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The last member known to the writer is that one given by $n = 8$, LP = 48, and

$$p = 9999999900000001, \text{ or}$$

$$p = 2^8 \cdot 5^8 \cdot \underset{(1)}{3^2} \cdot \underset{(2)}{11} \cdot \underset{(8)}{73} \cdot \underset{(4)}{101} \cdot \underset{(8)}{137} + 1.$$

It will be observed that, for Family IV, n is equal to the number of 9's in the prime number; moreover, that the largest period length of any of the factors of $(p - 1)$ is *also* equal to the number of 9's. Observe, too, that n is the exponent of 2 in the factored form of $(p - 1)$. The other factors have LP which are factors of n.

Yates [3] has made a special study of Family IV and has shown the number to be composite for $n = 10$.

There are, of course, many other "Golden" primes having LR << 1, but probably are not representable by such simple algebraic expressions as those generating Families I-IV. Probably neither do these others have such simple digital structure.

The above discussed relationships causes one to speculate as to the possibility of there being some sort of functional relationship between the digital structure of a prime number (its "anatomy") and the length, LP, of the period of its reciprocal; in fact, the writer has some evidence of this from the digital structure of other "Golden" primes.

References

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2. Samuel Yates, "Peculiar Properties of Repunits," *J. Recreational Math., 2*(3), pp. 139-146, July 1969.
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ABC Puzzles – Puzzle B

Each of three boxes contain two coins. One contains two dimes, another two nickels, and the third a dime and a nickel. The boxes are marked 10 cents, 15 cents, and 20 cents, but none contain the amount marked on it. Without looking in the boxes, you may draw out one coin at a time.

What is the fewest number of coins you must draw to determine the contents of each box?

Maxey Brooke, Sweeny Texas

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A GENERALIZATION OF THE GOLDEN RATIO

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Introduction

If two positive numbers a and b satisfy the equation $\frac{b}{a} = \frac{a+b}{b}$ it follows that $\left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right) - 1 = 0$ and hence $\frac{b}{a} = \frac{1+\sqrt{5}}{2} = 1.6180339989\ldots$ This irrational number, usually denoted by the Greek letter phi (ϕ), has so fascinated mathematicians since even before the time of Euclid (c.300 B.C.) that it has come to be called the *golden ratio*. One of its best-known properties involves the Fibonacci sequence $\{F_n\}$, defined by $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n (n \geq 1)$; namely, $\frac{F_{n+1}}{F_n} \to \phi$ as $n \to \infty$, a result which readily follows from the so-called Binet formula

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}, \quad n \geq 1. \tag{1}$$

Perhaps not so well-known is the fact that the more general sequence $\{a_n\}$, where a_1, a_2 are both arbitrary positive integers and $a_{n+2} = a_{n+1} + a_n (n \geq 1)$, also has the property $\frac{a_{n+1}}{a_n} \to \phi$ as $n \to \infty$. For example, Table 1 shows this convergence for the case where $a_1 = 26$ and $a_2 = 17$; the values of $\frac{a_{n+1}}{a_n}$ are accurate to 6 decimal places.

The Generalized Golden Ratio

The main purpose of this paper is to consider the still more general sequence $\{a_n\}$ defined by $a_1 = a, a_2 = b$, and

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$$a_{n+2} = ra_{n+1} + sa_n, \quad n \geqslant 1, \tag{2}$$

where a, b, r, s are all arbitrary positive integers. Putting

$$a_n = p\alpha^n + q\beta^n \tag{3}$$

(see [2]), it follows that

$$a_{n+2} = (\alpha + \beta)a_{n+1} - \alpha\beta a_n,$$

whence comparison with (2) yields $\alpha + \beta = r$ and $\alpha\beta = -s$. Thus α, β are roots of the equation $x^2 - rx - s = 0$; i.e. $\alpha, \beta = \dfrac{r \pm \sqrt{r^2 + 4s}}{2}$. Moreover, the equations

$$a_1 = p\alpha + q\beta = a$$
$$a_2 = p\alpha^2 + q\beta^2 = b$$

yield $p = \dfrac{b - a\beta}{\alpha(\alpha - \beta)}$ and $q = \dfrac{a\alpha - b}{\beta(\alpha - \beta)}$. Therefore (3) gives

$$a_n = \left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{n-1} - \left(\frac{b - a\alpha}{\alpha - \beta}\right)\beta^{n-1}, \quad n \geqslant 1, \tag{4}$$

as the generalization of Binet's formula (1); Binet's formula of course results from taking $a = b = r = s = 1$, in which case $\alpha, \beta = \dfrac{1 \pm \sqrt{5}}{2}$, $\alpha + \beta = 1$ and $\alpha - \beta = \sqrt{5}$.

