Simultaneous equations *

Jean-Marie Dufour †
McGill University

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† William Dow Professor of Economics, McGill University, Centre interuniversitaire de recherche en analyse des organisations (CIRANO), and Centre interuniversitaire de recherche en économie quantitative (CIREQ). Mailing address: Department of Economics, McGill University, Leacock Building, Room 519, 855 Sherbrooke Street West, Montréal, Québec H3A 2T7, Canada. TEL: (1) 514 398 8879; FAX: (1) 514 398 4938; e-mail: jean-marie.dufour@mcgill.ca . Web page: http://www.jeanmariedufour.com

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1. Examples of simultaneous equations models

1.1. Simple Keynesian model

$$
\begin{aligned}
C_t &= \alpha + \beta Y_t + u_t & (1.1)\\
Y_t &= C_t + I_t & (1.2)
\end{aligned}
$$

where I_t does not depend on u_t, Y_t or C_t. One sees easily that Y_t and u_t are not independent, for

$$Y_t = \alpha + \beta Y_t + u_t + I_t \,.$$

Consequently, we expect that OLS will not yield consistent estimators of α and β. Indeed, we can express C_t and Y_t as functions of I_t and u_t :

$$
\begin{aligned}
C_t &= \frac{\alpha}{1-\beta} + \frac{\beta}{1-\beta} I_t + \frac{1}{1-\beta} u_t \,, & (1.3)\\
Y_t &= \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I_t + \frac{1}{1-\beta} u_t \,, & (1.4)
\end{aligned}
$$

which shows that C_t and Y_t are jointly determined, given I_t and u_t. We call the last two equations the *reduced form* of the model. C_t and Y_t are the *endogenous variables* of the model, while I_t is an *exogenous variable*.

1.2. Supply and demand model

Consider now the following equations:

$$
\begin{aligned}
q_t &= a_1 + b_1 p_t + c_1 Y_t + u_{t1} \,, \text{ (demand function)} & (1.5)\\
q_t &= a_2 + b_2 p_t + c_2 R_t + u_{t2} \,, \text{ (supply function)} & (1.6)
\end{aligned}
$$

where

$q_t =$ quantity (at time t), $p_t =$ price, $Y_t =$ income, $R_t =$ rain volume,

u_{t1} and u_{t2} are random disturbances.

We can solve these equations to express q_t and p_t as functions of Y_t and R_t. On subtracting (1.5) from (1.6), we get:

$$a_2 - a_1 + (b_2 - b_1)\, p_t + c_2 R_t - c_1 Y_t + u_{t2} - u_{t1} = 0,$$

hence

$$p_t = \frac{a_1 - a_2}{b_2 - b_1} + \frac{c_1}{b_2 - b_1} Y_t - \frac{c_2}{b_2 - b_1} R_t + \frac{u_{t1} - u_{t2}}{b_2 - b_1},$$

$$\begin{aligned} q_t &= a_2 + \frac{b_2 (a_1 - a_2)}{b_2 - b_1} + \frac{b_2 c_1}{b_2 - b_1} Y_t - \frac{b_2 c_2}{b_2 - b_1} R_t + \frac{b_2}{b_2 - b_1} (u_{t1} - u_{t2}) + c_2 R_t + u_{t2} \\ &= \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1} + \frac{b_2 c_1}{b_2 - b_1} Y_t - \frac{b_1 c_2}{b_2 - b_1} R_t + \frac{b_2 u_{t1} - b_1 u_{t2}}{b_2 - b_1} \end{aligned}$$

or, in a more compact way,

$$q_t = \pi_1 + \pi_2 Y_t + \pi_3 R_t + v_{t1}, \tag{1.7}$$

$$p_t = \pi_4 + \pi_5 Y_t + \pi_6 R_t + v_{t2}, \tag{1.8}$$

with

$$\pi_1 = \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1}, \quad \pi_2 = \frac{b_2 c_1}{b_2 - b_1}, \quad \pi_3 = -\frac{b_1 c_2}{b_2 - b_1},$$

$$\pi_4 = \frac{a_1 - a_2}{b_2 - b_1}, \quad \pi_5 = \frac{c_1}{b_2 - b_1}, \quad \pi_6 = -\frac{c_2}{b_2 - b_1},$$

$$v_{t1} = \frac{b_2 u_{t1} - b_1 u_{t2}}{b_2 - b_1}, \quad v_{t2} = \frac{u_{t1} - u_{t2}}{b_2 - b_1}.$$

We easily see that

p_t and u_{t1} are not independent

p_t and u_{t2} are not independent.

OLS applied to (1.5) and (1.6) does not yield consistent estimators of the parameters of the equations.

