Seemingly unrelated regressions *

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1. Model

Consider m linear regressions of the form:

$$y_i = X_i\beta_i + u_i\;, \quad i = 1, \ldots, m \tag{1.1}$$

where

$$y_i \text{ and } u_i \text{ are } T \times 1 \text{ vectors,} \tag{1.2}$$
$$X_i \text{ is a } T \times k_i \text{ matrix,} \tag{1.3}$$
$$1 \leq \operatorname{rank}(X_i) = k_i < T\,, \tag{1.4}$$
$$E\left[u_i u_j'\right] = \sigma_{ij} I_T\,. \tag{1.5}$$

These m relations can be written:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & X_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \tag{1.6}$$

or

$$y = X\beta + u$$

where $y,\ X,\ \beta$ and u are vectors (or matrices) of dimensions $(Tm) \times 1$, $(Tm) \times k$, $k \times 1$ and $(Tm) \times 1$ respectively,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & X_m \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

$$k = k_1 + k_2 + \cdots + k_m\;,$$

$$V(u) \equiv \Sigma = \begin{bmatrix} \sigma_{11} I_T & \sigma_{12} I_T & \cdots & \sigma_{1m} I_T \\ \sigma_{21} I_T & \sigma_{22} I_T & \cdots & \sigma_{2m} I_T \\ \vdots & \vdots & & \vdots \\ \sigma_{m1} I_T & \sigma_{m2} I_T & \cdots & \sigma_{mm} I_T \end{bmatrix} = \Sigma_c \otimes I_T\;,$$

$$\Sigma_c = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} .$$

Note _ $\otimes$ represents Kronecker's product. If

$$A = [a_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \ , \ B = [b_{ij}]_{\substack{i=1,\dots,p \\ j=1,\dots,q}} \ ,$$

then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix} .$$

Properties of Kronecker's product:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \ , \tag{1.7}$$

$$(A \otimes B)' = A' \otimes B', \tag{1.8}$$

$$A \otimes (B + C) = A \otimes B + A \otimes C \,, \tag{1.9}$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{1.10}$$

2. Estimation

2.1. Generalized least squares

The generalized least squares estimator of β in model (1.6) is given by:

$$\hat{\beta}_Z = \left(X' \Sigma^{-1} X\right)^{-1} X' \Sigma^{-1} y \,, \tag{2.1}$$

$$V\left(\hat{\beta}_Z\right) = \left(X' \Sigma^{-1} X\right)^{-1} \ . \tag{2.2}$$

The idea of using generalized least squares to jointly estimate several regression was suggested by Zellner ("Seemingly unrelated regressions"). In general, $\hat{\beta}_Z$ is an estimator of β more efficient than OLS applied to each equation in (1.1).

2.2. Equivalence conditions with ordinary least squares

It is possible to identify cases where the two methods are equivalent.

1. $\sigma_{ij} = 0, \ \forall i \neq j$ (errors uncorrelated between equations).
   In this case,

$$\Sigma_c = \begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{mm} \end{bmatrix},$$

   hence

$$\Sigma^{-1} = \Sigma_c^{-1} \otimes I_T = \begin{bmatrix} \frac{1}{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_{mm}} \end{bmatrix} \otimes I_T = \begin{bmatrix} \frac{1}{\sigma_{11}} I_T & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{22}} I_T & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_{mm}} I_T \end{bmatrix},$$

$$X'\Sigma^{-1}X = \begin{bmatrix} \frac{1}{\sigma_{11}} (X_1'X_1) & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{22}} (X_2'X_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_{mm}} (X_m'X_m) \end{bmatrix},$$

$$X'\Sigma^{-1}y = \begin{bmatrix} \frac{1}{\sigma_{11}} X_1'y_1 \\ \frac{1}{\sigma_{22}} X_2'y_2 \\ \vdots \\ \frac{1}{\sigma_{mm}} X_m'y_m \end{bmatrix},$$

$$\left(X'\Sigma^{-1}X\right)^{-1} = \begin{bmatrix} \sigma_{11} (X_1'X_1)^{-1} & 0 & \cdots & 0 \\ 0 & \sigma_{22} (X_2'X_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{mm} (X_m'X_m)^{-1} \end{bmatrix},$$

   and

$$\hat{\beta}_Z = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}y = \begin{bmatrix} (X_1'X_1)^{-1} X_1'y_1 \\ (X_2'X_2)^{-1} X_2'y_2 \\ \vdots \\ (X_m'X_m)^{-1} X_m'y_m \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_m \end{bmatrix}.$$

2. $X_1 = X_2 = \cdots = X_m \equiv \bar{X}$ (same regressors in the m equations).