Table 1.

n	a_n	$\frac{a_{n+1}}{a_n}$
1	26	0.653846
2	17	2.529412
3	43	1.395349
4	60	1.716667
5	103	1.582524
6	163	1.631902
7	266	1.612782
8	429	1.620047
9	695	1.617266
10	1124	1.618327
11	1819	1.617922
12	2943	1.618077
.	.	.
.	.	.

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Note also that $\alpha = \dfrac{r + \sqrt{r^2 + 4s}}{2}$ is the *generalized golden ratio*, since

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^n - \left(\frac{b - a\alpha}{\alpha - \beta}\right)\beta^n}{\left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{n-1} - \left(\frac{b - a\alpha}{\alpha - \beta}\right)\beta^{n-1}}$$

$$= \frac{\alpha - \beta\left(\frac{b - a\alpha}{b - a\beta}\right)\left(\frac{\beta}{\alpha}\right)^{n-1}}{1 - \left(\frac{b - a\alpha}{b - a\beta}\right)\left(\frac{\beta}{\alpha}\right)^{n-1}} \to \alpha \text{ as } n \to \infty$$

since $-1 < \dfrac{\beta}{\alpha} < 0$. Clearly α depends on r and s, but not on a and b.

As an illustration of the above theory, Table 2 shows the early behavior of $\dfrac{a_{n+1}}{a_n}$ (evaluated to 6 decimal places) for two different sequences, both however satisfying $r = 3$ and $s = 2$. For the first sequence $a = 4$ and $b = 7$, while the second sequence has $a = 6$ and $b = 5$. Of course, in each case the limit

$$\alpha = \frac{3 + \sqrt{17}}{2} = 3.561552813\ldots$$

Table 2.

n	a_n	$\frac{a_{n+1}}{a_n}$	a_n	$\frac{a_{n+1}}{a_n}$
1	4	1.750000	6	0.833333
2	7	4.142857	5	5.400000
3	29	3.482759	27	3.370370
4	101	3.574257	91	3.593407
5	361	3.559557	327	3.556575
6	1285	3.561868	1163	3.562339
7	4577	3.561503	4143	3.561429
8	16301	3.561561	14755	3.561572
9	58057	3.561552	52551	3.561550
10	206773	3.561553	187163	3.561553
11	736433	3.561553	666591	3.561553
12	2622845	3.561553	2374099	3.561553
.	.	.	.	.
.	.	.	.	.

Particular Cases

The above results generalize the work of several earlier papers. For example, taking $a = b = r = 1$, s arbitrary, we obtain

$$a_n = \frac{\left(\frac{1+\sqrt{4s+1}}{2}\right)^n - \left(\frac{1-\sqrt{4s+1}}{2}\right)^n}{\sqrt{4s+1}}, \quad n \geqslant 1,$$

and $\alpha = \dfrac{1+\sqrt{4s+1}}{2}$, and several properties of this particular sequence appear in [1]. On the other hand Guest [2] considered the case $a = p$, $b = p + 1$ (p arbitrary), $r = s = 1$, and Horadam [3] the slightly more general case $a = p$, $b = p + q$ (p,q both arbitrary), $r = s = 1$. In [3] the formula corresponding to (4) appears (in my notation) as

$$a_n = \frac{1}{2\sqrt{5}}(l\alpha^n - m\beta^n), \quad n \geqslant 1,$$

where $l = 2(p - q\beta)$, $m = 2(p - q\alpha)$ and of course $\alpha, \beta = \dfrac{1 \pm \sqrt{5}}{2}$. This follows from (4) after some manipulation.

Integral Values of α

An interesting question concerns the possibility of the *generalized golden ratio* being integral. Taking $\alpha = n$ say, where n is an arbitrary positive integer, it follows that $\sqrt{r^2 + 4s} = 2n - r$, $r^2 + 4s = 4n^2 - 4nr + r^2$, and hence $s = n(n - r)$. Since r, s are both positive, r can thus take only the values $1, 2, \ldots, n - 1$. In other words, for each positive integer n there are exactly $n - 1$ combinations of r and s which yield $\alpha = n$.

As a final example, suppose we seek a generalized Fibonacci sequence for which $\alpha = 4$. In this case the equation $s = 4(4 - r)$ yields the 3 combinations $r = 1, s = 12$; $r = 2, s = 8$; $r = 3, s = 4$. As an illustration, Table 3 shows the early behavior of $\dfrac{a_{n+1}}{a_n}$ (evaluated to 6 decimal places) for the sequence with $a = 5$, $b = 2$, $r = 3$ and $s = 4$.

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Table 3.

n	a_n	$\frac{a_{n+1}}{a_n}$
1	5	0.400000
2	2	13.000000
3	26	3.307692
4	86	4.209302
5	362	3.950276
6	1430	4.012587
7	5738	3.996863
8	22934	4.000785
9	91754	3.999804
10	366998	4.000049
11	1468010	3.999988
12	5872022	4.000003
.	.	.
.	.	.

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About the Author

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