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## Automorphic Numbers

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An automorphic number is distinguished by having its square end with the number itself. To put this more formally, a number is called automorphic in the scale of notation with base  $B$  if all its powers end in the same digits in this scale of notation. If  $n$  is the number of digits appearing at the end of every power of the automorphic number  $N$ , we have  $N^x \equiv N \pmod{B^n}$ , for  $x > 0$ . If we let  $x = 2$  we have  $N^2 \equiv N \pmod{B^n}$ , which is sufficient to satisfy the formal definition given above.

In any scale of notation, base  $B$ , there are two trivial solutions ( $N = 0$  and  $N = 1$ ). In base 10, the remaining solutions end in 5 or 6.

If the automorphic numbers of  $n$  digits are known, the automorphic numbers of a lesser number of digits can be obtained by dropping the first digit, the first and second digits, etc. Thus, since 625 is automorphic, so also is 25 and 5. It is more difficult, however, to determine the automorphic numbers of  $n + 1$ ,  $n + 2$ ,  $\dots$ ,  $n'$  digits, although this was done by one of the authors of this article for  $n' = 100$  using nothing more complicated than a desk adding machine. These were first published in *The Fibonacci Quarterly*, Vol. 2, No. 3, page 230 (October 1964).

It is the purpose of this paper to state how the automorphic numbers can be expanded and to describe a practical method of doing this either with simple tools or with a computer.

If one of the automorphic numbers,  $N$ , of  $n$  digits is squared, the last  $n$  digits must (by definition) be the same as  $N$ . If  $N$  ends in 5, the next digit to the left is the first digit of  $N + 1$ ; if  $N$  ends in 6, the next digit to the left, when subtracted from 10, will produce the first digit of  $N + 1$ . For example,  $5^2 = 25$  so 25 is automorphic;  $6^2 = 36$  so 76 is automorphic ( $7 = 10 - 3$ ). Continuing,  $25^2 = 625$  so 625 is automorphic;  $76^2 = 5776$  so 376 is automorphic.

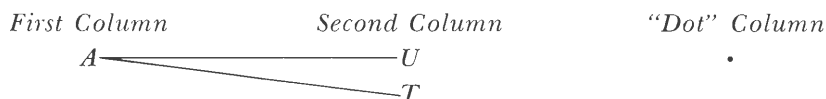
If the two automorphic numbers are written, one below the other, it will be seen that the sum of the digits in each column is 9 (in base 10) except for the units where it is 11. Therefore, it is only necessary to develop one of the numbers, and the number ending in 5 is somewhat simpler to work with.

One innovation which was found to be useful (both for computer and "hand" calculation) was to omit all working digits except those necessary to determine the next digit of the automorphic number. Take 2890625, for example. The normal method of squaring is as follows:

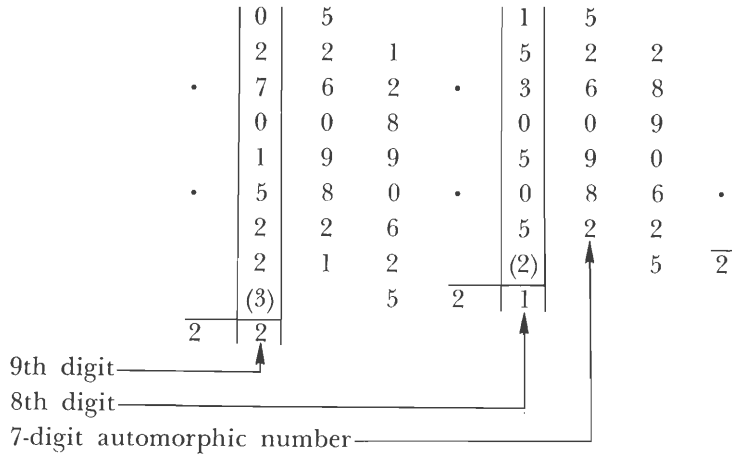
$$\begin{array}{r}
 2890625 \\
 \underline{2890625} \\
 14453125 \\
 5781250 \\
 17343750 \\
 260156250 \\
 23125000 \\
 5781250 \\
 \text{(carry)} \quad 11-122111- - - \\
 \underline{8355712890625} \\
 \text{13-digit automorphic} \quad \underline{9918212890625}
 \end{array}$$

The only digits in the work which are of any value are the ones in *italics*. The left-hand digits are useless—the right-hand digits are only needed to obtain the “carry.” As long as the “carry” can be determined from the previous calculations, the right-hand digits can be omitted.

