If $u \in C[0, 1]$ is convex, then v defined by

$$v(x) = \frac{1}{x} \int_0^x u(t)\, dt, \qquad 0 < x \leqq 1,$$

$$v(0) = u(0),$$

is also convex.

Problem 67-3, *A Definite Integral*, by M. LAWRENCE GLASSER (Battelle Memorial Institute).

Evaluate

$$I(z) = \int_0^1 \exp\{-aM(z, u)\}\, du,$$

where

$$M(z, u) = \{\coth zu + \coth z(1 - u)\}^{-1}.$$

The integral has arisen in a study of the magnetic susceptibility of an electron gas subject to a weak periodic potential.

Problem 67-4, *A Double Sum*, by L. CARLITZ (Duke University).

Show that

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \binom{r+s}{r}^2 \binom{m+n-r-s}{m-r}^2 = \frac{1}{2}\binom{2m+2n+2}{2m+1}.$$

Problem 67-5, *"Up-Down" Permutations*, by D. J. NEWMAN (Yeshiva University) and W. WEISSBLUM (AVCO Corporation).

Show that

$$\sec t + \tan t = \sum_{n=0}^{\infty} \frac{A_n t^n}{n!},$$

where A_n denotes the number of "up-down" permutations of $1, 2, \cdots, n$ (e.g., $A_4 = 5$ corresponding to 2143, 3142, 3241, 4132 and 4231).

SOLUTIONS

Problem 60-12, *A Sorting Problem*, by WALTER WEISSBLUM (AVCO Corporation).

The first step in a method commonly used in computing machines for sorting a sequence $x_1, x_2, \cdots, x_n$ of random numbers is to break the sequence into strings

$$(x_1, \cdots, x_{n_1}), (x_{n_1+1}, \cdots, x_{n_2}), \cdots$$

such that $(x_1, \cdots, x_{n_1})$ is the longest initial monotone string (either increasing or decreasing), $(x_{n_1+1}, \cdots, x_{n_2})$ is the longest subsequent monotone string, etc. In order to estimate the time required for the sorting process it is necessary to

$a^{b-c}\int_0^t J_{b+c}(at)\,dt$. The problem arose in the solution of the transient response of an m-derived electrical filter problem.

Problem 68-10, *Rank and Eigenvalues of a Matrix*, by SYLVAN KATZ (Aeronutronic Division, Philco-Ford Corporation) and M. S. KLAMKIN (Ford Scientific Laboratory).

Determine the rank and eigenvalues of the $n \times n$ ($n \geqq 3$) matrix $\| A_{rs} \|$, where $A_{rs} = \cos (r - s)\theta$ and $\theta = 2\pi/n$. This problem arose in a study of electromagnetic wave propagation.

SOLUTIONS

Problem 67-5, *"Up-Down" Permutations*, by D. J. NEWMAN (Yeshiva University) and W. WEISSBLUM (AVCO Corporation).

Show that

$$\sec t + \tan t = \sum_{n=0}^{\infty} \frac{A_n t^n}{n!},$$

where A_n denotes the number of "up-down" permutations of 1, 2, $\cdots$, n (e.g.' $A_4 = 5$ corresponding to 2143, 3142, 3241, 4132 and 4231).

Comment by H. W. GOULD (West Virginia University). The "up-down" permutations are nothing but the familiar alternating permutations which were enumerated in closed form by D. André as long ago as 1879. A detailed exposition of the method is given by E. Netto, *Lehrbuch der Combinatorik*, 1927, pp. 105–112, where various references in the literature are given. Netto shows how to find a suitable recurrence relation for A_n, the number of alternating permutations, how to set up a generating function, and finally how to obtain in fact (p. 112),

$$\sec t + \tan t = \sum_{n=0}^{\infty} \frac{A_n t^n}{n!}, \tag{1}$$

which is precisely the relation posed in Problem 67-5. However, since the secant power series has only even powers of t, the tangent series only odd powers of t, the expansion (1) is separated into two separate series by Netto. It is instructive to note (p. 111) that the expansion may also be written in the form (due to André, 1879)

$$\tan\left(\frac{\pi}{4} + \frac{t}{2}\right) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}. \tag{2}$$

