}(p) := \begin{cases} 1 & \text{if } p \text{ is true} \\ 0 & \text{if } p \text{ is false} \end{cases} . \tag{4.1}$$

4.2 Definition INDICATOR FUNCTION OF A SET. *Let A be a subset of* $\mathbb{R}$. *The indicator function of A is the function* $\mathbf{1} : \mathbb{R} \to \{0, 1\}$ *defined by*

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} . \tag{4.2}$$

4.3 Definition INDICATOR RANDOM VARIABLE. *Let X be a real random variable, and A an X-measurable event* (i.e., *a subset of* $\mathbb{R}$ *such that the probability* $\mathbb{P}[X \in A]$ *is defined*). *The indicator random variable of A is defined as*

$$\mathbf{1}\{X \in A\} := \mathbf{1}_A(X) = \begin{cases} 1 & \text{if } X \in A \\ 0 & \text{if } X \notin A \end{cases} . \tag{4.3}$$

4.4 Proposition MEAN AND VARIANCE INDICATOR RANDOM VARIABLE. *Let X be a real random variable, and A an X-measurable event. Then*

$$\mathbb{E}[\mathbf{1}_A(X)] = \mathbb{P}(A), \tag{4.4}$$

$$\mathbb{V}[\mathbf{1}_A(X)] = \mathbb{P}(A)[1 - \mathbb{P}(A)]. \tag{4.5}$$

From Definition 4.3, we can write:

$$\mathbb{E}[\mathbf{1}_{(-\infty,x]}(X)] = F(x),\ x \in \mathbb{R}, \tag{4.6}$$

$$\mathbb{E}[\mathbf{1}_{[x,\infty)}(X)] = G(x),\ x \in \mathbb{R}, \tag{4.7}$$

$$\mathbb{E}[\mathbf{1}_{(-\infty,x)}(X)] = \mathbb{P}[X < x] = F(x) - \mathbb{P}[X = x],\ x \in \mathbb{R}, \tag{4.8}$$

$$\mathbb{E}[\mathbf{1}_{(x,\infty)}(X)] = \mathbb{P}[X > x] = G(x) - \mathbb{P}[X = x],\ x \in \mathbb{R}. \tag{4.9}$$

The above indicator variables also satisfy:

$$\mathbb{V}[\mathbf{1}_{(-\infty,x]}(X)] = \mathbb{P}[X \le x]\mathbb{P}[X > x] = F(x)[1-F(x)]\,, \tag{4.10}$$

$$\mathbb{V}[\mathbf{1}_{[x,\infty)}(X)] = \mathbb{P}[X \ge x]\mathbb{P}[X < x] = G(x)[1-G(x)]\,, \tag{4.11}$$

$$\begin{aligned} \mathbb{V}[\mathbf{1}_{(-\infty,x)}(X)] &= \mathbb{P}[X < x]\mathbb{P}[X \ge x] \\ &= [1-G(x)]G(x) = \mathbb{V}[\mathbf{1}_{[x,\infty)}(X)]\,, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathbb{V}[\mathbf{1}_{(x,\infty)}(X)] &= \mathbb{P}[X > x]\mathbb{P}[X \le x] \\ &= [1-F(x)]F(x) = \mathbb{V}[\mathbf{1}_{(-\infty,x]}(X)]\,, \end{aligned} \tag{4.13}$$

$$\mathbb{E}[\mathbf{1}_{(-\infty,x]}(X)\mathbf{1}_{[x,\infty)}(X)] = \mathbb{P}[X = x]\,, \tag{4.14}$$

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{(-\infty,x]}(X)\mathbf{1}_{(x,\infty)}(X)] &= \mathbb{E}[\mathbf{1}_{(-\infty,x)}(X)\mathbf{1}_{[x,\infty)}(X)] \\ &= \mathbb{E}[\mathbf{1}_{(-\infty,x)}(X)\mathbf{1}_{[x,\infty)}(X)] = 0\,, \end{aligned} \tag{4.15}$$

$$\mathsf{C}[\mathbf{1}_{(-\infty,x]},\, \mathbf{1}_{[x,\infty)}(X)] = \mathbb{P}[X = x] - \mathbb{P}[X \le x]\mathbb{P}[X \ge x] \tag{4.16}$$

$$= \mathbb{P}[X = x] - F(x)G(x)\,, \tag{4.17}$$

$$\begin{aligned} \mathsf{C}[\mathbf{1}_{(-\infty,x]}(X),\, \mathbf{1}_{(x,\infty)}(X)] &= -\mathbb{P}[X \le x]\mathbb{P}[X > x] \\ &= -F(x)[1-F(x)] = -\mathbb{V}[\mathbf{1}_{(-\infty,x]}(X)]\,, \end{aligned} \tag{4.18}$$

$$\begin{aligned} \mathsf{C}[\mathbf{1}_{(-\infty,x)}(X),\, \mathbf{1}_{[x,\infty)}(X)] &= -\mathbb{P}[X < x]\mathbb{P}[X \ge x] \\ &= -[1-G(x)]G(x) = -\mathbb{V}[\mathbf{1}_{[x,\infty)}(X)]\,. \end{aligned} \tag{4.19}$$

5. Sign function

5.1 Definition SIGN FUNCTION. *The sign of $x \in \mathbb{R}$ is the function defined by*

$$\mathbf{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}\,. \tag{5.1}$$

It is easy to see that

$$\mathbf{sgn}(x) = \mathbf{1}_{[0,\infty)}(x) - \mathbf{1}_{(-\infty,0]}(x) = \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x)\,. \tag{5.2}$$