A method of eliminating all but the necessary digits is shown below. It is based on the Trachtenberg Speed System. Write the largest known automorphic number (ending in 5) vertically with the units digit at the top. In the column immediately to the right, write the same number with the units digit at the bottom, but one line lower on the page than the other column. In the next column to the right indicate with a dot (•) if there is a “carry” from the previous work (this will never exceed 1). Then multiply in accordance with the following rule:



Multiply the digit *A* by the digit *U* and jot down the *units* digit of the product. Multiply the digit *A* by the digit *T* and jot down the *tens* digit of the product. Add these two digits and increase the sum by 1 if there is a dot (“carry”) on the same line as *A*. Write this sum on the same line as *A* in the column to the *left* of the first column. If the sum exceeds 9, write the units digit and a “dot” to indicate a “carry” for determining the next but one digit. After multiplying all the digits of the first column (*A*) by the appropriate digits of the second column (*U* and *T*), put at the bottom of the column the “column carry” from the previous calculation and then add. The units digit of the sum is the next digit of the automorphic number and the remaining digits are the “column carry.” However, if the units digit is odd, the “column carry” must be increased by 1.



When using a computer, the normal method of storing results cannot be employed because of the necessity of developing the product column-by-column rather than line-by-line; which, in turn, requires multiplying individual digits of a number, rather than the number as a whole. However, an IBM 360/30 (64K) computer was able to develop and check 500 digits. The checking was done after the required number of digits of the "5" number had been obtained. The corresponding "6" number was determined as explained previously and squared. However, it was squared column-by-column, and after each column was added, the units digit was compared with corresponding digit of the number, before proceeding with the next column.

As mentioned, these numbers were first expanded to 100 digits by "hand." With a desk adding machine available, it was found most satisfactory to develop six digits at a time. This involved some special procedures which would require too much space to describe in an article of this length. After each six digits had been determined the results were verified by using the number ending in 6.

The non-trivial 1000-digit automorphic numbers in bases 6, 10, and 12 are printed out in full, after the following commentary by the editor of this journal.

*Editor's Comments:* Almost simultaneously with the receipt of the manuscript of this article, the editor received a number of computer-calculated automorphic numbers. The authors of this article had submitted 500-digit pairs in bases 6 and 10. Then M. F. Jones, of the Computer Automation Systems, Ltd., in London, sent a pair of base-10 1500-digit automorphs. Larry Dean Morse of Rolla, Missouri sent in 1000-digit pairs in bases 6 and 12, and 2000-digit pairs in base 10. The climax (for the present, at least) was reached when the editor received, through the courtesy of Martin Gardner, the monstrous 22,300-digit automorphic number ending in 5. This had been calculated by Jerry Feinberg and Terry Moore of the California Institute of Technology with the help of a computer with a sense of humor: the print-out asks the reader to "Enjoy, Enjoy, Enjoy" and to "Square Me If You Can!"

Automorphic numbers have been studied for many years, but the recent interest has come only after a number of years of apparent lack of interest. Maurice Kraitchik, in his *Mathematical Recreations* (W. W. Norton & Co., Inc., N. Y., 1942) gave the 16-digit pair in base 10. This same pair is quoted in a number of books on mathematical recreations—with no attempt to add even one extra digit. Then, in 1964, J. A. H. Hunter's 17-digit pair was published in *Recreational Mathematics Magazine* (No. 14, Jan-Feb 1964, page 19). R. A. Fairbairn took up the challenge immediately, publishing the 100-digit pair in *The Fibonacci Quarterly*.

More recently, Martin Gardner quoted one of the 100-digit number in his "Mathematical Games" column in the January 1968 issue of *Scientific American* (pages 124-127). That produced the spate of 1000-and-more-digit endeavors.