2. Notations and hypotheses

The above two models are special cases of simultaneous equations models. The general form of a linear simultaneous equations model:

$$\Sigma_{j=1}^{G} b_{jg} y_{tj} + \Sigma_{k=1}^{K} \gamma_{kg} x_{tk} = u_{tj}, \; g = 1, \ldots, G, \;\; t = 1, \ldots, T, \tag{2.1}$$

or, in matrix notation,

$$B' Y_t + \Gamma' X_t = U_t \quad t = 1, \ldots, T, \tag{2.2}$$

where

$$Y_t = (y_{t1}, \ldots, y_{tG})', \; X_t = (x_{t1}, \ldots, x_{tK})', \; U_t = (u_{t1}, \ldots, u_{tG})', \tag{2.3}$$

$$B = \left[b_{gj}\right]_{\substack{g=1,\ldots,G \\ j=1,\ldots,G}}, \quad \Gamma = \left[\gamma_{kg}\right]_{\substack{g=1,\ldots,G \\ k=1,\ldots,K}}. \tag{2.4}$$

The g-th columns of the matrices B and Γ contain the coefficients of the g-th equation ($g = 1, \ldots, G$). More compactly, we can write:

$$B'Y' + \Gamma'X' = U',$$

and

$$YB + X\Gamma = U, \text{ (structural form)} \tag{2.5}$$

where

$$\begin{array}{lll} Y = [Y_1, \ldots, Y_T]', & X = [X_1, \ldots, X_T]', & U = [U_1, \ldots, U_T]', \\ Y : T \times G, & X : T \times K, & U : T \times G. \end{array}$$

To each row of Y, X and U corresponds an observation and to each column corresponds an equation. We call the equivalent equations (2.1), (2.2) or (2.5) the *structural form* of the model. Concerning random disturbances, we suppose that

$$\mathsf{E}\left[u_{si}u_{tj} \,|\, X_t\right] = \sigma_{ij}\delta_{st} = \begin{cases} \sigma_{ij} & \text{if} \quad s = t \\ 0 & \text{if} \quad s \neq t \end{cases} \tag{2.6}$$

or equivalently,

$$\begin{aligned} \mathsf{E}\left[U_s U_t' \,|\, X_t\right] &= \Sigma = \left[\sigma_{ij}\right], \quad \text{if} \quad s = t \\ &= 0 \qquad\qquad\quad, \quad \text{if} \quad s \neq t. \end{aligned} \tag{2.7}$$

Finally, we suppose that

$$\det(B) \neq 0, \tag{2.8}$$

i.e., the matrix B is invertible. This assumption must be satisfied, in particular, if we suppose that

$$\det(\Sigma) \neq 0, \tag{2.9}$$

i.e., if the covariance matrix nonsingular. Indeed, if this were not the case, we could find a fixed vector $a \neq 0$ with dimension $G \times 1$ such that $Ba = 0$, hence

$$a'B'Y_t + a'\Gamma'X_t = a'\Gamma'X_t = a'U_t$$

and

$$a'\Sigma a = V[a'U_t \,|\, X_t] = V[a'\Gamma'X_t \,|\, X_t] = 0,$$

which means that Σ is a singular matrix $[\det(\Sigma) = 0]$. Note further that the invertibility of B entails that each row and each column of B must differ from zero, so that there is at least one endogenous variable in each equation.

If we multiply by B^{-1}, we get

$$Y = -X\Gamma B^{-1} + UB^{-1} = X\Pi + V \,, \text{ (reduced form)} \tag{2.10}$$

where

$$\Pi = -\Gamma B^{-1} \,,$$
$$V = UB^{-1} = [V_1, \ldots, V_T]' = [U_1, \ldots, U_T]' B^{-1} \,,$$
$$V_t' = U_t' B^{-1}, V_t = (B^{-1})' U_t \,,$$

$$\begin{aligned} \mathsf{E}\left[V_s V_t'\right] &= \left(B^{-1}\right)' \Sigma \left(B^{-1}\right), \text{ if } \quad s = t \\ &= 0 \qquad\qquad\qquad , \text{ if } \quad s \neq t \,. \end{aligned}$$

Equation (2.10) is called the *reduced form* of the model.

3. The identification problem

3.1. Special cases

Let us go back to the model of supply and demand in section 1. On using the reduced form

$$\begin{aligned} q_t &= \pi_1 + \pi_2 Y_t + \pi_3 R_t + v_{t1} \,, \\ p_t &= \pi_4 + \pi_5 Y_t + \pi_6 R_t + v_{t2} \,, \end{aligned}$$

we can estimate $\pi_1, \ldots, \pi_6$ by OLS, for Y_t and R_t are independent of v_{t1} and v_{t2}. Further, we can express $a_1, b_1, c_1, a_2, b_2, c_2$ as functions of $\pi_1, \ldots, \pi_6$:

$$\begin{aligned} b_1 &= \frac{\pi_3}{\pi_6}, \quad b_2 = \frac{\pi_2}{\pi_5}, \quad c_2 = -\pi_6 (b_2 - b_1) \,, \\ c_1 &= \pi_5 (b_2 - b_1) \,, \quad a_1 = \pi_1 - b_1 \pi_6 \,, \quad a_2 = \pi_1 - b_2 \pi_4 \,. \end{aligned}$$

On replacing π_1 by $\hat{\pi}_1$, etc., we can obtain estimates of the structural parameters $a_1, b_1, \ldots$ (indirect least squares method).

