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David Singmaster
David Fielker
NJ AS

Correspondence, 5 pages
August 1979

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Neil Sloane — I don't find M, N or N_n of p. 5 in your list — see Miller for the details. David.

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2 August 1979

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Dear David Fielker,

You may find the following too technical but I thought I would write out what I had found out.

David Singmaster
David Singmaster

The question raised by Geoff Millan in MT 86 (Mar 1979) 13 is quite appropriate. The answer $[\frac{N+8}{3}]$ is certainly not generally correct - but the general solution is not known. There are actually several forms of the problem. I will summarize these and what is known about them below. The references provide more details.

A transformer has n output connections having distinct non-negative integral voltages which we order as $V_1 = 0 < V_2 < V_3 < \ldots < V_n = N$. Can we achieve each integral voltage 1, 2, ..., N by using some pair of output connections? If so, the integers $V_1, V_2, \ldots, V_n$ are called a <u>restricted difference basis</u>.

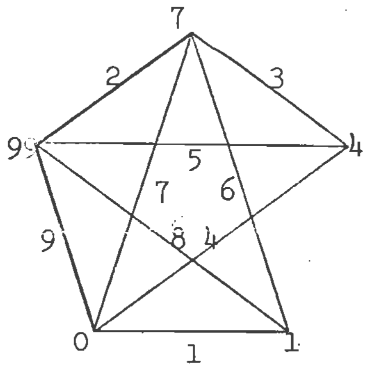
Viewing 1, 2, ..., N as vertices of a graph, we label the vertex i with V_i and, if there is an edge ij, we label the edge with $|V_i - V_j|$. Then we shall say the graph is <u>numbered</u>. A <u>graceful</u> numbering has the edge labels taking on all the values 1, 2, ..., N once each. Thus we must have $N = V_n$ edges in the graph. Such a graph is a graceful graph on n vertices or a graceful subgraph of K_n, the complete graph on n vertices. Each restricted difference basis gives such a graph by drawing just one edge for each of the distinct differences $|V_i - V_j|$.

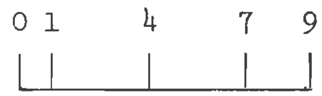
We can also view V_i as the markings on a lazy ruler-marker's ruler. We can measure every integral distance from 1 to N if the ruler has markings at lengths V_i from one end. This shows that there is a clear symmetry - if V_i are a restricted difference basis, then $N-V_i$ are also a restricted difference basis.

The basic problem is to maximize V_n for a given n over all restricted difference bases. The problem referred to by Millan is the equivalent problem of minimizing n for a given N. I find the first form algebraically easier to deal with.

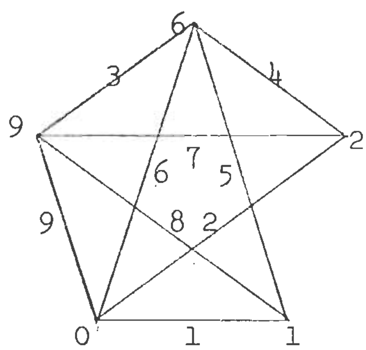
The two optimal cases for $n = 5$ are shown below to illustrate the different forms of the problem.

0, 1, 4, 7, 9





0, 1, 2, 6, 9



The missing edges in the graphs correspond to repeated differences and illustrate the fact that the complete graph K_n cannot be gracefully numbered for $n \geq 5$, i.e. $N < \binom{n}{2}$ for $n \geq 5$.

If we consider the n-1 differences $d_i = V_{i+1} - V_i$, $i = 1, 2, \ldots, n-1$ for the first few values of n, we discover some nice patterns. It is easy to see that the differences 1, 3, 3, ..., 3, 2 give a restricted difference basis with $V_n = 3(n-3) + 3 = 3n - 6$, for $n \geq 3$. If this were the maximal value of V_n, we would have $N \leq 3n-6$ so $n \geq (N+6)/3$ and this is the same as $n \geq [\frac{N+8}{3}]$, which is the result referred to by Millan. However, we can generalize this easily by taking differences 1, 1, ..., 1, k, k, ..., k, k-1 (with k-2 1's and n-k k's). This gives a restricted difference basis with $V_n = k(n-k) + 2k - 3$. We denote this expression by N_k (thinking of n as given). A short table of values is given below.

