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Dr. Neil J.A. Sloane,
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Dear Neil,

1. In my last letter I referred to the missing Theorem 10 from p.474 of Conway & Sloane as a footnote. I hastily combined your note about the foot of the page, but as soon as I investigated, I got it right.

2. [Don't bother to read this section: I'm mainly talking to myself. However, keep on file for second edition of *Handbook*, should this ever materialize.] I have now found *Ann. Eugenic* in our Medical Library. I can now explain (some of) Fisher's differences with (some of) the rest of us, though I still have some lack of understanding.

It would be better to describe S.455 as RINGS & BRANCHES, A2861
which is Fisher's terminology. Coxeter, in his review (M.R.4, 183-184) notes that Fisher uses "branch" in a nonstandard way, but in fact his *numbers* of branches agree exactly with the numbers of rooted unlabelled trees (S.454), though, if you check the rank of the members A81
of his sequence, he has one more edge. This can be explained by "planting" the trees, adding an edge at the root, and transferring the root to the other end, so that planted trees all have a single trunk. Fisher's RINGS & BRANCHES are the sum of PROPER RINGS and BRANCHES, though he calculates them *in toto* (S.455) and then subtracts off the A2861
BRANCHES to give the PROPER RINGS (S.547, which would be better labelled A2862
in Sloane as PROPER RINGS: note that you use the same label, but with oh en ee instead of digit one, for S.568, i.e. what Riordan and most A1429
of us would call connected graphs with one cycle, unicyclic graphs, or connected graphs with n vertices and n edges). Here is the relationship (note the confusion: if you take one edge away from his BRANCHES, giving rooted trees, the sequence 454 lines up, at least for five terms, A81
with 455; the displacement is undetectable under the Sloane convention A2861
about initial ones: see also remarks about column 6, below):

number of edges		1	2	3	4	5	6	7	8	9	10	
S.454	BRANCHES	1	1	2	4	9	20	48	115	286	719	A81
S.547	PROPER RINGS	0	1	2	5	11	31	77	214	576	1592	A2862
S.455	RINGS & BRANCHES	1	2	4	9	20	51	125	329	862	2311	A2861

where the columns are addition sums. Coxeter (MR 4, 183-184) confirms column 4 in his review [he also calculates (presumably!) one more member (n=17) of S.455, which, 3799624, could be added to Sloane]. Column 5 is confirmed by a picture in Fisher's article. But I can't confirm column 6. The first thing to notice is that Fisher includes rings of *two* vertices and edges. See enclosed sheet of drawings of (what I think are) all the proper rings for $1 \le n \le 6$. I agree with Fisher up to n=5, for which he draws a picture. However, he gives 31 as the number of PROPER RINGS, whereas I can find only 29. If we subtract off the tadpoles (graphs with a 2-cycle), then my 29 - 16 = 13 agrees with S.568, and I would have thought that the number of tadpoles was easy to calculate, by convolving the sequence for branches (when the number of edges is even, say $2e$, the last term is $\binom{b_{e-1}+1}{2}$ i.e. the number of choices of 2 branches, each with e-1 edges with repetitions allowed:

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2: 1.1, or $\binom{1+1}{2}$?!

3: 1.1,

4: 2.1 + $\binom{1+1}{2}$

5: 4.1 + 2.1,

6: 9.1 + 4.1 + $\binom{2+1}{2}$

7: 20.1 + 9.1 + 4.2,

8: 48.1 + 20.1 + 9.2 + $\binom{4+1}{2}$

9: 115.1 + 48.1 + 20.2 + 9.4,

10: 286.1 + 115.1 + 48.2 + 20.4 + $\binom{9+1}{2}$

11: 719.1 + 286.1 + 115.2 + 48.4 + 20.9,...

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i.e. 1, 1, 3, 6, 16, 37, 96, 239, 622, 1607, 4235, 11185, 29862, 80070, 216176, 586218, 1597578, ...

a new (?) sequence which you can have for what it is worth!