R. A. Fairbairn points out that the number of automorphic numbers (of any given number of digits, and including the trivial 0 and 1) in any base can be calculated. The number of automorphic numbers in base  $b$  is  $2^p$  where  $p$  is the number of distinctly different primes in the factorization of  $b$ . For example,  $6 = (2)(3)$ , so there are  $2^2$  or four automorphic numbers in base 6;  $12 = (2^2)(3)$ , so there are also  $2^2$  or four automorphic numbers in base 12. In base 2310, since  $2310 = (2)(3)(5)(7)(11)$ , there are  $2^5$  or 32 automorphic numbers: those ending in 0 and 1, and those pairs ending in the following "digits" (each of the four-digit integers listed is actually the terminal "digit"—in base 2310—of the 15 pairs of automorphic numbers):

0210	0231	0330	0385	0441	0540	0561	0595
2101	2080	1981	1926	1870	1771	1750	1716
	0616	0715	0771	0826	0925	0946	1155
	1695	1596	1540	1485	1386	1365	1156

The sum of the terminal digits of the non-trivial automorphic number pairs in any base  $b$  is  $b + 1$ ; the sum of the other corresponding digit-pairs in base  $b$  is  $b - 1$ .

Although this editor will always welcome larger and larger automorphic numbers in any base, he cannot anticipate publishing any in JRM after this issue. More extensive evaluations must be published elsewhere—perhaps in a tentatively-planned *Mathematical Tables for Recreational Mathematicians*.

*Vernon deGuerre*, employed by the Borough of North York, Metropolitan Toronto, as computer programmer is presently attending a computer course at York University, Toronto, under the direction of Dr. P. Rajagopal, who was most helpful in suggesting improvements which have been incorporated in this article.

*R. A. Fairbairn* is a retired engineer of The Bell Telephone Company of Canada.

The Non-Trivial Automorphic Numbers, Base 6 to 1000 Digits

53543 30200 25150 34425 10341 01350 23342 41452 33333 31105  
 34111 55403 15010 23233 43551 23230 22333 40243 11541 22344  
 21113 52055 41520 15332 44243 43504 32151 53040 22415 32512  
 24544 45230 42031 52324 52305 20534 54435 01511 54243 41555  
 41515 42020 10525 30044 32311 25235 15131 13023 30451 30505

14041 01311 20315 51200 10322 20520 23145 10432 14140 50535  
 23504 01225 10550 10523 02442 35530 22214 44500 21331 02340  
 51234 24054 22230 51333 44421 24103 14143 53401 41354 42500  
 30531 01252 22105 21154 03112 15254 12432 34344 02305 15013  
 35243 53233 53152 22220 14214 34152 03515 42235 52433 30341

50022 31551 34244 50343 42111 20010 23210 13344 33004 30533  
 15540 22225 42052 05211 53345 05111 15105 54321 31130 51220  
 32550 55135 43205 45214 23350 23004 15424 41322 22024 34242  
 00424 25345 30353 25312 41521 02513 23435 25543 13405 54040  
 40035 14254 21450 34304 10120 53532 54030 33554 22054 31315

10155 33524 23551 41224 53451 24435 24205 45402 43203 41021  
 41412 10414 04315 20555 21325 13552 31551 52531 31015 25051  
 22450 12353 30231 00403 35415 52351 24250 22532 55332 35332  
 33125 13545 02150 45245 31544 51010 42321 00300 12522 35130  
 34535 14224 51331 52214 20003 05000 34241 40040 33342 05344

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02012 25355 30405 21130 45214 54205 32213 14103 22222 24450  
 21444 00152 40545 32322 12004 32325 33222 15312 44014 33211  
 34442 03500 14035 40223 11312 12051 23404 02515 33140 23043  
 31011 10325 13524 03231 03250 35021 01120 54044 01312 14000  
 14040 13535 45030 25511 23244 30320 40424 42532 25104 25050