Consider now the model:

$$q_t = a_1 + b_1 p_t + c_1 Y_t + u_{t1} \,, \quad \text{(demand function)} \tag{3.1}$$
$$q_t = a_2 + b_2 p_t + u_{t2} \,, \quad \text{(supply function).} \tag{3.2}$$

The reduced form is then

$$\begin{aligned} q_t &= \pi_1 + \pi_2 Y_t + v_{t1}\,, \\ p_t &= \pi_4 + \pi_5 Y_t + v_{t2}\,, \end{aligned}$$

where

$$\pi_1 = \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1}\,, \quad \pi_2 = \frac{c_1 b_2}{b_2 - b_1}\,, \; \pi_4 = \frac{a_1 - a_2}{b_2 - b_1}\,, \quad \pi_5 = \frac{c_1}{b_2 - b_1}\,,$$

$$v_{t1} = \frac{b_2 u_{t1} - b_1 u_{t2}}{b_2 - b_1}\,, \quad v_{t2} = \frac{u_{t1} - u_{t2}}{b_2 - b_1}\,,$$

hence the equations

$$b_2 = \frac{\pi_2}{\pi_5}\,, \quad a_2 = \pi_1 - b_2 \pi_4$$

from which we can estimate b_2 and a_2. In this case, there is no unique solution for a_1, b_1 and c_1. Only the supply function can be estimated. We then say that the demand function is not identified (or is *underidentified*). If we wish to get a unique solution, we must add constraints.

Similarly, if we consider the equations

$$\begin{aligned} q_t &= a_1 + b_1 p_t + c_1 Y_t + d_1 R_t + u_{t1}\,, \quad \text{(demand function)} \\ q_t &= a_2 + b_2 p_t + u_{t2}\,, \quad \text{(supply function)} \end{aligned}$$

the reduced form becomes:

$$\begin{aligned} q_t &= \pi_1 + \pi_2 Y_t + \pi_3 R_t + v_{t1}\,, \\ p_t &= \pi_4 + \pi_5 Y_t + \pi_6 R_t + v_{t2}\,, \end{aligned}$$

where

$$\pi_1 = \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1}\,, \quad \pi_2 = \frac{c_1 b_2}{b_2 - b_1}\,, \quad \pi_3 = \frac{d_1 b_2}{b_2 - b_1}\,, \quad v_{t1} = \frac{b_2(u_{t1} - u_{t2})}{b_2 - b_1}$$

$$\pi_4 = \frac{a_1 - a_2}{b_2 - b_1}\,, \quad \pi_5 = \frac{c_1}{b_2 - b_1}\,, \quad \pi_6 = \frac{d_1}{b_2 - b_1}\,, \quad v_{t2} = \frac{u_{t1} - u_{t2}}{b_2 - b_1}\,.$$

Here, we can compute b_2 in two different ways:

$$b_2 = \frac{\pi_2}{\pi_5}\,, \quad b_2 = \frac{\pi_3}{\pi_6}\,.$$

Consequently, we also have:

$$\begin{aligned} \frac{\pi_2}{\pi_5} &= \frac{\pi_3}{\pi_6}, \\ a_2 &= \pi_1 - b_2\pi_4 = \pi_1 - \left(\frac{\pi_2}{\pi_5}\right)\pi_4 = \pi_1 - \left(\frac{\pi_3}{\pi_6}\right)\pi_5 . \end{aligned}$$

We then say that the equation is *overidentified*. The overidentification of the equation entails restrictions on the parameters of the reduced form. Further, we can easily verify that the demand equation is not identified.

Another way of studying the identification problem consists in examining linear combinations of the equations. Consider again:

$$q_t = a_1 + b_1 p_t + c_1 Y_t + u_{t1}, \tag{3.3}$$

$$q_t = a_2 + b_2 p_t + u_{t2} . \tag{3.4}$$

Take a linear combination of the two previous equations:

$$\begin{aligned} q_t &= w(a_1 + b_1 p_t + c_1 Y_t + u_{t1}) + (1-w)(a_2 + b_2 p_t + u_{t2}) \\ &= [wa_1 + (1-w)a_2] + [wb_1 + (1-w)b_2]\, p_t \\ &\quad + wc_1 Y_t + [wu_{t1} + (1-w)u_{t2}] \\ &= a_1^* + b_1^* p_t + c_1^* Y_1 + u_{t1}^* . \end{aligned} \tag{3.5}$$

Equation (3.5) cannot be distinguished from (3.3).