From Proposition 4.4, it is easy that:

$$\mathbb{E}[\mathbf{1}_{[0,\infty)}(X)] = \mathbb{P}[X \geq 0]\,, \quad \mathbb{E}[\mathbf{1}_{(-\infty,0]}(X)] = \mathbb{P}[X \leq 0]\,, \tag{5.3}$$

$$\mathbb{E}[\mathbf{1}_{(0,\infty)}(X)] = \mathbb{P}[X > 0]\,, \quad \mathbb{E}[\mathbf{1}_{(-\infty,0)}(X)] = \mathbb{P}[X < 0]\,, \tag{5.4}$$

$$\begin{aligned} \mathbb{V}[\mathbf{1}_{[0,\infty)}(X)] &= \mathbb{P}[X \geq 0]\mathbb{P}[X < 0] = \mathbb{P}[X > 0]\mathbb{P}[X < 0] + \mathbb{P}[X = 0]\mathbb{P}[X < 0] \\ &= \mathbb{V}[\mathbf{1}_{(0,\infty)}(X)]\,, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \mathbb{V}[\mathbf{1}_{(-\infty,0]}(X)] &= \mathbb{P}[X \leq 0]\mathbb{P}[X > 0] = \mathbb{P}[X > 0]\mathbb{P}[X < 0] + \mathbb{P}[X = 0]\mathbb{P}[X > 0] \\ &= \mathbb{V}[\mathbf{1}_{(-\infty,0]}(X)]\,, \end{aligned} \tag{5.6}$$

$$\mathsf{C}[\mathbf{1}_{[0,\infty)}(X), \mathbf{1}_{(-\infty,0]}(X)] = \mathbb{P}[X > 0] - \mathbb{P}[X \geq 0]\mathbb{P}[X \leq 0]\,, \tag{5.7}$$

$$\mathsf{C}[\mathbf{1}_{(0,\infty)}(X), \mathbf{1}_{(-\infty,0)}(X)] = -\mathbb{P}[X > 0]\mathbb{P}[X < 0]\,. \tag{5.8}$$

5.2 Proposition MEAN AND VARIANCE OF SIGN FUNCTIONS. *Let X be a real random variable. Then*

$$\begin{aligned} \mathbb{E}[\mathbf{sgn}(X)] &= \mathbb{P}[X \geq 0] - \mathbb{P}[X \leq 0] = \mathbb{P}[X > 0] - \mathbb{P}[X < 0] \\ &= 2\mathbb{P}[X \geq 0] - \mathbb{P}[X \neq 0] = \mathbb{P}[X \neq 0] - 2\mathbb{P}[X \leq 0]\,, \end{aligned} \tag{5.9}$$

$$\mathbb{E}[\big(\mathbf{sgn}(X)\big)^2] = \mathbb{P}[X > 0] + \mathbb{P}[X < 0] = 1 - \mathbb{P}[X = 0]\,, \tag{5.10}$$

$$\begin{aligned} \mathbb{V}[\mathbf{sgn}(X)] &= \mathbb{P}[X > 0] + \mathbb{P}[X < 0] - (\mathbb{P}[X > 0] - \mathbb{P}[X < 0])^2 \\ &= 1 - \mathbb{P}[X = 0] - (\mathbb{P}[X > 0] - \mathbb{P}[X < 0])^2. \end{aligned} \tag{5.11}$$

$\mathbb{V}[\mathbf{sgn}(X)]$ decreases as the asymmetry around zero (the difference $|\mathbb{P}[X > 0] - \mathbb{P}[X < 0]|$) or the mass at zero $\mathbb{P}[X = 0]$ increases. If $\mathbb{P}[X = 0] = 0$, it follows from (5.11) that

$$\begin{aligned} \mathbb{V}[\mathbf{sgn}(X)] &= 1 - (\mathbb{P}[X > 0] - \mathbb{P}[X < 0])^2 = 1 - [1 - F(0) - F(0)]^2 \\ &= 1 - [1 - 2F(0)]^2 = 1 - [1 - 4F(0) + 4F(0)^2] = 4F(0) - 4F(0)^2 \\ &= 4F(0)[1 - F(0)]\,. \end{aligned} \tag{5.12}$$

If $\mathbb{P}[X > 0] = \mathbb{P}[X < 0]$, we have

$$\mathbb{V}[\mathbf{sgn}(X)] = 1 - \mathbb{P}[X = 0] \tag{5.13}$$

6. Quantile functions

6.1 Definition QUANTILE FUNCTION. *Let $F(x)$ be a distribution function. The quantile function associated with F is the generalized inverse of F*, i.e.

$$F^{-1}(q) \equiv F^{-}(q) = \inf\{x : F(x) \geq q\}\ ,\ 0 < q < 1\ . \tag{6.1}$$

6.2 Remark $F^{-1}(q)$ may also be defined for $q = 0$ and $q = 1$, if we allow $F^{-1}(0) = -\infty$ and $F^{-1}(1) = +\infty$. More precisely,

$$F^{-1}(0) = -\infty \Leftrightarrow F(x) > 0\,,\ \forall x \in \mathbb{R}\,, \tag{6.2}$$

$$F^{-1}(1) = \infty \Leftrightarrow F(x) < 1\,,\ \forall x \in \mathbb{R}\,. \tag{6.3}$$

If $F^{-1}(0) = m$ where m is a finite real number, this means X has a finite lower bound (almost surely), *i.e.*

$$\mathbb{P}[X < m] = 0 \text{ and } \mathbb{P}[X \geq x] > 0 \text{ for all } x > m\,. \tag{6.4}$$

If $F^{-1}(1) = M$ where M is a finite real number, this means X has a finite upper bound (almost surely), *i.e.*

$$\mathbb{P}[X > M] = 0 \text{ and } \mathbb{P}[X \geq x] > 0 \text{ for all } x < M\,. \tag{6.5}$$

In general, irrespective whether $F^{-1}(0)$ and $F^{-1}(1)$ are finite, we can write:

$$\mathbb{P}[X < F^{-1}(0)] = 0 \text{ and } \mathbb{P}[X \geq x] > 0 \text{ for all } x > F^{-1}(0)\,, \tag{6.6}$$

$$\mathbb{P}[X > F^{-1}(1)] = 0 \text{ and } \mathbb{P}[X \geq x] > 0 \text{ for all } x < F^{-1}(1)\,. \tag{6.7}$$

6.3 Theorem PROPERTIES OF QUANTILE FUNCTIONS. *Let $F(x)$ be a distribution function. Then the following properties hold:*

(a) *for each $q \in (0,1)$, there is a unique real number a such that $a = F^{-1}(q)$;*

