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More on Automorphic Numbers

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In "Automorphic Numbers" in the July 1968 issue of *JRM* (pp. 173-179) Vernon deGuerre and the author described a method of expanding the automorphic numbers in the usual scale of notation (base $B = 10$). Tables showing the 1000-digit automorphic numbers in bases six, ten, and twelve were published, along with a listing of the final digits of the thirty non-trivial automorphic numbers in base 2310.

It is the purpose of this paper to give a general method of determining the single digit automorphic numbers (or automorphic digits) for any base B, and of expanding these digits into larger automorphic numbers. The method of squaring automorphic numbers described in the previous article may be employed.

To Find the Automorphic Digits (i. e., the units' digits) to any Base B

Let $B = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots \times p_r^{\alpha_r}$, where $p_1, p_2, \cdots, p_r$ are different primes and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers. The number of *pairs* of automorphs is $2^{r-1} - 1$, and this is the number of ways of resolving the composite number "B" into two factors which are prime to each other. (Unity, of course, is not a prime number.)

Let $B = mn$, with $1 < m < n$, m and n being relatively prime integers. Now there is some positive integer x $(< m)$ for which $nx = my + 1$. Then "nx" is an automorphic digit and "$B + 1 - nx$" is the other member of the pair. By using all relatively prime values of m and n, the total number of pairs of automorphic digits $(2^{r-1} - 1)$ is obtained.

For example, if $B = 24 = 3 \cdot 8$; $m = 3$ and $n = 8$. Then $8x = 3y + 1$. Integral values of x and y can be determined in accordance with the methods of solving indeterminate equations. In this case $x = 2$ and $y = 5$. Therefore, one automorphic digit is $2 \cdot 8 = 16$ and the other member of the pair is $24 + 1 - 16 = 9$. Since there is only one set of relatively prime values of m and n, these are the only automorphic digits.

Let the smaller one of the pair belong to the class "A" and the larger to the class "A′".

To Develop the Coefficient of "B" in an Automorphic Number (i.e. the second digit from the right)

Square the automorphic digit. The right-hand digit will, of necessity, be the same as the automorphic digit. Let the left-hand digit be s_1. Double the automorphic digit and let u be the units' digit (Modulus B) of this product. If a_1 is the second digit from the right of the automorphic number it is necessary and sufficient to find a_1 and y such that:

$$a_1 = (By - s_1) / (u - 1)$$

a_1 and y being positive integers and $a_1 < B$. (Note: If $u = 0$, then $y = 0$ and $a_1 = s_1$).

The second digit of the other automorphic number of the pair a_1' can be obtained from the equation

$$a' = B - 1 - a.$$

For example, if $B = 24$, the automorphic digits (A and A') are "9" and "16". Consider "9" (in base 24):

$$9^2 = 3\text{-}09: \quad s_1 = 3; \quad u = 2 \cdot 9 \pmod{24} = 18;$$
$$a_1 = (24y - 3) / (18 - 1) \quad (a_1 < 24); \quad y = 15;$$
$$a_1 = 21; \quad a_1' = (24 - 1) - 21 = 2$$

Therefore the two-digit automorphic numbers are: 21-09 and 2-16.

To Develop Additional Digits in an Automorphic Number (i.e., the coefficients of B^2, B^3 . . .)

This can best be accomplished by first establishing the *expansion factors* (E and E') for the pair of automorphic digits. These expansion factors may be defined as follows:

$$a_r = Es_r \quad \text{and} \quad a_r' = E's_r'.$$

The values of a_1, s_1, and a_1' are known. Square the other automorphic digit of the pair (A') to obtain s_1'. Develop e and e' from the following formulae:

$$e = (Bz + a_1) / s_1 \qquad e' = (Bz' + a_1') / s_1'$$

where z and z' have the integer values which will produce all the integral values of e ($< B$) and e' ($< B$). From these values of e and e', select the pair (E and E') which will result in

$$E + E' = B.$$

For example, if $B = 24$; $A = 9$ and $A' = 16$. Then $s_1 = 3$, $a_1 = 21$, and $a'_2 = 2$.