Explicit values of A_n then follow from the well-known expansions for secant and tangent using Euler and Bernoulli numbers. These may be found in Nörlund's *Differenzenrechnung*, for example. Thus, we have in fact,

$$A_{2n} = (-1)^n E_{2n}, \quad A_{2n-1} = \frac{(-1)^{n-1}\, 2^{2n}(2^{2n} - 1)}{2n} B_{2n}, \tag{3}$$

where E_{2n} and B_{2n} are the Euler and Bernoulli numbers, respectively.

A short table of values of A_n is as follows:

n	0	1	2	3	4	5	6	7	8	9	10
A_n	1	1	1	2	5	16	61	272	1385	7936	50521

Recently R. C. Entringer has written a paper, *A combinatorial interpretation of the Euler and Bernoulli numbers* (published in the Nieuw Archief voor Wiskunde (3), 14(1966), pp. 241–246), in which he obtains the explicit result (3) above giving a detailed treatment using recurrence relations and generating functions. Apparently he was unaware of the work of André and of Netto, for his only reference is to a paper by A. J. Kempner (*On the shape of polynomial curves,* Tôhoku Math. J., 37 (1933), pp. 347–362) in which the problem of alternating permutations had been posed and only the recurrence relation developed.

Also solved by J. W. Bessman (Scope Inc.), L. Carlitz (Duke University), L. Carlitz and R. A. Scoville (Duke University), P. G. Comba (IBM Corp.), R. C. Entringer (University of New Mexico), S. S. Kapur and D. R. Chand (Lockheed-Georgia Co.), U. Ljungh (Research Institute of National Defense), A. Milewski (IBM, France) and the proposers.

Problem 67-6, *A Trigonometric Inequality,* by J. N. Lyness (Argonne National Laboratory) and C. B. Moler (University of Michigan).

For all real x, show that

$$\sum_{n=1}^{\infty} (-1)^{n+1}\left\{\frac{\sin \pi n x}{n}\right\}^{2r} \geqq 0, \qquad r = 1, 2, 3, \cdots .$$

Solution by F. W. Steutel (Technische Hogeschool Twente, Enschede, Netherlands).

Setting

$$S_r(x) = \sum_{n=1}^{\infty} (-1)^{n-1}\left\{\frac{\sin \pi n x}{\pi n x}\right\}^{2r} \quad \text{and} \quad T_r(x) = \sum_{n=1}^{\infty}\left\{\frac{\sin \pi n x}{\pi n x}\right\}^{2r},$$

we then have $S_r(x) = T_r(x) - 2T_r(2x)$. Noting that $\{(\sin t)/t\}^{2r}$ is the cosine transform of $h_r(y)$, the $2r$-fold convolution of the uniform density on $(-1, 1)$, we can use Poisson's formula[2] for $T_r(x)$. This yields

$$T_r(x) + \frac{1}{2} = \frac{1}{x}\left\{\frac{1}{2} h_2(0) + \sum_{n=1}^{\infty} h_r\left(\frac{n}{x}\right)\right\}$$

and hence,

$$S_r(x) = \frac{1}{2} - \frac{1}{x}\sum_{n=1}^{\infty}\left\{h_r\left(\frac{n}{x}\right) - h_r\left(\frac{2n}{x}\right)\right\}.$$

Since $h_r(y)$ is decreasing in $(0, \infty)$,[3] it follows that $S_r(x) \leqq \frac{1}{2}$. Since $h_r(y) = 0$ for $|y| \geqq 2r$, it also follows that $S_r(x) = \frac{1}{2}$ for $|x| \leqq 1/2r$. Interpreting

[2] E. C. Titchmarsh, *The Theory of Fourier Integrals*, Oxford University Press, London, 1948, p. 60.

[3] W. Feller, *An Introduction to Probability Theory and its Applications*, John Wiley, New York, 1966, p. 164.