(b) *$a = F^{-1}(q)$ iff the two following conditions hold:*

  (1) *$F(a) \geq q$;*
  (2) *$x < a \Rightarrow F(x) < q$;*

(c) *$F^{-1}(q) = \inf\{x : \mathbb{P}[X < x] \leq q \leq \mathbb{P}[X \leq x]\}\ ,\ 0 < q < 1$;*

(d) *$F^{-1}(q) = \sup\{x : F(x) < q\}\ ,\ 0 < q < 1$;*

(e) *$F^{-1}(q)$ is nondecreasing and left continuous;*

(f) *$F(x) \geq q \Leftrightarrow x \geq F^{-1}(q)$, for all $x \in \mathbb{R}$ and $q \in (0,1)$;*

(g) *$F(x) < q \Leftrightarrow x < F^{-1}(q)$, for all $x \in \mathbb{R}$ and $q \in (0,1)$;*

(h) *$F[F^{-1}(q)-] \leq q \leq F[F^{-1}(q)]$, for all $q \in (0,1)$;*

(*i*) $F^{-1}[F(x)] \leq x \leq F^{-1}[F(x)+]$, *for all* $x \in \mathbb{R}$;

(*j*) *if* F *is continuous at* $x = F^{-1}(q)$, *then* $F[F^{-1}(q)] = q$;

(*k*) *if* F^{-1} *is continuous at* $q = F(x)$, *then* $F^{-1}[F(x)] = x$;

(*l*) *for* $q \in (0, 1)$, $F[F^{-1}(q)] = q \Leftrightarrow q \in F[\mathbb{R}]$;

(*m*) $F[F^{-1}(q)] = q$ *for all* $q \in (0, 1)$ $\Leftrightarrow (0, 1) \subseteq F[\mathbb{R}]$
$\Leftrightarrow F$ *is continuous*
$\Leftrightarrow F^{-1}$ *is strictly increasing*;

(*n*) *for any* $x \in \mathbb{R}$, $F^{-1}[F(x)] = x \Leftrightarrow F(x-\varepsilon) < F(x)$ *for all* $\varepsilon > 0$;

(*o*) *for any* $x \in \mathbb{R}$, $\mathbb{P}[X = x] > 0 \Rightarrow F^{-1}[F(x)] = x$;

(*p*) $F^{-1}[F(x)] = x$ *for all* $x \in \mathbb{R}$ $\Leftrightarrow F$ *is strictly increasing*
$\Leftrightarrow F^{-1}$ *is continuous*;

(*q*) F *is continuous and strictly increasing* $\Leftrightarrow F^{-1}$ *is continuous and strictly increasing ;*

(*r*) $F^{-1} \circ F \circ F^{-1} = F^{-1}$ *or, equivalently,*

$$F^{-1}\left(F\left[F^{-1}(q)\right]\right) = F^{-1}(q), \text{ for all } q \in (0, 1) ;$$

(*s*) $F \circ F^{-1} \circ F = F$ *or, equivalently,*

$$F\left(F^{-1}[F(x)]\right) = F(x), \text{ for all } x \in \mathbb{R} .$$

6.4 Theorem CHARACTERIZATION OF DISTRIBUTIONS BY QUANTILE FUNCTIONS. *If* $G(x)$ *is a real-valued nondecreasing left continuous function with domain* $(0, 1)$, *there is a unique distribution function* F *such that* $G = F^{-1}$.

6.5 Theorem DIFFERENTIATION OF QUANTILE FUNCTIONS. *Let* $F(x)$ *be a distribution function. If* F *has a positive continuous* $f(x)$ *density* f *in a neighborhood of* $F^{-1}(q_0)$, *where* $0 < q_0 < 1$, *then the derivative* $dF^{-1}(q)/dq$ *exists at* $q = q_0$ *and*

$$\left.\frac{dF^{-1}(q)}{dq}\right|_{q_0} = \frac{1}{f(F^{-1}(q_0))} . \tag{6.8}$$

6.6 Proposition *Let* X *be a real-valued random variable with distribution function* $F(x) = \mathbb{P}[X \leq x]$ *and survival function* $G(x) = \mathbb{P}[X \geq x]$. *Then, for any* $q \in (0, 1)$,

(*a*) $\mathbb{P}[X \leq F^{-1}(q)] \geq q$ *and* $\mathbb{P}[X \geq F^{-1}(q)] \geq 1-q$;

(*b*) $\mathbb{P}[X < F^{-1}(q)] \leq q$ *and* $\mathbb{P}[X > F^{-1}(q)] \leq 1-q$.

7. Quantile sets and generalized quantile functions

7.1 Notation *X is a random variable with distribution function $F_X(x) = \mathbb{P}[X \le x]$. $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is the set of the extended real numbers.*

7.2 Definition QUANTILE OF RANDOM VARIABLE. *A* quantile *of order q (or a q-*quantile*) of the random variable X is any number $m_q \in \bar{\mathbb{R}}$ such that $\mathbb{P}[X \le m_q] \ge q$ and $\mathbb{P}[X \ge m_q] \ge 1-q$, where $0 \le q \le 1$. In particular, $m_{0.5}$ is a* median *of X, $m_{0.25}$ is a* first *(or* lower*)* quartile *of X, and $m_{0.75}$ is a* third *(or* upper*)* quartile *of X.*

7.3 Remark For $q = 0$, $m_q = -\infty$ always satisfies the quantile condition. If there is a finite number d_L such that $\mathbb{P}[X \le d_L] = 0$, then any x such that $x \le d_L$ is a quantile of order 0. Similarly, for $q = 1$, $m_q = \infty$ always satisfies the quantile condition. If there is a finite number d_U such that $\mathbb{P}[X \le d_U] = U$, then any x such that $x \ge d_U$ is a quantile of order 1.

8. Distribution and quantile transformations

8.1 Notation *$U(0, 1)$ a uniform random variable on the interval $(0, 1)$.*

8.2 Theorem QUANTILES OF TRANSFORMED RANDOM VARIABLES. *Let X be a real-valued random variable with distribution function $F_X(x) = \mathbb{P}[X \le x]$. If $g(x)$, $x \in \mathbb{R}$, is a nondecreasing left continuous function, then*

$$F^{-1}_{g(X)}(q) = g\left(F_X^{-1}(q)\right), \quad \text{for all } 0 < q < 1, \tag{8.1}$$

where $F_{g(X)}(x) = \mathbb{P}[g(X) \le x]$ and $F^{-1}_{g(X)}(q) = \inf\{x : F_{g(X)}(x) \ge q\}$.