3.2. Identification conditions for equations with omitted variables

Let us now study the general equation system:

$$B'Y_t + \Gamma' X_t = U_t\,, \quad t = 1, \ldots, T \ .$$

Let β' the first row of B, γ' the first row of Γ, and u_{t1} the first element of U_t. The first equation of the system can be written:

$$\beta' Y_t + \gamma' X_t = u_{t1}\,, \tag{3.6}$$

where $\beta \neq 0$ (by the invertibility of B), $U_t = (u_{t1}, U_{t2}')'$ and U_{t2} is a vector with dimension $(G-1)\times 1$. We will now study the case where there are G_1 endogenous variables and K_1 exogenous variables in this equation.

To do this, we consider the following partitions of the variable and parameter vectors:

$$Y_t = \left(Y'_{t1}, Y'_{t2}\right)', \quad X_t = \left(X'_{t1}, X'_{t2}\right)',$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

where

$$\begin{aligned} \beta_1 &: \text{ coefficients of the } G_1 \text{ endogenous variables (in the equation),} \\ \beta_2 &: \text{ coefficients of the } G_2 \text{ excluded endogenous variables,} \\ \gamma_1 &: \text{ coefficients of the } K_1 \text{ included exogenous variables,} \\ \gamma_2 &: \text{ coefficients of the } K_2 \text{ excluded exogenous variables,} \\ G &= G_1 + G_2, K = K_1 + K_2. \end{aligned}$$

Further, if B and Γ are partitioned conformably with β and γ, *i.e.*,

$$B = \begin{bmatrix} \beta_1 & B_1 \\ \beta_2 & B_2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1 & \Gamma_1 \\ \gamma_2 & \Gamma_2 \end{bmatrix}, B_i : G_i \times (G-1), \Gamma_i : G_i \times (G-1), i = 1, 2,$$

we see that the system (3.6) can be rewritten:

$$\beta_1' Y_{t1} + \beta_2' Y_{t2} + \gamma_1' X_{t1} + \gamma_2' X_{t2} = u_{t1}, \; t = 1, \ldots, T, \tag{3.7}$$

$$B_1' Y_{t1} + B_2' Y_{t2} + \Gamma_1' X_{t1} + \Gamma_2' X_{t2} = U_{t2}, \; t = 1, \ldots, T. \tag{3.8}$$

We suppose that $\beta_2 = 0$ and $\gamma_2 = 0$, so that equation (3.7) has the form

$$\beta_1' Y_{t1} + \gamma_1' X_{t1} = u_{t1}, \; t = 1, \ldots, T. \tag{3.9}$$

The parameters of the reduced form are linked to the structural parameters by the formula:

$$\Pi = -\Gamma B^{-1}$$

or

$$\Pi B = -\Gamma. \tag{3.10}$$

Since β and γ are the first columns of B and Γ respectively, we can write:

$$\Pi \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = -\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

hence

$$\Pi \begin{pmatrix} \beta_1 \\ 0 \end{pmatrix} = -\begin{pmatrix} \gamma_1 \\ 0 \end{pmatrix} .$$

If we partition Π conformably with β,

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix} , \ \Pi_{ij} : K_i \times G_j \ , i, j = 1, 2 ,$$

we must have:

$$\begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix} \begin{pmatrix} \beta_1 \\ 0 \end{pmatrix} = -\begin{pmatrix} \gamma_1 \\ 0 \end{pmatrix}$$

or, equivalently,

$$\Pi_{11}\beta_1 = -\gamma_1 , \tag{3.11}$$

$$\Pi_{21}\beta_1 = 0 . \tag{3.12}$$

For the first row of the equation to be identified, we must be able to solve in a unique way the first two equations for β_1 and γ_1. Equation (3.11) only allows one to get γ_1 from β_1 and Π_{11}. Consequently, equation (3.12) determines β_1. Since $\beta_1 \neq 0$ and the equation (3.12) is homogeneous, we must have:

$$0 \leq \text{rank}(\Pi_{21}) \leq G_1 - 1 .$$

If we had $\text{rank}(\Pi_{21}) = G_1$, $\beta_1 = 0$ would be the only solution. The set of the solutions of the equation $\Pi_{21}\beta_1 = 0$ is a vector subspace of $\mathbb{R}^{G_1}$ whose dimension is equal to $G_1 - \text{rank}(\Pi_{21})$. This set corresponds to a unique vector up to a multiplicative constant if and only if the solution space has dimension 1, *i.e.*, if $\text{rank}(\Pi_{21}) = G_1 - 1$. We thus get the following condition which is necessary and sufficient for β_1 to be uniquely determined up to a multiplicative constant:

$$\text{rank}(\Pi_{21}) = G_1 - 1 \quad \text{(rank condition for identification).} \tag{3.13}$$