$$(A')^2 = (16)^2 = 10\text{-}16: \qquad s_1' = 10;$$
$$e = (24z + 21) / 3 = 8z + 7 \qquad e' = (24z' + 2) / 10 = (12z' + 1) / 5$$
$$e\ (< 24) = 7,\ 15,\ 23 \qquad e'\ (< 24) = 5,\ 17.$$

But $E + E'$ must equal B (24), therefore $E = 7$ and $E' = 17$. These expansion digits, 7 and 17, are used to develop a_2, a_3, . . . and a_2', a_3', ...

For example: $(21\text{-}09)^2 = 00\text{-}21\text{-}09 \pmod{24^3}$, so $s_2 = 0$, $a_2 = 0$; $(2\text{-}16)^2 = 07\text{-}02\text{-}16 \pmod{24^3}$, so $s_2' = 7$, $a_2' = 23$. Therefore the three-digit automorphic numbers ($B = 24$) are 0-21-09 (trivial with initial "0") and 23-02-16.

Note: Either one can be obtained from the other ($0 = 24 - 1 - 23$), but it is advisable to check the work as development progresses. Even with expansion factors, automorphic numbers can be developed only one digit at a time.

Some Observations

Automorphic digits must be greater than $\sqrt{B}$ to eliminate trivial values. For every base B, s_1 increases as A increases. Also s_1 is always less than A.

When B is even but not divisible by 4, there is one automorphic number whose expansion factor (E) equals 1 ($A = B/2$), and it is only in these cases that $E = 1$. When there are more than two automorphic digits for a given base, the expansion factors (E) decrease in value for the first half of the automorphic digits and also decrease in value for the second half. (It is assumed that the automorphic digits have been arranged in order of ascending magnitude.) When 2 (but not 4) is a factor of B, the middle two expansion factors are 1 and $B - 1$, and the expansion factors after the middle two are the same as those before the middle two.

Consider the automorphic number of r digits in base 10, i.e., . . . 625, $= K\cdot 5^r$. It has been observed that if $r > 5$ and is even, $K = 2^3\cdot H + 1$; and if $r \geqq 5$ and is odd, $K = 2^3\cdot H' + 5$ (5 being $B/2$); where H and H' are also integers. A more limited test indicates that with $B = 6$ the automorphic number ending in 3 ($B/2$) follows the same pattern. It would be interesting to know if this always applies to an automorphic number ending in $B/2$. Also do other automorphic numbers have similar properties? To explore these avenues without adequate tools would involve a vast amount of labor.

TABLE 1—Automorphic Numbers

Scale of Notation Base = B	*Factor Resolution*	*Automorphic Digit Pairs* A	A'	*Expansion Factors* E	E'	*Expanded Automorphic Numbers*	
6	2·3	3	4	1	5	2221350213	3334205344
10	2·5	5	6	1	9	8212890625	1781709376
12	3·4	04	09	05	07	03-08-05-04	08-03-06-09
14	2·7	07	08	01	13	00-12-03-07	13-01-10-08
15	3·5	06	10	04	11	13-10-08-06	01-04-06-10
18	2·9	09	10	01	17	01-02-04-09	16-15-13-10
20	4·5	05	16	11	09	11-06-11-05	08-13-08-16
24	3·8	09	16	07	17	13-00-21-09	10-23-02-16
30	5·6	06	25	19	11	11-02-19-06	18-27-10-25
	3·10	10	21	11	19	17-01-03-10	12-28-26-21
	2·15	15	16	01	29	01-26-07-15	28-03-22-16

TABLE 1 (Continued)