3. The twinge of conscience during the writing of section 2, which caused me to insert the preliminary bracket, also sent me to the Supplement to the Handbook, and to the last two years of our (my!) correspondence. The only things I found are that in the middle of my (comparatively short) 86-12-10 letter, *Math. Mag.* should have been *Math. Gaz.*, together with items 4. to 6. below.

4. In my 86-09-25 letter, I mentioned the middle sequence of the three below (and again as item 10 in 87-02-04). It was mentioned as being related to S.93 (the first of the three). It's even nearer to S.93.2 (the third of the three). All three start

N93=607 1, 2, 2, 3, 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 19, 23, and then

N93=607 26,30,35,40,46,52,60,67,77,87,98,111,124,140,157,175,197,219,

5171 26,31,35,41,46,54,60,69,78,89,99,113,126,143,159,179,199,224, ← *

3114 26,31,35,41,46,54,61,70,79,91,102,117,131,149,167,189,211,239,

N93=607 244,272,302,336,372,413,...

5171 248,277,307,343,378,....

3114 266,299,333,374,415,465,...

*In the middle sequence, 224 was originally misprinted as 244.

5. In 87-02-04, I suggested the "hex numbers", but they *are* in the Supplement as S.1827.5

6. Enclosure 1 (dated 87-05-19, but enclosed with a more recent letter) wasn't very clear. It quoted MR. I now enclose the relevant *Math. Mag.* page, suitably annotated.

7. You occasionally list values of N for which something or other is prime. You also list $N^2 + 1$ when it's prime. But you don't list the values of N which make $N^2 + 1$ prime:

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1, 2, 4, 6, 10, 14, 16, 20, 24, 26, 36, 40, 54, 56, 66, 74, 84, 90, 94, 110, 116,120,124,126,130,134,146,150,156,160,170,176,180,184,204,206,210, 224,230,236,240,250,256 ~~(a good place to stop, but perhaps two lines are already filled)~~, 260,264,270,280,284,300,306,314,....

2 lines in 8/10 T

That's quite enough for now.

Best wishes,

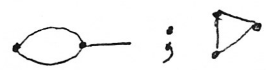
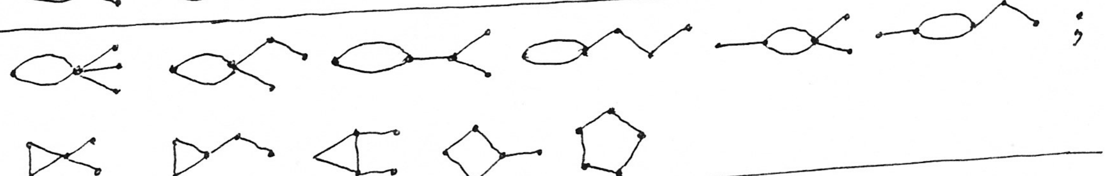
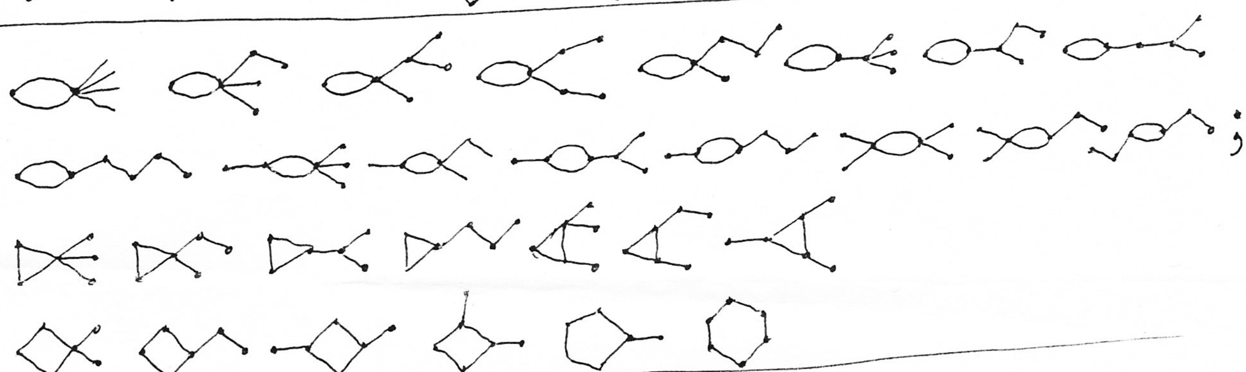
Yours sincerely,

Richard.