8.3 Corollary QUANTILES OF A LINEAR TRANSFORMATION. *Let X be a real-valued random variable with distribution function $F_X(x) = \mathbb{P}[X \le x]$, and let a and b be two real constants. If $a > 0$, then $F^{-1}_{aX+b}(q) = aF_X^{-1}(q) + b, \quad \text{for } 0 < q < 1$.*

8.4 Theorem TRANSFORMATION BY A DISTRIBUTION FUNCTION. *Let X be a real-valued random variable with distribution function $F_X(x) = \mathbb{P}[X \le x]$, $F_0(x)$ a distribution function, and $U = F_0(X)$. Then, for all $u \in (0, 1)$,*

$$U \le u \Leftrightarrow F_0(X) \le u \Leftrightarrow X \le F_0^{-1}(u) \tag{8.2}$$

and

$$\mathbb{P}[U \le u] = \mathbb{P}[X \le F_0^{-1}(u)] = F_X[F_0^{-1}(u)]. \tag{8.3}$$

8.5 Definition RELATIVE DISTRIBUTION. *Let X be a real-valued random variable with distribution function $F_X(x) = \mathbb{P}[X \le x]$, and $F_0(x)$ a distribution function. The distribution of $U = F_0(X)$ is called the* relative distribution *of X with respect to F_0.*

8.6 Proposition QUANTILES OF THE RELATIVE DISTRIBUTION TRANSFORMATION. *Let X be a real-valued random variable, $F_0(x)$ and $F_1(x)$ two distribution functions, and $U = F_0(X)$. Then*

$$F^{-1}_{F_1^{-1}(U)} = F_1^{-1}\left(F_U^{-1}\right) . \tag{8.4}$$

8.7 Theorem PROPERTIES OF QUANTILE TRANSFORMATION. *Let $F(x)$ be a distribution function, and U a random variable with distribution $F_0(x)$ such that $F_0(0) = 0$ and $F_0(1) = 1$. If $X = F^{-1}(U)$, then, for all $x \in \mathbb{R}$,*

$$X \leq x \Leftrightarrow F^{-1}(U) \leq x \Leftrightarrow U \leq F(x) \tag{8.5}$$

or, equivalently,

$$\mathbf{1}\{X \leq x\} = \mathbf{1}\{F^{-1}(U) \leq x\} = \mathbf{1}\{U \leq F(x)\} , \tag{8.6}$$

and

$$\mathbb{P}[X \leq x] = \mathbb{P}[F^{-1}(U) \leq x] = \mathbb{P}[U \leq F(x)] = F_0\left(F(x)\right) ; \tag{8.7}$$

further,

$$\mathbf{1}\{X < x\} = \mathbf{1}\{F^{-1}(U) < x\} = \mathbf{1}\{U \leq F(x-)\} \text{ with probability } 1 \tag{8.8}$$

and

$$\mathbb{P}[X < x] = \mathbb{P}[F^{-1}(U) < x] = \mathbb{P}[U \leq F(x-)] . \tag{8.9}$$

In particular, if U follows a uniform distribution on the interval $(0, 1)$, i.e. $U \sim U(0, 1)$, the distribution function of $F^{-1}(U)$ is F :

$$\mathbb{P}[F^{-1}(U) \leq x] = \mathbb{P}[X \leq x] = \mathbb{P}[U \leq F(x)] = F(x)\,,\ \forall x \in \mathbb{R}. \tag{8.10}$$

8.8 Corollary QUANTILE TRANSFORMATION OF $U[0,1]$ VARIABLE. *Let $F(x)$ be a distribution function, $\bar{U} \sim U[0, 1]$ and $\bar{X} = F^{-1}(\bar{U})$. Then,*

$$\mathbb{P}[\bar{X} = -\infty] = \mathbb{P}[\bar{X} = \infty] = 0\,, \tag{8.11}$$

$$\mathbb{P}[\bar{X} \leq x] = F(x)\,,\ \forall x \in \mathbb{R}. \tag{8.12}$$

8.9 Theorem PROPERTIES OF DISTRIBUTION TRANSFORMATION. *Let X be a real-valued random variable with distribution function $F(x) = \mathbb{P}[X \leq x]$. Then the following properties hold:*

(a) $\mathbb{P}[F(X) \leq u] \leq u$, *for all $u \in [0, 1]$;*

(b) $\mathbb{P}[F(X) \leq u] = u \Leftrightarrow u \in \mathrm{cl}\{F(\mathbb{R})\}$,
where $\mathrm{cl}\{F(\mathbb{R})\}$ is the closure of the range of F;

(c) $\mathbb{P}[F(X) \leq F(x)] = \mathbb{P}[X \leq x] = F(x)$, *for all $x \in \mathbb{R}$;*

(d) $F(X) \sim U(0, 1) \Leftrightarrow F$ *is continuous;*

(e) *for all x,* $\mathbf{1}\{F(X) \leq F(x)\} = 1\{X \leq x\}$ *with probability* 1;

(f) $F^{-1}(F(X)) = X$ *with probability* 1.

8.10 Theorem QUANTILES AND P-VALUES. *Let* X *be a real-valued random variable with distribution function* $F(x) = \mathbb{P}[X \le x]$ *and survival function* $G(x) = \mathbb{P}[X \ge x]$. *Then, for any* $x \in \mathbb{R}$,

$$\begin{aligned} G(x) &= \mathbb{P}[G(X) \ge G(x)] \\ &= \mathbb{P}\big[X \ge F^{-1}\big((F(x) - p_F(x))^{+}\big)\big] \\ &= \mathbb{P}\big[X \ge F^{-1}\big((1 - G(x))^{+}\big)\big] \end{aligned} \tag{8.13}$$

where $p_F(x) = \mathbb{P}[X = x] = F(x) - F(x-)$.