Scale of Notation Base = B	*Factor Resolution*	*Automorphic Digit Pairs*		*Expansion Factors*		*Expanded Automorphic Numbers*	
		A	*A′*	*E*	*E′*		
60	4·15	16	45	29	31	44-45-56-16	15-14-03-45
	3·20	21	40	19	41	50-22-13-21	09-37-46-40
	5·12	25	36	11	49	24-51-50-25	35-08-09-36
105	7·15	015	091	076	029	032-047-015	072-057-091
	5·21	021	085	064	041	076-046-021	028-058-085
	3·35	036	070	034	071	003-093-036	101-011-070
120	8·15	016	105	089	031	058-016	061-105
	5·24	025	096	071	049	115-025	004-096
	3·40	040	081	041	079	053-040	066-081
210	14·15	015	196	181	029	181-105	028-196
	10·21	021	190	169	041	128-021	081-190
	6·35	036	175	139	071	204-036	005-175
	3·70	070	141	071	139	163-070	046-141
	5·42	085	126	041	169	134-085	075-126
	7·30	091	120	029	181	081-091	128-120
	2·105	105	106	001	209	052-105	157-106
2310	11·210	0210	2101	1891	0419	1279-0210	1030-2101
	10·231	0231	2080	1849	0461	0947-0231	1362-2080
	7·330	0330	1981	1651	0659	1367-0330	0942-1981
	6·385	0385	1926	1541	0769	1604-0385	0705-1926
	21·110	0441	1870	1429	0881	2226-0441	0083-1870
	30·77	0540	1771	1231	1079	0336-0540	1973-1771
	33·70	0561	1750	1189	1121	0004-0561	2305-1750
	35·66	0595	1716	1121	1189	0573-0595	1736-1716
	15·154	0616	1695	1079	1231	1396-0616	0914-1695
	42·551	0715	1596	0881	1429	0661-0715	1648-1596
	3·770	0771	1540	0769	1541	1283-0771	1026-1540
	14·165	0826	1485	0659	1651	2045-0826	0264-1485
	5·462	0925	1386	0461	1849	1940-0925	0369-1386
	22·105	0946	1365	0419	1891	0453-0946	1856-1365
	2·1155	1155	1156	0001	2309	0577-1155	1732-1156

TABLE 2—Automorphic Numbers

Scale of Notation Base = B	Squares			
	S_1	Automorphic Digit	S_2	Automorphic Number
6	1	3	2	1-3
	2	4	3	4-4
10	2	5	6	2-5
	3	6	7	7-6
12	01	04	04	05-04
	06	09	09	06-09
14	03	07	12	03-07
	04	08	13	10-08
15	02	06	02	08-06
	06	10	03	06-10
18	04	09	02	04-09
	05	10	03	13-10
20	01	05	06	11-05
	12	16	07	08-16
24	03	09	00	21-09
	10	16	07	02-16
30	01	06	08	19-06
	03	10	11	03-10
	07	15	26	07-15
	08	16	27	22-16
	14	21	22	26-21
	20	25	27	10-25
60	04	16		
	07	21		
	10	25		
	21	36		
	26	40		
	33	45		
105	002	015		
	004	021		
	012	036		
	046	070		
	068	085		
	078	091		
210	001	015		
	002	021		
	006	036		
	023	070		
	034	085		

Scale of Notation Base = B	Squares	
	S_1	Automorphic Digit
	039	091
	052	105
	053	106
	068	120
	075	126
	094	141
	145	175
	171	190
	182	196
2310	0019	0210
	0023	0231
	0047	0330
	0064	0385
	0084	0441
	0126	0540
	0136	0561
	0153	0595
	0164	0616
	0221	0715
	0257	0771
	0295	0826
	0370	0925
	0387	0946
	0577	1155
	0578	1156
	0806	1365
	0831	1386
	0954	1485
	1026	1540
	1102	1596
	1241	1695
	1274	1716
	1325	1750
	1357	1771
	1513	1870
	1605	1926
	1698	1981
	1872	2080
	1910	2101