RKG:1

Richard K. Guy.

encl: diagrams
Math. Mag. 59 (1986)87.

# of edges	proper rings
1	
2	
3	
4	
5	
6	

	# of edges	1	2	3	4	5	6	7	8	9	10	11	
S.547	PROPER RINGS	0	1	2	5	11	31?	77?	214?	576?	1592?	4375?	A28862
	# of "tadpoles"	0	1	1	3	6	16?	37?					A27852
S.568	Unicyclic graphs	0	0	1	2	5	13	33	89	240	657	1806	A1429

− 1) as n becomes
$\leqslant I(71^n) < 71/70$, a
In other words, the
having 71 as a factor
urs as a factor of N.
es P larger than 41,
the sequence to 8
vals) are disjoint for

one of its factors in
ximum increase, the
se in the index will
$50360 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$
e best choice since
st, given $L = 90 = 2$
$7L) = 104/35$.
ization of N to the
rem, which follows

he least upper bound

(5)

ven integer but not a
P is any odd prime

heorem 1,

n $(PQ)^n$ is deficient
odd primes, we only

es deficient, but its

ave a 1913 theorem
prime factors. The
ominant role played
y either 3 or 5, then
ime factors—a fact

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✓SJS ✓SJS

which can be verified in a couple of minutes using Theorem 4 and a calculator.

It is an interesting historical anomaly that the first four perfect numbers (6, 28, 496, 8128) were all known by A.D. 100, and probably much earlier, but that the 13th century found scholars still unaware that odd abundant numbers such as 945 existed. The difference is partly explained by the fact that Euclid published in his *Elements* a formula for even perfect numbers.

Using formula (4), it is a simple matter to extend Dickson-type results—especially with the aid of a computer. We have prepared TABLE A, which gives, in tabular form, pairs of numbers J and K for the following statement: *Every number N with $I(N) \geqslant J$ must have at least K distinct prime factors.*

$I(N) = \dfrac{\sigma(N)}{N}$

Susanne 2 to enter! A5579 A5580

new

N Even		N Odd	
J	K	J	K
2	2	2	3
3	3	3	8
4	4	4	21
5	6	5	54
6	9	6	141
7	14	7	372
8	22	8	995
9	35	9	2697
10	55	10	7397
11	89	11	20502
12	142	12	57347
13	230		
14	373		
15	609		
16	996		
17	1637		
18	2698		
19	4461		
20	7398		
21	12301		
22	20503		
23	34253		
24	57348		

1 1 2 3 5 8 13 20 34 53 88 143 236 393 641 1061 1763 2937 4903 8202 13750 23095

387

67

differences: close to Fibs, but only to begin with.

TABLE A. Every number N with $I(N) \geqslant J$ must have at least K distinct prime factors.

Given J, the number K is computed as follows:

$$\text{for } n \text{ even:} \quad K = \min\left\{ n: \prod_{i=1}^{n} \frac{P_i}{P_i - 1} > J \right\},$$
$$\text{for } n \text{ odd:} \quad K = \min\left\{ n: \prod_{i=2}^{n} \frac{P_i}{P_i - 1} > J \right\} \tag{6}$$

where P_i is the ith prime in the natural ordering of the primes ($P_1 = 2$, $P_2 = 3$, $P_3 = 5$, etc.).

The first few entries in TABLE A were known to R. D. Carmichael [4] in 1907; his paper explicitly states formula (5). Also Paul Poulet [19] in 1929 gave the first seven entries in the "even" table. Recently, computers have been called upon to generate this and similar tables (see, for example, [18]).

While compiling TABLE A for N even, it became apparent that there was a pattern in the

Math. Mag.

Neil, Same name etc as 5347