In this case,

$$X = \begin{pmatrix} \bar{X} & 0 & \cdots & 0 \\ 0 & \bar{X} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \bar{X} \end{pmatrix} = I_m \otimes \bar{X} \ ,$$

hence

$$\begin{aligned} \hat{\beta}_Z &= \left[\left(I_m \otimes \bar{X}\right)' \left(\Sigma_c^{-1} \otimes I_T\right) \left(I_m \otimes \bar{X}\right)\right]^{-1} \left(I_m \otimes \bar{X}\right)' \left(\Sigma_c^{-1} \otimes I_T\right) y \\ &= \left[\Sigma_c^{-1} \otimes \left(\bar{X}'\bar{X}\right)\right]^{-1} \left(\Sigma_c^{-1} \otimes \bar{X}'\right) y \\ &= \left[\Sigma_c \otimes \left(\bar{X}'\bar{X}\right)^{-1}\right] \left(\Sigma_c^{-1} \otimes \bar{X}'\right) y = \left[I_m \otimes \left(\bar{X}'\bar{X}\right)^{-1} \bar{X}'\right] y \\ &= \begin{pmatrix} \left(\bar{X}'\bar{X}\right)^{-1} \bar{X}' & 0 & \cdots & 0 \\ 0 & \left(\bar{X}'\bar{X}\right)^{-1} \bar{X}' & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \left(\bar{X}'\bar{X}\right)^{-1} \bar{X}' \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \\ &= \begin{bmatrix} \left(\bar{X}'\bar{X}\right)^{-1} \bar{X}'y_1 \\ \left(\bar{X}'\bar{X}\right)^{-1} \bar{X}'y_2 \\ \vdots \\ \left(\bar{X}'\bar{X}\right)^{-1} \bar{X}'y_m \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_m \end{bmatrix} \ . \end{aligned}$$

2.3. Estimation of the covariance matrix

In practice, Σ_c must be estimated. Two main methods have been proposed to do this.

1. Method based on OLS applied to individual equations.
   Given

$$\hat{u}_i = y_i - X_i\hat{\beta}_i \ ,$$
$$\hat{\beta}_i = \left(X_i'X_i\right)^{-1} X_i y_i \ , \quad i = 1, \ldots, m \ ,$$

   we can compute the estimators

$$\hat{\sigma}_{ij} = \hat{u}_i'\hat{u}_j/T \ , \quad i, j = 1, \ldots, m \ ,$$

$$\hat{\Sigma}_c = \left[\hat{\sigma}_{ij}\right]_{i,j=1,\ldots,m} \ , \ \hat{\Sigma} = \hat{\Sigma}_c \otimes I_T \ ,$$

which yields the following estimator of β :

$$b_Z = \left(X'\hat{\Sigma}^{-1}X\right)^{-1}X'\hat{\Sigma}^{-1}y \ .$$

For T large,

$$b_Z \overset{app}{\sim} N\left[\beta, \left(X'\Sigma^{-1}X\right)^{-1}\right] \ .$$

2. Iterative procedure.

   (a) $\hat{\Sigma}$ is estimated by OLS: $\hat{\beta}_0 \equiv b_Z$;

   (b) we reestimate Σ with the new residuals: this yields

$$\hat{\Sigma}_1 \rightarrow \hat{\beta}_1 = \left(X'\hat{\Sigma}_1^{-1}X\right)^{-1}X'\hat{\Sigma}_1^{-1}y \ ; \tag{2.3}$$

   (c) we reestimate Σ with the new residuals: this yields

$$\hat{\Sigma}_2 \rightarrow \hat{\beta}_2 = \left(X'\hat{\Sigma}_2X\right)^{-1}X'\hat{\Sigma}_2^{-1}y \, ; \tag{2.4}$$

   (d) and so on up to convergence.

This iterative procedure is equivalent to maximum likelihood and the resulting estimator is asymptotically normal.

3. Chronological list of references

1. Zellner (1962)
2. Zellner (1963)
3. Kmenta and Gilbert (1968)
4. Oberhofer and Kmenta (1974)
5. Revankar (1974)
6. Revankar (1976)
7. Mehta and Swamy (1976)
8. Kunitomo (1977)
9. Buse (1979)
10. Srivastava and Dwivedi (1979)
11. Breusch (1980)
12. Kariya (1981*b*)
13. Kariya (1981*a*)
14. Harvey and Phillips (1982)
15. Kariya, Fujikoshi and Krishnaiah (1984)
16. Rothenberg (1984)
17. Phillips (1985)
18. Hillier (1987)
19. Srivastava and Giles (1987)
20. Shiba and Tsurumi (1988)
21. Kiviet, Phillips and Schipp (1995)
22. Rilstone and Veall (1996)

23. van Garderen (1997)

24. Dufour and Torrès (1998)

25. Dufour and Khalaf (2001)

26. Dufour and Khalaf (2002*b*)

27. Dufour and Khalaf (2002*a*)

28. Holgersson and Shukur (2001)

References

Breusch, T. S. (1980), 'Useful invariance results for generalized regression models', *Journal of Econometrics* **13**, 327–340.