9. Relations between moments and quantiles

9.1 Theorem QUANTILE REPRESENTATION OF THE MEAN. *If* $\mathbb{E}(|X|) < \infty$, *we have:*

$$\mathbb{E}(X) = \int_0^1 F_X^{-1}(u)\,du = \int_0^1 F_X^{+}(u)\,du. \tag{9.1}$$

9.2 Lemma EXPANSION OF THE EXPECTED ABSOLUTE DEVIATION. *For any* m *and* c,

$$\begin{aligned} \mathbb{E}(|X - c|) &= \mathbb{E}(|X - m|) + (c - m)[\mathbb{P}(X \le m) - \mathbb{P}(X > m)] \\ &\quad + 2\int_{(m,\,c)} (c - x)\,dF_X(x), \quad \text{if } m \le c, \\ &= \mathbb{E}(|X - m|) + (m - c)[\mathbb{P}(X \ge m) - \mathbb{P}(X < m)] \\ &\quad + 2\int_{(c,m)} (x - c)\,dF_X(x), \quad \text{if } m > c. \end{aligned} \tag{9.2}$$

PROOF OF LEMMA 9.2 Let $m \le c$. We can write :

$$\mathbb{E}(|X - m|) = \int_{(-\infty,m]} (m - x)\,dF_X(x) + \int_{(m,c]} (x - m)\,dF_X(x) + \int_{(c,\infty)} (x - m)\,dF_X(x), \tag{9.3}$$

$$\mathbb{E}(|X - c|) = \int_{(-\infty,m]} (c - x)\,dF_X(x) + \int_{(m,c]} (c - x)\,dF_X(x) + \int_{(c,\infty)} (x - c)\,dF_X(x). \tag{9.4}$$

Subtracting (9.3) from (9.4), we get :

$$\begin{aligned} \mathbb{E}(|X - c|) &- \mathbb{E}(|X - m|) \\ &= \int_{(-\infty,m]} (c - m)\,dF_X(x) + \int_{(m,c]} (c + m - 2x)\,dF_X(x) \end{aligned}$$

$$
\begin{aligned}
& + \int_{(c,\infty)} (m-c)\, dF_X(x) \\
= \;& (c-m)\left\{\mathbb{P}[X \le m] - \mathbb{P}[X > c]\right\} \\
& + (c+m)\,\mathbb{P}[m < X \le c] - 2\int_{(m,c]} x\, dF_X(x) \\
= \;& (c-m)\left\{\mathbb{P}[X \le m] - \mathbb{P}[X > m] + \mathbb{P}[m < X \le c]\right\} \\
& + (c+m)\,\mathbb{P}[m < X \le c] - 2\int_{(m,c]} x\, dF_X(x) \\
= \;& (c-m)\left\{\mathbb{P}[X \le m] - \mathbb{P}[X > m]\right\} \\
& + 2c\,\mathbb{P}[m < X \le c] - 2\int_{(m,c]} x\, dF_X(x) \\
= \;& (c-m)\left\{\mathbb{P}[X \le m] - \mathbb{P}[X > m]\right\} + 2\int_{(m,c]} (c-x)\, dF_X(x) \ge 0\,.
\end{aligned}
$$

Now, let $c < m$. We can write:

$$
\mathbb{E}(|X-m|) = \int_{(-\infty,c)} (m-x)\, dF_X(x) + \int_{[c,m)} (m-x)\, dF_X(x) + \int_{[m,\infty)} (x-m)\, dF_X(x)\,, \tag{9.5}
$$

$$
\mathbb{E}(|X-c|) = \int_{(-\infty,c)} (c-x)\, dF_X(x) + \int_{[c,m)} (x-c)\, dF_X(x) + \int_{[m,\infty)} (x-c)\, dF_X(x)\,. \tag{9.6}
$$

Subtracting (9.5) from (9.6), we get:

$$
\begin{aligned}
\mathbb{E}(|X-c|) \;-\; & \mathbb{E}(|X-m|) \\
= \;& \int_{(-\infty,c)} (c-m)\, dF_X(x) + \int_{[c,m)} (2x-c-m)\, dF_X(x) + \int_{[m,\infty)} (m-c)\, dF_X(x) \\
= \;& (c-m)\left\{\mathbb{P}[X < c] - \mathbb{P}[X \ge m]\right\} - (c+m)\,\mathbb{P}[c \le X < m] + 2\int_{[c,m)} x\, dF_X(x) \\
= \;& (c-m)\left\{\mathbb{P}[X < m] - \mathbb{P}[c \le X < m] - \mathbb{P}[X \ge m]\right\} \\
& - (c+m)\,\mathbb{P}[c \le X < m] + 2\int_{[c,m)} x\, dF_X(x) \\
= \;& (m-c)\left\{\mathbb{P}[X \ge m] - \mathbb{P}[X < m]\right\} - 2c\,\mathbb{P}[c \le X < m] + 2\int_{[c,m)} x\, dF_X(x)
\end{aligned}
$$

$$= \quad (m-c)\left\{\mathbb{P}[X \geq m] - \mathbb{P}[X < m]\right\} + 2\int_{[c,m)} (x-c)\, dF_X(x) \geq 0\,.$$

□

9.3 Theorem OPTIMALITY OF MEDIANS FOR ABSOLUTE ERROR. *Let m be any median of X,* i.e. $\mathbb{P}(X \leq m) \geq 0.5$ *and* $\mathbb{P}(X \geq m) \geq 0.5$. *Then,*

$$\mathbb{E}(|X-m|) \leq \mathbb{E}(|X-c|) \text{ for any } c\,. \tag{9.7}$$

9.4 Corollary *Let m_1 and m_2 be two medians of X. Then*

$$\mathbb{E}(|X-m_1|) = \mathbb{E}(|X-m_2|) \tag{9.8}$$

and the function $\mathbb{E}(|X-c|)$ has a minimal value with respect to c given by $\mathbb{E}(|X-m_1|)$.