For this condition ton be satisfied, it is also necessary (but not sufficient) that

$$\begin{aligned} & K_2 \geq G_1 - 1 \\ \Leftrightarrow \quad & G_2 + K_2 \geq G_2 + G_1 - 1 \\ \Leftrightarrow \quad & G_2 + K_2 \geq G - 1 \quad \text{(order condition for identification).} \end{aligned} \tag{3.14}$$

In other words, the number of excluded exogenous variables in the equation must be at least equal to the number of included endogenous variables less one, or the total number of excluded variables must be at least equal to the total number of endogenous variables in the system less one. If $K_2 + G_2 = G - 1$, we say that the equation is *exactly identified*. If $K_2 + G_2 > G - 1$, we say it is *overidentified*.

Finding the rank of Π_{12} is however difficult. Consider

$$B = \begin{bmatrix} \beta_1 & B_1 \\ 0 & B_2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1 & \Gamma_1 \\ 0 & \Gamma_2 \end{bmatrix},$$

as well as the $(G_2 + K_2) \times G$ matrix of structural coefficients on the endogenous and exogenous excluded from the first equation but appearing in the other equations of the model:

$$D = \begin{bmatrix} 0 & \Gamma_2 \\ 0 & B_2 \end{bmatrix}.$$

By equation (3.10), we see easily that

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} \beta_1 & B_1 \\ 0 & B_2 \end{bmatrix} = - \begin{bmatrix} \gamma_1 & \Gamma_1 \\ 0 & \Gamma_2 \end{bmatrix}$$

hence

$$\Pi_{21}\beta_1 = 0,$$
$$\Pi_{21}B_1 + \Pi_{22}B_2 = -\Gamma_2$$

and

$$D = \begin{bmatrix} 0 & \Gamma_2 \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} -\Pi_{21} & -\Pi_{22} \\ 0 & I_{G_2} \end{bmatrix} \begin{bmatrix} \beta_1 & B_1 \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} -\Pi_{21} & -\Pi_{22} \\ 0 & I_{G_2} \end{bmatrix} B.$$

Since matrix B is nonsingular, we can conclude that

$$\text{rank}(D) = \text{rank}\left(\begin{bmatrix} -\Pi_{21} & -\Pi_{22} \\ 0 & I_{G_2} \end{bmatrix} \right) = \text{rank}(\Pi_{21}) + G_2$$

and

$$\text{rank}(\Pi_{21}) = G_1 - 1 \Leftrightarrow \text{rank}(D) = G_1 - 1 + G_2 = G - 1.$$

By the rank condition (3.13), equation (3.6) is thus exactly identified if and only if

$$\text{rank}\,(D) = G - 1 \text{ (structural rank condition).} \tag{3.15}$$

3.3. Identification conditions based on general linear constraints

The rank condition

$$\text{rank}(\Pi_{21}) = G_1 - 1$$

can be generalized to general linear restrictions of the form as follows. We have

$$\Pi'\beta + \gamma = 0 \tag{3.16}$$

or

$$[\Pi' \,, I_K]\begin{pmatrix}\beta\\ \gamma\end{pmatrix} = \overline{\Pi}\begin{pmatrix}\beta\\ \gamma\end{pmatrix} = 0 \tag{3.17}$$

where $\overline{\Pi} = [\Pi' \,, I_K]$ is a $K \times (G+K)$ matrix. In general, equation (3.17) does not have a unique solution (even to a multiplicative factor). To have a unique solution, we must add r_1 constraints

$$\Phi_1\begin{pmatrix}\beta\\ \gamma\end{pmatrix} = c_1 \ , \tag{3.18}$$

where Φ_1 is a $r_1 \times (G+K)$ matrix and c_1 is a $r_1 \times 1$ vector. On consolidating (3.17) and (3.18), we get the system:

$$\begin{bmatrix}\overline{\Pi}\\ \Phi_1\end{bmatrix}\begin{pmatrix}\beta\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\ c_1\end{pmatrix} . \tag{3.19}$$

This system has a unique solution for $(\beta', \gamma')'$ if and only if

$$\text{rank}\left(\begin{bmatrix}\overline{\Pi}\\ \Phi_1\end{bmatrix}\right) = G + K \ \ \text{(generalized rank condition).}$$

This condition entails, in particular, that

$$r_1 \geq G_1 \ , \ \ \text{(generalized order condition)}$$

i.e., the number of constraints must be at least equal to the number of endogenous variables in the system.