Table of $N_k = k(n-k) + 2k - 3$

$n\backslash k$	2	3	4	5	6	7	8	9	10
2	1								
3	3	3							
4	5	6	5						
5	7	9	9	7					
6	9	12	13	12	9				
7	11	15	17	17	15	11			
8	13	18	21	22	21	18	13		
9	15	21	25	27	27	25	21	15	
10	17	24	29	32	33	32	29	24	17

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(I find the relationships in this table quite intriguing. After some transformation it yields the simpler table for $k(n-k)$ and this turns out to be a transform of the ordinary multiplication table!) Clearly we can maximize N_k as a function of k and this occurs when $k = [\frac{n+2}{2}]$ or $[\frac{n+3}{2}]$ yielding a maximal value $M = [(\frac{n+2}{2})^2 - 3]$. We can first do better than use $k = 3$ at $n = 6$ where we can achieve $N = 13$ by using 5 differences 1, 1, 4, 4, 3 giving 6 values 0, 1, 2, 6, 10, 13. Since M is asymptotic to $n^2/4$ while $\binom{n}{2}$ is asymptotic to $n^2/2$, we see that M is asymptotic to $\frac{1}{2}\binom{n}{2}$, that is, we can achieve about one half of the maximum possible value.

Remarkably, it is possible to do quite a bit better. Miller lists all solutions for various N and its minimal n and gives the number of solutions with minimal n for N up to 68. The first case where the above M fails to be maximal is at $n = 8$, where $N = V_n = 23$ can be achieved from two sets of differences: 1,1,9,4,3,3,2 and 1,3,6,6,2,3,2. The known maximal values of $N = V_n$ are given below along with $\binom{n}{2}$ and the above given M for comparison, along with two other values to be discussed shortly. The values marked ? are conjectural - solutions with these values are known, but it is not known if they are best possible. It has been shown that one can always construct solutions with $N \geq (n-3)^2/3$, i.e. we can achieve about 2/3 of the maximum expected value.

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have / unrestricted difference bases

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$= 1003.5$

n	$\binom{n}{2}$	M	N	N_u	N_d
2	1	1	1	1	1
3	3	3	3	3	3
4	6	6	6	6	6
5	10	9	9	9	11
6	15	13	13	13	17
7	21	17	17	18	25
8	28	22	23	24?	34
9	36	27	29	29?	44
10	45	33	36	37?	55
11	55	39	43	45?	72
12	66	46	50	51?	
13	78	53	58	61?	
14	91	61	68	70?	
15	105	69	79?	79?	
16	120	78	90?	93?	
17	136	87	101?	101?	
18	153	97	112?	113?	
19	171	107	123?	127?	
20	190	118	138?		
21	210	129	153?		
22	221	141	168?		

|| have

too close

There are two variants of the basic problem which have been considered. For an <u>unrestricted difference basis</u>, we drop the requirement that $N = V_n$. That is, we want the differences $V_i - V_j$ to take on all the values 1, 2, ..., N and then possibly some larger values up to V_n. The first case where this gives a larger value of N for a given n occurs for $n = 7$, where the following four sets of differences have $N = 18$: 6,3,1,7,5,2; 8,1,3,6,5,2; 14,1,3,6,2,5; 13,1,2,5,4,6 , where V_n is 24, 25, 31, 31 respectively. We denote the maximal N for unrestricted difference sets by N_u. Known values are given in the table.

For a <u>distinct difference basis</u>, we want all the $\binom{n}{2}$ differences $|V_i - V_j|$ to be distinct. We then want to minimize $N = V_n$ over all such bases for given n. For example, with $n = 5$, the values 0, 1, 4, 9, 11 have the ten distinct differences from 1 to 11 omitting 6. We denote this minimum by N_d. Known values are in the table. Bermond conjectures that $N + N_d \geq 2\binom{n}{2}$.

Finally, it is an outstanding conjecture that every tree can be gracefully numbered.

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If you do decide to publish this, my current affiliation is: Faculty of Mathematics, The Open University, Walton Hall, Milton Keynes, but the letterhead address is better for post.