9.5 Corollary *Let m be any median of X. Then*

$$\mathbb{E}(|X-m|) = \mathbb{E}\left(\left|X - F_X^{-1}(0.5)\right|\right) \leq \mathbb{E}(|X-c|) \text{ for any } c\,. \tag{9.9}$$

9.6 Corollary *Let m be any median of X. Then,*

$$\mathbb{E}(|X-m|) \leq \mathbb{E}(|X-\mu_X|) \leq \sigma_X\,. \tag{9.10}$$

9.7 Theorem OPTIMALITY OF QUANTILES. *Let*

$$L(c) = a(X-c)_+ + b(X-c)_- \tag{9.11}$$

where $a > 0$ and $b > 0$, let $q = a/(a+b)$ and let m_q be any quantile of order q of X. Then,

$$\mathbb{E}[L(m_q)] = \mathbb{E}[L(F_X^{-1}(q))] \leq \mathbb{E}[L(c)] \text{ for any } c\,. \tag{9.12}$$

9.8 Theorem CONCENTRATION CONDITION FOR VARIANCE DOMINANCE. *Let X and Y be two random variables with finite means μ_X and μ_Y and finite variances σ_X^2 and σ_Y^2. If*

$$\mathbb{P}\left[\,|X-\mu_X| \leq x\right] \geq \mathbb{P}\left[\,|Y-\mu_Y| \leq x\right] \text{ for all } x, \tag{9.13}$$

then $\sigma_X^2 \leq \sigma_Y^2$.

9.9 Theorem MEAN-QUANTILE INEQUALITY. *Let m_q* a quantile *of order q of the random variable X. Then, if $\mathbb{E}(|X|) < \infty$,*

$$\begin{aligned} \mathbb{E}(X) - \sigma_X[(1-q)/q]^{1/2} \quad &\leq \quad \mathbb{E}(X\,|\,X \leq m_q) \leq m_q \\ &\leq \quad \mathbb{E}(X\,|\,X \geq m_q) \leq \mathbb{E}(X) + \sigma_X[q/(1-q)]^{1/2} \end{aligned} \tag{9.14}$$

where $\sigma_X = \left[\mathbb{E}(X-\mathbb{E}X)^2\right]^{1/2}$, *and*

$$\left|m_q - \mathbb{E}(X)\right| \leq \sigma_X \max\left\{[(1-q)/q]^{1/2},\, [q/(1-q)]^{1/2}\right\}. \tag{9.15}$$

9.10 Corollary MEAN-MEDIAN INEQUALITY. *Let m be any median of X. Then, if $\mathbb{E}(|X|) < \infty$,*

$$|m - \mathbb{E}(X)| \leq \sigma_X. \tag{9.16}$$

9.11 Theorem SYMMETRIZATION INEQUALITIES. *Let X_1 and X_2 two i.i.d. random variables, let m be any median of X, and set $\widetilde{X} = X_1 - X_2$ Then, for any ε and a,*

$$\mathbb{P}[X - m \geq \varepsilon] \leq 2\,\mathbb{P}\big[\widetilde{X} \geq \varepsilon\big] \tag{9.17}$$

and

$$\mathbb{P}\big[\,|X-m| \geq \varepsilon\big] \leq 2\,\mathbb{P}\big[|\widetilde{X}| \geq \varepsilon\big] \leq 4\,\mathbb{P}\big[|X-a| \geq \varepsilon/2\big]. \tag{9.18}$$

9.12 Theorem RANGE-STANDARD DEVIATION INEQUALITY. *If $Q_{\min}$ and $Q_{\max}$ are two real numbers such that $\mathbb{P}[Q_{\min} \leq X \leq Q_{\max}] = 1$, then*

$$\mathbb{E}(|X-\mu_X|) \leq \sigma_X \leq [Q_{\max} - Q_{\min}]/2. \tag{9.19}$$

PROOF OF THEOREM 9.2 If $d = |Q_{\max} - Q_{\min}| = +\infty$, the result holds trivially. Let $d < +\infty$, which means that $Q_{\max}$ and $Q_{\min}$ are both finite. Setting $\nu = [Q_{\min} + Q_{\max}]/2$, we see that $|X - \nu| \leq d/2$ with probability one. Using the fact that the mean μ_X minimizes $E[(X-c)^2]$ with respect to c, it follows that

$$\sigma_X^2 = \mathbb{E}[(X-\mu_X)^2] \leq \mathbb{E}[(X-\nu)^2] \leq d^2/4 \tag{9.20}$$

and $\sigma_X \leq [Q_{\max} - Q_{\min}]/2$. □

9.13 Theorem RANGE-MEAN ABSOLUTE DEVIATION INEQUALITY. *If $Q_{\min}$ and $Q_{\max}$ are two real numbers such that $\mathbb{P}[Q_{\min} \leq X \leq Q_{\max}] = 1$ and if m is a median of X, then*

$$\mathbb{E}(|X-m|) \leq \mathbb{E}(|X-\mu_X|) \leq [Q_{\max} - Q_{\min}]/2. \tag{9.21}$$

PROOF OF THEOREM 9.13 If $d = |Q_{\max} - Q_{\min}| = +\infty$, the result holds trivially. Let $d < +\infty$, which means that $Q_{\max}$ and $Q_{\min}$ are both finite. Setting $\nu = [Q_{\min} + Q_{\max}]/2$, we see that $|X - \nu| \leq d/2$ with probability one. Using the fact that the median m minimizes $\mathbb{E}[|X-c|]$ with respect to c, it follows that

$$\mathbb{E}(|X-m|) \leq \mathbb{E}(|X-\mu_X|) \leq \mathbb{E}(|X-\nu|) \leq d/2. \tag{9.22}$$

□

10. Multivariate generalizations

10.1 Notation CONDITIONAL DISTRIBUTION FUNCTIONS. *Let* $X = (X_1, \ldots, X_k)'$ *a* $k \times 1$ *random vector in* $\mathbb{R}^k$. *Then we denote as follows the following set of conditional distribution functions:*

$$\begin{aligned} F_{1|\cdot}(x_1) &= F_1(x_1) = \mathbb{P}[X_1 \le x_1]\,, & (10.1)\\ F_{2|\cdot}(x_2|x_1) &= \mathbb{P}[X_2 \le x_2\,|\,X_1 = x_1]\,, & (10.2)\\ &\vdots \\ F_{k|\cdot}(x_k\,|\,x_1, \ldots, x_{k-1}) &= \mathbb{P}[X_k \le x_k\,|\,X_1 = x_1, \ldots, X_{k-1} = x_{k-1}]\,. & (10.3) \end{aligned}$$