In this context, we can also formulate a rank condition similar to (3.15) which is expressed in terms of the matrices B and Γ. The equation

$$B'Y_t + \Gamma' X_t = U_t$$

can be written:

$$AZ_t = U_t \, ,$$

with

$$A = \left[B', \Gamma'\right], Z_t = \begin{bmatrix} Y_t \\ Z_t \end{bmatrix}.$$

Let α_1' be the first row of A (parameters of the first equation). The restrictions on the first equation may then be written

$$\alpha_1' \phi = 0$$

or

$$\left(\ell_1' A\right) \phi = 0,$$

where $\alpha_1' = \ell_1' A$ and $\ell_1 = (1, 0, \ldots, 0)'$. If we multiply $AZ_t = U_t$ by a nonsingular matrix, the first equation satisfies the same restrictions: the transformed system

$$FAZ_t = FU_t,$$

must satisfy

$$F_1' A\phi = 0, \quad F_1 = c \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

A necessary and sufficient condition for this to hold is:

$$\text{rank}(A\phi) = G - 1.$$

4. Estimation: limited information methods

For the estimation of simultaneous equations, one typically distinguishes between two types of methods:

1. limited-information methods: the parameters of a single equation are estimated, without taking into account the information contained in the other equations;

2. full-information methods: the parameters of all the equations are estimated jointly, taking into account restrictions entailed by different equations.

There are several limited-information methods. The simplest and most widely used is two-stage least squares (2SLS). Let the equation

$$\begin{aligned} y &= Y_1\beta + X_1\gamma + u \\ &= Z_1\delta + u \end{aligned} \tag{4.1}$$

where

$$\begin{array}{rcl} y & : & T\times 1 \text{ observation vector on the endogenous dependent variable,} \\ Y_1 & : & T\times \overline{G}_1 \text{ matrix of observations on the other endogenous variables,} \\ X_1 & : & T\times K_1 \text{ matrix of observations on included exogenous variables,} \\ X & = & [X_1, X_2] : T\times K \text{ matrix of observations on all the exogenous variables,} \end{array}$$

β and δ are parameters vectors to estimate,

$$u : T\times 1 \text{ vector of random disturbances,}$$
$$Z_1 = [Y_1,\, X_1]\,, \delta = \begin{pmatrix} \beta \\ \delta \end{pmatrix},$$
$$\mathsf{E}\left(uu'\right) = \sigma^2 I_T\,,$$

$$\begin{array}{rcl} G_1 & = & \overline{G}_1 + 1 = \text{number of exogenous in the equation,} \\ G & = & G_1 + G_2 = \text{total number of endogenous variables,} \\ K & = & K_1 + K_2 = \text{total number of endogenous variables.} \end{array}$$

We suppose that the equation (4.1) is identified, which entails that

$$G_2 + K_2 \geq G - 1 \quad \text{(order condition for identification).}$$

Let us multiply (4.1) by $X^{'}$:

$$\begin{aligned} X^{'}y &= X'Z_1\delta + X'u \\ &= X'Z_1\delta + v \end{aligned} \tag{4.2}$$

where

$$\mathsf{E}\left[vv'\right] = \sigma^2 X'X\,.$$

If we apply GLS to the transformed equation (4.2), we obtain:

$$\hat{\delta}_{2S} = \left[\left(Z_1'X\right)\left(X'X\right)^{-1}X^{'}Z_1\right]^{-1}\left(Z_1'X\right)\left(X'X\right)^{-1}X^{'}y$$

which is called the two-stage least squares (2SLS). If we consider the reduced form for Y_1, we get an expression of the form:

$$Y_1 = X\Pi_1 + V_1\,,$$

hence

$$\mathsf{E}(Y_1) = X\Pi_1 \; .$$

Consequently,

$$\begin{aligned} y &= Y_1\beta + X_1\gamma + u \\ &= [\mathsf{E}(Y_1) + V_1]\,\beta + X_1\gamma + u \\ &= \mathsf{E}(Y_1)\,\beta + X_1\gamma + (u + V_1\beta) \\ &= \mathsf{E}(Y_1)\,\beta + X_1\gamma + u^* \end{aligned}$$

where

$$u^* = u + V_1\beta \; .$$

If we knew $\mathsf{E}(Y_1)$,we could estimate β and γ by OLS. We can estimate $\mathsf{E}(Y_1)$ by

$$\hat{Y}_1 = X\hat{\Pi}_1 = X\left(X'X\right)^{-1}X'Y_1 \; .$$

If we write

$$\hat{V}_1 = Y_1 - \hat{Y}_1 = M_X Y_1\,, M_1 = I - X\left(X'X\right)^{-1}X^1\,,$$

then

$$\begin{aligned} y &= \left(\hat{Y}_1 + \hat{V}_1\right)\beta + X_1\gamma + u \\ &= \hat{Y}_1\beta + X_1\gamma + u^{**} \\ &= \hat{Z}_1\delta + u^{**}\,, \end{aligned} \tag{4.3}$$

where

$$u^{**} = u + \hat{V}_1\beta,\; \hat{Z}_1 = \left(\hat{Y}_1, X_1\right) \; .$$

We can then apply OLS to equation (4.3):

$$\hat{\delta}_{IV} = \left(\hat{Z}_1'\hat{Z}_1\right)^{-1}\hat{Z}_1'y$$

hence the name “two-stage least squares”.