Buse, A. (1979), 'Goodness-of-fit in the seemingly unrelated regressions model: A generalization', *Journal of Econometrics* **10**, 109–113.

Dufour, J.-M. and Khalaf, L. (2001), Finite sample tests in seemingly unrelated regressions, *in* D. E. A. Giles, ed., 'Computer-Aided Econometrics', Marcel Dekker, New York. Forthcoming.

Dufour, J.-M. and Khalaf, L. (2002*a*), 'Exact tests for contemporaneous correlation of disturbances in seemingly unrelated regressions', *Journal of Econometrics* **106**(1), 143–170.

Dufour, J.-M. and Khalaf, L. (2002*b*), 'Simulation based finite and large sample tests in multivariate regressions', *Journal of Econometrics* **111**(2), 303–322.

Dufour, J.-M. and Torrès, O. (1998), Union-intersection and sample-split methods in econometrics with applications to SURE and MA models, *in* A. Ullah and D. E. A. Giles, eds, 'Handbook of Applied Economic Statistics', Marcel Dekker, New York, pp. 465–505.

Harvey, A. C. and Phillips, G. D. A. (1982), 'Testing for contemporaneous correlation of disturbances in systems of regression equations', *Bulletin of Economic Research* **34(2)**, 79–81.

Hillier, G. H. (1987), 'Classes of similar regions and their properties for econometric testing problems', *Econometric Theory* **3**, 1–44.

Holgersson, H. E. T. and Shukur, G. (2001), 'Some aspects of non-normality tests in systems of regression equations', *Communications in Statistics, Simulation and Computation* **30**(2), 291–310.

Kariya, T. (1981*a*), 'Bounds for the covariance matrices of Zellner's estimator in the SUR model and 2SAE in a heteroscedastic model', *Journal of the American Statistical Association* **76**, 975–979.

Kariya, T. (1981*b*), 'Tests for the independence between two seemingly unrelated regression equations', *The Annals of Statistics* **9(2)**, 381–390.

Kariya, T., Fujikoshi, Y. and Krishnaiah, P. R. (1984), 'Tests for independence between two multivariate regression equations with different design matrices', *Journal of Multivariate Analysis* **15**, 383–407.

Kiviet, J. F., Phillips, G. D. A. and Schipp, B. (1995), 'The bias of OLS, GLS and ZEF in dynamic seemingly unrelated regression models', *Journal of Econometrics* **69**, 241–266.

Kmenta, J. and Gilbert, R. F. (1968), 'Small sample properties of alternative estimators of seemingly unrelated regressions', *Journal of the American Statistical Association* **63**, 1180–1200.

Kunitomo, N. (1977), 'A note on the efficiency of Zellner's estimator for the case of two seemingly unrelated regression', *Economic Studies Quarterly* **28**, 73–77.

Mehta, J. S. and Swamy, P. A. V. B. (1976), 'Further evidence on the relative efficiencies of Zellner's seemingly unrelated regressions estimator', *Journal of the American Statistical Association* **71**, 634–639.

Oberhofer, W. and Kmenta, J. (1974), 'A general procedure for obtaining maximum likelihood estimates in generalized regression models', *Econometrica* **42**, 579–590.

Phillips, P. C. B. (1985), 'The exact distribution of the SUR estimator', *Econometrica* **53**, 745– 756.

Revankar, N. S. (1974), 'Some finite sample results in the context of two seemingly unrelated regressions', *Journal of the American Statistical Association* **68**, 187–190.

Revankar, N. S. (1976), 'Use of restricted residuals in SUR systems: Some finite sample results', *Journal of the American Statistical Association* **71**, 183–188.

Rilstone, P. and Veall, M. (1996), 'Using bootstrapped confidence intervals for improved inferences with seemingly unrelated regression equations', *Econometric Theory* **12**, 569–580.

Rothenberg, T. J. (1984), 'Hypothesis testing in linear models when the error covariance matrix is nonscalar', *Econometrica* **52**, 827–842.

Shiba, T. and Tsurumi, H. (1988), 'Bayesian and non-Bayesian tests of independence in seemingly unrelated regressions', *International Economic Review* **29**, 377–389.

Srivastava, V. K. and Dwivedi, T. D. (1979), 'Estimation of seemingly unrelated regression equations: A brief survey', *Journal of Econometrics* **10**, 15–32.

Srivastava, V. K. and Giles, D. E. (1987), *Seemingly Unrelated Regression Equations Models. Estimation and inference*, Marcel Dekker, New York.

van Garderen, K. J. (1997), 'Curved exponential models in econometrics', *Econometric Theory* **13**, 771–790.

Zellner, A. (1962), 'An efficient method for estimating seemingly unrelated regressions and tests for aggregate bias', *Journal of the American Statistical Association* **57**, 348–368.

Zellner, A. (1963), 'Estimators for seemingly unrelated regressions: Some exact finite sample results', *Journal of the American Statistical Association* **58**, 977–992.