Further, we define the following transformations of $X_1, \ldots, X_k$:

$$\begin{aligned} Z_1 &= F_1(X_1)\,, & (10.4)\\ Z_2 &= F_{2|\cdot}(X_2\,|\,X_1)\,, & (10.5)\\ &\vdots \\ Z_k &= F_{k|\cdot}(X_k\,|\,X_1, \ldots, X_{k-1})\,. & (10.6) \end{aligned}$$

10.2 Theorem TRANSFORMATION TO *i.i.d.* $U(0,1)$ VARIABLES (ROSENBLATT). *Let* $X = (X_1, \ldots, X_k)'$ *be a* $k \times 1$ *random vector in* $\mathbb{R}^k$ *with an absolutely continuous distribution function* $F(x_1, \ldots, x_k) = \mathbb{P}[X_1 \le x_1, \ldots, X_k \le x_k]$. *Then the random variables* $Z_1, \ldots, Z_k$ *are independent and identically distributed according to a* $U(0, 1)$ *distribution.*

11. Proofs and additional references

1.4 - 1.5 Rudin (1976), Chapter 4, pp. 95-97, and Chung (1974), Section 1.1. For (a)-(b), see Phillips (1984), Sections 9.1 (p. 243) and 9.3 (p. 253).

1.6 - 1.9 Ramis, Deschamps, and Odoux (1982), Section 4.3.2, p.121.

1.10 Chung (1974), Section 1.1, p. 4.

1.19 Kolmogorov and Fomin (1975), Section 32.

1.21 Royden (1968, Chapter 5, Section 2, Lemma 3).

1.25 Protter and Morrey (1991, Chapter 12, Theorem 12.8), Kolmogorov and Fomin (1975, Section 32, Theorem 3).

1.27 Devinatz (1968, Chapter 5, Theorem 5.5.4).

1.30 Kolmogorov and Fomin (1975, Section 32, Theorem 4), Royden (1968, Chapter 5, Section 2, Theorem 4).

1.31 The equivalence follows from the combination of Theorems 1.19 and 1.30.

1.33 Devinatz (1968, Chapter 5, Theorem 5.5.3).

1.38 Kolmogorov and Fomin (1975), Section 33.2 (Theorems 2 and 4).

1.39 Kolmogorov and Fomin (1975), Section 31.1, Theorem 1.

1.40 Kolmogorov and Fomin (1975), Section 31.1, Theorem 5.

1.41 Haaser and Sullivan (1991), Section 9.3; Riesz and Sz.-Nagy (1955/1990), Chapter 1; Kolmogorov and Fomin (1975), Section 31.2, Theorem 1.

1.42 Kolmogorov and Fomin (1975), Section 32 (Corollary 1).

1.43 Kolmogorov and Fomin (1975), Section 31.3 (Theorems 7 and 8), and Section 33.2 (Theorem 5). For (c), see Ross (1980), Chapter 6, Theorem 34.3.

1.44 Kolmogorov and Fomin (1975), Section 33.1 (Theorem 1).

1.4 Kolmogorov and Fomin (1975), Section 33.2.

1.38 Kolmogorov and Fomin (1975), Section 33.2 (Theorem 2).

1.45 Kolmogorov and Fomin (1975), Section 33.2 (Theorem 6).

1.46 Kolmogorov and Fomin (1975), Section 33.2 (Remark to Theorem 6).

2.3 (2.4) is proved by Reiss (1989, Appendix 1, Lemma A.1.1). (2.5) and (2.6) are also given by Gleser (1985, Lemma 1, p. 957).

2.4 Reiss (1989), Appendix 1, Lemma A.1.3.

2.5 Reiss (1989), Appendix 1, Lemma A.1.2.

3.2 (f) Lehmann and Casella (1998), Problem 1.7 (for the case $q = 1/2$).

6.3 (b) is mentioned by Hosseini (2009, 2010). (c) is mentioned by Reiss (1989, Lemma 1.5.4). For (d), see Williams (1991, Section 3.12 (p. 34).). (o) is stated by Hosseini (2009, 2010).

8.2 Parzen (1980) and Shorack and Wellner (1986, page 9, Exercise 3) state this result without proof. For a proof, see Hosseini (2009, 2010).

8.6 This follows directly from the observation that the quantile function $F_1^{-1}(q)$ is nondecreasing and left continuous.

8.4–8.5 For discussion of relative distributions, see Handcock and Morris (1999) and Thas (2010).

8.9 (a)-(b) Shorack and Wellner (1986), Chapter 1, Proposition 2.

9.1 See the literature on Lorenz curves: Arnold and Villaseñor (1987), Shaked and Shantikumar (1994, equation (2.A.17) and Theorem 3.C.4).

9.2 This result is stated by Gnedenko (1969, Section 30, page 194) for the case where $\mathbb{P}(X \leq m) = \mathbb{P}(X > m)$ and by Lehmann and Casella (1998, Chapter 1, Problem 1.8, p. 62) for the case where $F_X^+(0.5) < c$ with $\mathbb{P}(X \leq m) \geq 0.5$ and $\mathbb{P}(X \geq m) \geq 0.5$. We give here a complete proof.

9.3 See Ferguson (1967, Section 1.8, Problem 2, page 51), Gnedenko (1969, Section 30, page 194) and Lehmann and Casella (1998, Chapter 1, Problem 1.8, p. 62).

9.7 See Ferguson (1967, Section 1.8, Problem 2, page 51) and Gilat and Hill (1993).

9.8 See Rao (1973, Section 2b.2, page 96).

9.9 See Mallows and Richter (1969, Section 4) and Dharmadhikari (1991). The outer inequalities in (9.15) have also been obtained by Moriguti (1953). The symmetric inequality (9.15) follows in a straightforward way from (9.15). It is also mentioned by O'Cinneide (1990); for an alternative derivation, see David (1991).