We thus have apparently two estimators:

$$\begin{aligned} \hat{\delta}_{2S} &= \left[\left(Z_1'X\right)\left(X'X\right)^{-1}\left(X'Z_1\right)\right]^{-1}\left(Z_1'X\right)\left(X'X\right)^{-1}X'y\,, \\ \hat{\delta}_{IV} &= \left(\hat{Z}_1'\hat{Z}_1\right)^{-1}\hat{Z}_1'y\,. \end{aligned}$$

We will now show that $\hat{\delta}_{2S} = \hat{\delta}_{IV}$:

$$\begin{aligned} \hat{\delta}_{IV} &= \left[\begin{pmatrix} \hat{Y}_1' \\ X_1' \end{pmatrix} (\hat{Y}_1, X_1) \right]^{-1} \begin{pmatrix} \hat{Y}_1' \\ X_1' \end{pmatrix} y , \\ &= \begin{bmatrix} \hat{Y}_1'\hat{Y}_1 & \hat{Y}_1' X_1 \\ X_1'\hat{Y}_1 & X_1' X_1 \end{bmatrix}^{-1} \begin{pmatrix} \hat{Y}_1' y \\ X_1' y \end{pmatrix} , \end{aligned}$$

where

$$\begin{aligned} Z_1' X \left(X'X\right)^{-1} X' &= \begin{bmatrix} \hat{Y}_1' \\ X_1' \end{bmatrix} X \left(X'X\right)^{-1} X' \\ &= \begin{bmatrix} Y_1' X \left(X'X\right)^{-1} X' \\ X_1' X \left(X'X\right)^{-1} X' \end{bmatrix} = \begin{pmatrix} \hat{Y}_1' \\ X_1' \end{pmatrix} = \hat{Z}_1' , \end{aligned}$$

for

$$X = (X_1 \, , X_2) \, ,$$

$$\begin{aligned} X' X \left(X'X\right)^{-1} X' &= X' = \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} X \left(X'X\right)^{-1} X' , \\ &= \begin{pmatrix} X_1' X \left(X'X\right)^{-1} X' \\ X_2' X \left(X'X\right)^{-1} X' \end{pmatrix} = \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} , \end{aligned}$$

and

$$\begin{aligned} Z_1' X \left(X'X\right)^{-1} X' Z_1 &= Z_1' X \left(X'X\right)^{-1} X' X \left(X'X\right)^{-1} X' Z_1 \\ &= \begin{pmatrix} \hat{Y}_1' \\ X_1' \end{pmatrix} [Y_1, Z_1] = \hat{Z}_1' \hat{Z}_1 \, . \end{aligned}$$

Thus

$$\hat{\delta}_{2S} = \left(\hat{Z}_1'\hat{Z}_1\right)^{-1} \hat{Z}_1' y = \hat{\delta}_{1V} \, .$$

Suppose

$$\begin{aligned} \underset{T\to\infty}{p\lim} \frac{X'X}{T} &= Q_x \, , \det(Q_x) > 0 \, , \\ p\lim \frac{Z_1' X}{T} &= Q_1 \, , \, \mathrm{rank}(Q_1) = \overline{G}_1 + K_1 \, , \\ p\lim \frac{X' u}{T} &= 0 \, . \end{aligned}$$

Then

$$
\begin{aligned}
\hat{\delta}_{2S} &= \left[\left(Z_1'X_1\right)\left(X'X\right)^{-1}\left(X'Z_1\right)\right]^{-1}\left(Z_1'X\right)\left(X'X\right)^{-1}X'\left(Z_1\delta+u\right) \\
&= \delta + \left[\left(Z_1'X\right)\left(X'X\right)^{-1}\left(Y'Z_1\right)\right]^{-1}\left(Z_1'X\right)\left(X'X\right)^{-1}X'u \\
& p\lim\left(\hat{\delta}_{2S}-\delta\right) \\
&= p\lim\left[\left(\frac{Z_1'X}{T}\right)\left(\frac{X'X}{T}\right)^{-1}\left(\frac{Y'Z_1}{T}\right)\right]^{-1}\left(\frac{Z_1'X}{T}\right)\left(\frac{X'X}{T}\right)^{-1}\frac{X'u}{T} \\
&= 0\,,
\end{aligned}
$$

i.e., $\hat{\delta}_{2S}$ is a consistent estimator of δ. To estimate σ^2, we use

$$
\hat{\sigma}^2_{2S} = \left(y - Y_1\hat{\beta} - X_1\hat{\gamma}\right)'\left(y - Y_1\hat{\beta} - X_1\hat{\gamma}\right)/\left(T - G_1 - k_1\right)\,.
$$