41514 54244 35240 04355 45233 35035 32410 45123 41415 05020  
 32051 54330 45005 45032 53113 20025 33341 11055 34224 53215  
 04321 31501 33325 04222 11134 31452 41412 02154 14201 13055  
 25024 54303 33450 34401 52443 40301 43123 21211 53250 40542  
 20312 02322 02403 33335 41341 21403 52040 13320 03122 25214

05533 24004 21311 05212 13444 35545 32345 42211 22551 25022  
 40015 33330 13503 50344 02210 50444 40450 01234 24425 04335  
 23005 00420 12350 10341 32205 32551 40131 14233 33531 21313  
 55131 30210 25202 30243 14034 53042 32120 30012 42150 01515  
 15520 41301 34105 21251 45435 02023 01525 22001 33501 24240

45400 22031 32004 14331 02104 31120 31350 10153 12352 14534  
 14143 45141 51240 35000 34230 42003 24004 03024 24540 30504  
 33105 43202 25324 55152 20140 03204 31305 33023 00223 20223  
 22430 42010 53405 10310 24011 04545 13234 55255 43033 20425  
 21020 41331 04224 03341 35552 50555 21314 15515 22213 50213

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The Non-Trivial Automorphic Numbers, Base 10 to 1000 Digits

12781	25400	13369	00860	34889	08436	40238	75765	93682	19796
26181	91783	35204	92704	19932	48752	37825	86714	82789	05344
89744	01426	12317	03569	95484	19499	44461	06081	46207	25403
65599	98271	58835	60350	49327	79554	07419	61849	28095	20937
53026	85239	09375	62839	14857	16123	67351	97060	92242	42398
77700	75749	55787	27155	97674	13458	99753	76955	15862	71888
79415	16307	56966	88163	52155	04889	82717	04378	50802	84340
84412	64412	68218	48514	15772	99160	34497	01789	23357	96684
99144	73895	66001	93254	58276	78000	61832	98544	26232	82725
75561	10733	16069	70158	64984	22229	12554	85729	87933	71478
66323	17240	55157	56102	35254	39949	99345	60808	38011	90741
53006	00560	55744	81870	96927	85099	77591	80500	75416	42852
77081	62011	35024	68060	58163	27617	16767	65260	93752	80568
44214	48619	39604	99834	47280	67219	06670	41724	00942	34466
19781	24266	90787	53594	46166	98508	06463	61371	66384	04902
92193	41881	90958	16595	24477	86184	61409	12878	29843	84317
03248	17342	88865	72737	66314	65191	04988	02944	79608	14673
76050	39571	96893	71467	18013	75619	05546	29968	14764	26390
39530	07319	10816	98029	38509	89006	21665	09580	86381	10005
57423	42323	08961	09004	10661	99773	92256	25991	82128	90625
87218	74599	86630	99139	65110	91563	59761	24234	06317	80203
73818	08216	64795	07295	80067	51247	62174	13285	17210	94655
10255	98573	87682	96430	04515	80500	55538	93918	53792	74596
34400	01728	41164	39649	50672	20445	92580	38150	71904	79062
46973	14760	90624	37160	85142	83876	32648	02939	07757	57601
22299	24250	44212	72844	02325	86541	00246	23044	84137	28111
20584	83692	43033	11836	47844	95110	17282	95621	49197	15659
15587	35587	31781	51485	84227	00839	65502	98210	76642	03315
00855	26104	33998	06745	41723	21999	38167	01455	73767	17274
24438	89266	83930	29841	35015	77770	87445	14270	12066	28521
33676	82759	44842	43897	64745	60050	00654	39191	61988	09258
46993	99439	44255	18129	03072	14900	22408	19499	24583	57147
22918	37988	64975	31939	41836	72382	83232	34739	06247	19431
55785	51380	60395	00165	52719	32780	93329	58275	99057	65533
80218	75733	09212	46405	53833	01491	93536	38628	33615	95097
07806	58118	09041	83404	75522	13815	38590	87121	70156	15682
96751	82657	11134	27262	33685	34808	95011	97055	20391	85326
23949	60428	03106	28532	81986	24380	94453	70031	85235	73609
60469	92680	89183	01970	61490	10993	78334	90419	13618	89994
42576	57676	91038	90995	89338	00226	07743	74008	17871	09376