9.11 See Loève (1977, Section 18.1, p. 257).

9.12 For the case of a discrete distribution, this inequality was given by Thompson (1935), without proof, and by Guterman (1962) and Sher (1979) with simple proofs. See also Page and Murty (1982, 1983).

9.13 This result has not apparently been stated elsewhere.

10.2 See Rosenblatt (1952).

References

ARNOLD, B. C., AND J. A. VILLASEÑOR (1987): *Majorization and the Lorenz Order: A Brief Introduction*. Springer-Verlag, New York.

CHUNG, K. L. (1974): *A Course in Probability Theory*. Academic Press, New York, second edn.

DAVID, H. A. (1991): "Mean Minus Median: A Comment on O'Cinneide," *The American Statistician*, 45, 257.

DEVINATZ, A. (1968): *Advanced Calculus*. Holt, Rinehart and Winston, New York.

DHARMADHIKARI, S. (1991): "Bounds on Quantiles: A Comment on O'Cinneide," *The American Statistician*, 45, 257–258.

FERGUSON, T. S. (1967): *Mathematical Statistics: A Decision Theoretic Approach*. Academic Press, New York.

GILAT, D., AND T. P. HILL (1993): "Quantile-Locating Functions and the Distance Between the Mean and Quantiles," *Statistica Neerlandica*, 47, 279–283.

GLESER, L. J. (1985): "Exact Power of Goodness-of-Fit Tests of Kolmogorov Type for Discontinuous Distributions," *Journal of the American Statistical Association*, 80, 954–958.

GNEDENKO, B. V. (1969): *The Theory of Probability*. MIR Publishers, Moscow.

GUTERMAN, H. E. (1962): "An Upper Bound for the Sample Standard Deviation," *Technometrics*, 4, 134–135.

HAASER, N. B., AND J. A. SULLIVAN (1991): *Real Analysis*. Dover Publications, New York.

HANDCOCK, M. S., AND M. MORRIS (1999): *Relative Distribution Methods in the Social Sciences*. Springer, New York.

HOSSEINI, R. (2009): "Statistical Models for Agroclimate Risk Analysis," Ph.D. thesis, Department of Statistics, University of Bristish Columbia, Vancouver, Canada.

——— (2010): "Quantiles Equivariance," Discussion paper, University of British Columbia, Vancouver, Canada, arXiv:1004.0533v1.

KOLMOGOROV, A. N., AND S. V. FOMIN (1975): *Introductory Real Analysis*. Dover, Mineola, New York, Translated from Russian and edited by Richard A. Silverman.

LEHMANN, E. L., AND G. CASELLA (1998): *Theory of Point Estimation*, Springer Texts in Statistics. Springer-Verlag, New York, second edn.

LOÈVE, M. (1977): *Probability Theory, Volumes I and II*. Springer-Verlag, New York, 4th edn.

MALLOWS, C. L., AND D. RICHTER (1969): "Inequalities of Chebyshev Type Involving Conditional Expectations," *Annals of Mathematical Statistics*, 40, 1922–1932.

MORIGUTI, S. (1953): "A Modification of Schwarz's Inequality with Applications to Distributions," *Annals of Mathematical Statistics*, 24, 107–113.

O'CINNEIDE, C. A. (1990): "The Mean is Within One Standard Deviation of Any Median," *The American Statistician*, 44, 292–293, Acknowledgement, 45 (1991), 257-258.

PAGE, W., AND V. N. MURTY (1982): "Nearness Relations Among Measures of Central Tendency and Dispersion: Part 1," *Two Year College Mathematics Journal*, 13, 315–327.

——— (1983): "Nearness Relations Among Measures of Central Tendency and Dispersion: Part 2," *Two Year College Mathematics Journal*, 14, 8–17.

PARZEN, E. (1980): "Quantile Functions, Convergence in Quantile, and Extreme Value Distributions," Discussion Paper B-3, Statistical Institute, Texas A & M University, College Station, Texas.

PHILLIPS, E. R. (1984): *An Introduction to Analysis and Integration Theory*. Dover Publications, New York.

PROTTER, M. H., AND C. B. MORREY (1991): *A First Course in Real Analysis*, Undergraduate Texts in Mathematics. Springer-Verlag, New York, second edn.

RAMIS, E., C. DESCHAMPS, AND J. ODOUX (1982): *Cours de mathématiques spéciales 3: topologie et éléments d'analyse*. Masson, Paris, second edn.

RAO, C. R. (1973): *Linear Statistical Inference and its Applications*. John Wiley & Sons, New York, second edn.

REISS, H. D. (1989): *Approximate Distributions of Order Statistics with Applications to Nonparametric Statistics*, Springer Series in Statistics. Springer-Verlag, New York.

RIESZ, F., AND B. SZ.-NAGY (1955/1990): *Functional Analysis*. Dover Publications, New York, second edn.

ROSENBLATT, M. (1952): "Remarks on a Multivariate Transformation," *Annals of Mathematical Statistics*, 23, 470–472.

ROSS, K. A. (1980): *Elementary Analysis: The Theory of Calculus*, Undergraduate Texts in Mathematics. Springer-Verlag, New York.

ROYDEN, H. L. (1968): *Real Analysis*. MacMillan, New York, second edn.

RUDIN, W. (1976): *Principles of Mathematical Analysis, Third Edition*. McGraw-Hill, New York.

SHAKED, M., AND J. G. SHANTIKUMAR (1994): *Stochastic Order and their Applications*. Academic Press, Boston.

SHER, L. (1979): "The Range of the Standard Deviation," *Two Year College Mathematics Journal*, 10, 33.

SHORACK, G. R., AND J. A. WELLNER (1986): *Empirical Processes with Applications to Statistics*. John Wiley & Sons, New York.

THAS, O. (2010): *Comparing Distributions*. Springer, Berlin.

THOMPSON, W. R. (1935): "On a Criterion for the Rejection of Observations and the Distribution of the Ratio of Deviation to Sample Standard Deviation," *Annals of Mathematical Statistics*, 6, 214–219.

WILLIAMS, D. (1991): *Probability with Martingales*. Cambridge University Press, Cambridge, U.K.