Under the same conditions, we can show that

$$
\sqrt{T}\left(\hat{\delta}_{2S}-\delta\right) \to N\left[0,\sigma^2 Q_{2S}\right]
$$

where

$$
\begin{aligned}
Q_{2S} &= p\lim\left[\left(\frac{Z_1'X}{T}\right)\left(\frac{X'X}{T}\right)^{-1}\left(\frac{Y'Z_1}{T}\right)\right]^{-1} \\
&= \sigma^2\left[\left(p\lim\frac{Z_1'X}{T}\right)\left(p\lim\frac{X'X}{T}\right)^{-1}\left(p\lim\frac{X'Z_1}{T}\right)^{-1}\right]\,.
\end{aligned}
$$

We can estimate Q_{2S} by

$$
\hat{\sigma}^2_{2S}\left[\left(\frac{Z_1'X}{T}\right)\left(\frac{Y'X}{T}\right)^{-1}\left(\frac{Y'Z_1}{T}\right)\right]^{-1} = \hat{\sigma}^2_{2S}\left[\frac{1}{T}\hat{Z}_1'\hat{Z}_1\right]^{-1}\,.
$$

5. Estimation: full-information methods

Full-information methods use information contained in all the equations. The simplest of these is three-stage least squares. We consider G structural equations:

$$
y_i = Y_i\beta_i + X_i\gamma_i + u_i\,,\quad i = 1,\ldots,G,
$$

where $X_i : T \times K_i$, $Y_i : T \times G_i$, and $G - G_i + K + K_i \geq G - 1$. Write

$$y_i = Z_i \delta_i + u_i\,, \quad i = 1, \ldots, G,$$

where

$$Z_i = [Y_i, X_i]\,, \delta_i = \begin{bmatrix} \beta_i \\ \gamma_i \end{bmatrix},$$
$$\mathsf{E}\left[u_i u_j'\right] = \sigma_{ij} I_T\ .$$

Then

$$X' y_i = X' Z_i \delta_i + X' u_i\,, \quad i = 1, \ldots, G\,,$$

$$\begin{bmatrix} X'y_1 \\ X'y_2 \\ \vdots \\ X'y_G \end{bmatrix} = \begin{bmatrix} X'Z_1 & 0 & \ldots & 0 \\ 0 & X'Z_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & X'Z_G \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_G \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_G \end{bmatrix},$$

and

$$\left(I \otimes X'\right) y = \left(I \otimes X'\right) Z\delta + \left(I \otimes X'\right) u$$

where

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_G \end{pmatrix}, Z = \begin{pmatrix} Z_1 & 0 & \ldots & 0 \\ 0 & Z_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & Z_G \end{pmatrix}, u = \begin{pmatrix} u_1 \\ \vdots \\ u_G \end{pmatrix},$$

$$V\left[\left(I \otimes X'\right) u\right] = \Sigma \otimes \left(X'X\right), \Sigma = \left[\sigma_{ij}\right]_{i,j=1,\ldots,G}\ .$$

If we knew Σ, we could use the GLS-type estimator:

$$\hat{\delta}_{3S} = \left(Z'\left[\Sigma^{-1} \otimes X\left(X'X\right)^{-1} X'\right] Z\right)^{-1} Z'\left[\Sigma^{-1} \otimes X\left(X'X\right)^{-1} X'\right] y\,.$$

Since Σ is unknown, we can estimate it from 2SLS residuals:

$$\hat{\Sigma} = \left[\hat{\sigma}_{ij}\right]\,,\ \hat{\sigma}_{ij} = \hat{u}_{ii}' \hat{u}_{jj}/T\,, \quad \hat{u}_i = y_i - Z_i \hat{\delta}_{i2S}\,,\ \ i\,, j = 1, \ldots, G,$$

which yields the estimator

$$\hat{\delta}_{3S} = \left(Z'\left[\hat{\Sigma}^{-1} \otimes X\left(X'X\right)^{-1} X'\right] Z\right)^{-1} Z'\left[\hat{\Sigma}^{-1} \otimes X\left(X'X\right)^{-1} X'\right] y\,.$$

Under general conditions, we can show that

$$\begin{aligned}\sqrt{T}\left(\hat{\delta}_{3S}-\delta\right) &\to N\left[0,\Sigma_{3S}\right], \\ \Sigma_{3S} &= p\lim\left[\frac{1}{T}Z'\left(\Sigma^{-1}\otimes X\left(X'X\right)^{-1}X'\right)Z\right]^{-1}.\end{aligned}$$

6. Sources and chronological list of references

1. Maddala (1977)
2. Chow (1983)
3. Hausman (1983)
4. Hsiao (1983)

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