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The Non-Trivial Automorphic Numbers, Base 12 to 1000 Digits

02638 40220 689A9 212B3 31244 84B1B 15866 73751 B4214 B0337  
 A5B4A 67299 A2409 57340 13578 23BA8 1628B 29304 47917 57518  
 03B80 9154A 19303 24718 10916 8629B 4067A 32043 221B5 B0294  
 25086 A5721 58B82 17B61 55475 B8B80 BB82B BA2B0 67419 26623  
 32056 51B63 2756A 65428 673A0 80213 B3BA1 89B76 0A134 6BAA7

6842B 5893B 5A2A7 44720 13415 BB842 A4138 9314A 24113 64156  
 7300A 59B5A A31A5 79041 BB6A8 40758 93874 52544 41250 34952  
 8B825 1025A 33B56 A1538 0A933 98270 29239 03B69 390A1 6BB15  
 0B709 01208 6308A 3A159 A8744 B6014 9BAB7 56628 75B2B B5564  
 A0149 46929 61364 A3474 69919 55778 B7642 6A509 89761 27955

AA3B6 82639 B99A6 40525 26169 3B1AA 6A655 9A029 09A78 A7A51  
 B6475 22A97 87775 26213 7BAA4 91655 B8383 12B16 A3111 5620B  
 8425A 746B0 B0607 B021B 17B39 B2643 102A8 41559 0B84A 64BAB  
 72943 763B4 31757 13386 24146 78702 30478 62478 63268 13990  
 B6A84 48895 23778 00861 94532 99042 12315 55A08 99A24 50318

70A45 0A253 0769A 08682 62B14 00167 78442 30904 79B6B 21001  
 3736A 3B4B9 843B2 42538 67A44 A8A03 38504 32547 97647 B6B71  
 B0024 4855A 5960B 39854 AA763 13BA1 A5163 79014 20433 5A1AA  
 78B11 3A4B2 A1B2B 37062 787B5 02745 66720 B2A15 25678 B7901  
 35998 30402 B6468 A133A 81A3A 16986 B267B 3452B 21B61 B3854

A259 468

B9583 7B99B 53212 9A908 8A977 370A0 A6355 4846A 079A7 0B884  
 16071 54922 197B2 6487B A8643 98013 A5930 928B7 742A4 646A3  
 B803B 2A671 A28B8 974A3 AB2A5 33920 7B541 89B78 99A06 0B927  
 96B35 1649A 63039 A405A 66746 0303B 00390 0190B 547A2 95598  
 89B65 6A058 94651 56793 5481B 3B9A8 0801A 32045 B1A87 50114

53790 63280 61914 7749B A87A6 00379 17A83 28A71 97AA8 57A65  
 48BB1 62061 18A16 42B7A 00513 7B463 28347 69677 7A96B 87269  
 30396 AB961 88065 1A683 B1288 2394B 92982 B8052 82B1A 500A6  
 B04B2 BA9B3 58B31 81A62 13477 05BA7 20104 65593 46090 06657  
 1BA72 75292 5A857 18747 522A2 66443 04579 516B2 3245A 94266

11805 39582 02215 7B696 95A52 80A11 51566 21B92 B2143 1416A  
 05746 99124 34446 959A8 40117 2A566 03838 A90A5 18AAA 659B0  
 37961 4750B 0B5B4 0B9A0 A4082 09578 AB913 7A662 B0371 57010  
 49278 45807 8A464 A8835 97A75 434B9 8B743 59743 58953 A822B  
 05137 73326 98443 BB35A 27689 22B79 A98A6 661B3 22197 6B8A3

4B176 B1968 B4521 B3539 590A7 BBA54 43779 8B2B7 42050 9ABBA  
 84851 80702 37809 79683 54177 131B8 836B7 89674 24574 0504A  
 0BB97 73661 625B0 82367 11458 A801A 16A58 42BA7 9B788 61A11  
 430AA 81709 1A090 84B59 43406 B9476 5549B 091A6 96543 042BA  
 86223 8B7B9 05753 1A881 3A181 A5235 09540 87690 9A05A 08369

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