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A007969: Proof of a Conjecture Related to the 1-Happy Numbers

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Abstract

Conway's 1-happy numbers A007969 are proved to coincide with the discriminants d of the Pell equation $x^2 - d\,y^2 = +1$ for which the positive fundamental solution $(x_0,\, y_0)$ has even y_0.

Conway [1] proposed three sequences, obtained from three types of sequences of couples called 0-happy couples (A,A), 1-happy couples (B, C) and 2-happy couples (D, E). By taking products of each couple one obtains three sequences that are given in OEIS [3] A000290 (the squares), A007969 and A007970, respectively. It is stated as a theorem, with the proof left to the reader, that each positive integer appears in exactly one of these three sequences. Here we consider the numbers $d = B\ C$ of the 1-happy couples. They are defined if the following indefinite binary quadratic form is soluble with positive integers B and C, where $B \geq 1$ and $C \geq 2$, and (without loss of generality) positive integers S and R (obviously $S = 0$ is excluded, and $R \neq 0$ because of $C > 1$).

$$C\,S^2 - B\,R^2 = +1\ , \tag{1}$$

The discriminant of this quadratic form is $D = 4\,C\,B = 4\,d$. Obviously $\gcd(C,\, B) = 1 = \gcd(S,\, R) = 1 = \gcd(C,\, R) = 1 = \gcd(S,\, B)$. The case of d a square is excluded because $B = C \neq 1$ contradicts $\gcd(C,\, B) = 1$, and if $C = c^2$ and $B = b^2$ with $c \neq b$ and $b > 1$ then $c\,S = 1$ and $b\,R = 0$ is the only solution, which is excluded because $c \geq 2$ from $C \geq 2$ and also from $R > 0$. Therefore, $D = 4\,d = 4\,B\,C$ is not a square. The B and C numbers are found under A191854, A191855, respectively.

We will prove that the sequence A007969 consists of those positive integers $D \equiv 0\,(mod\,4)$, D not a square, such that the (generalized) *Pell* equation

$$v^2 - D\,w^2 = +4 \tag{2}$$

has only improper solutions. (Improper solutions exit for each D not a square from the existing proper solutions of the standard *Pell* equation $x^2 - D\,y^2 = +1$, see *e.g.*, [2] Theorem 104, p. 197 - 198.) This indefinite binary quadratic form has discriminant $4\,D$.

This claim is equivalent to the statement that the sequence A007969 coincides with all positive integers d, d not a square, such that the *Pell* equation

$$x^2 - d\,y^2 = 1 \tag{3}$$

has positive fundamental solution $(x_0,\, y_0)$ with even y_0, $y_0 = 2\,Y_0$. The proof uses the fact that v has to be even, $v = 2\,x$ and that $D = 4\,d$, d not a square. Then put $w = y$. Of course, there are only proper solutions of this *Pell* equation: $\gcd(x,\, y) = 1$. But $\gcd(v,\, w) = \gcd(2\,x,\ y) = g > 1$ for all solutions because eq. (2) has to have only improper solutions. Now y can not be odd because then $\gcd(2\,x,\ y)$ would be $\gcd(x,\ y)$ which is 1 not $g > 1$. Therefore $y = 2\,Y$. Because any solution of eq. (3) can be obtained from the positive fundamental solution $(x_0,\ y_0)$, the one with the smallest positive y value, and

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all solutions will have even y if y_0 is even, (see, e.g., [2], Theorem 104, eq. (8), p. 198) the equivalence of the statements in connection with eqs. (2) and (3) is proved.

This equation, in turn, can be written as

$$X_0\,(X_0\,+\,1)\,=\,d\,Y_0^2\;, \tag{4}$$

where $x_0\,=\,2\,X_0\,+\,1$. Note that the above *gcd* condition is satisfied because $g\,\geq\,2$.

This follows because in eq. (3) x has to be odd, $x\,=\,2\,X\,=\,1$, because $y\,=\,2\,Y$. Then $x_0^2\,=\,8\,T(X_0)+1$, with the triangular numbers $T\,=$A000217. This produces the (even) pronic number $X_0\,(X_0\,+\,1)$ (see A002378) after division by 4. Remember for later use that $\gcd(n, n+1)\,=\,1$ for integers n.

Proposition:

i) Any solution of eq. (4) leads to a solution of eq.(1) with $B\,=\,\gcd(d,\,X_0)$, $C\,=\,\frac{d}{B}$, $R\,=\,R_0\,=\,\gcd(X_0,\,Y_0)$, and $S\,=\,S_0\,=\,\frac{Y}{R_0}$. This will provide the positive fundamental (proper) solution of eq. (1).

ii) The positive fundamental solution $(R_0,\,S_0)$ of eq. (1) leads to the solution of eq. (4) with $d\,=\,C\,B$, $X_0\,=\,B\,R_0^2$ and $Y_0\,=\,R_0\,S_0$. This leads to the positive proper fundamental solution $(2\,X_0\,+\,1, 2\,Y_0)$ of eq. (3).

The $x0$, X_0 and Y_0 numbers for d from A007969 are found under A262024, A262025 and A261250, respectively. The R_0 and S_0 numbers are found under A263006 and A263007. See also the Table.

Proof:

i) Given the fundamental solution d, X_0 and Y_0 of eq. (4) with d not a square, we note for a later redefinition the scaling freedom in $d\,=\,C\,B$ and $Y_0\,=\,R_0\,S_0$. Instead of C, B and R_0, S_0 one can take $B(n)\,=\,\frac{B}{n}$, $C(n)\,=\,n\,C$ and $R_0(m)\,=\,\frac{R_0}{m}$, $S_0(m)\,=\,m\,S$ with arbitrary positive integers n and m to be determined later.

We define $B\,:=\,\gcd(d,\,X_0)\,\geq\,1$. Then $C\,:=\,\frac{d}{B}$ is a positive integer not equal to B. Define $R_0\,:=\,\gcd(X_0,\,Y_0)\,\geq\,1$. Then $S_0\,:=\,\frac{Y_0}{R_0}$ is a positive integer. By definition B and R_0 divide X_0. Because $\gcd(X_0, X_0+1)\,=\,1$ (see the remark above), B and R_0 cannot divide X_0+1. From the *r.h.s.* (right-hand side) of eq. (4) which is $C\,B\,(S_0\,R_0)^2$ it follows therefore that $X_0\,=\,B\,R_0^2\,a$ with some positive integer a, and then $a\,(X_0+1)\,=\,C\,S_0^2$. The scaling freedom allows us to replace C, B and R_0, S_0 by their n and m-dependent counterparts, leading to $X_0\,=\,\frac{a}{n\,m^2}\,B\,R_0^2$ and $X_0+1\,=\,\frac{n\,m^2}{a}\,C\,S_0^2$. Choosing $n\,m^2\,=\,a$, *i.e.*, $n\,=\,n(a)\,=\,sqfp(a)\,=$A007913$(a)$ (the squarefree part of a) and $m\,=\,m(a)\,=\,\sqrt{\frac{a}{n(a)}}\,=$A000188$(a)$, we obtain

$$X_0\,=\,B\,R_0^2,\quad\text{and}\quad X_0\,+\,1\,=\,C\,S_0^2\,. \tag{5}$$

Elimination of X_0 leads to eq. (1) as $B\,R_0^2\,+\,1\,=\,C\,S_0^2$.

Now assume that there is a solution $(R_*,\,S_*)$ with positive but smaller values than $(R_0,\,S_0)$ then this would imply from the definition of R that there is a smaller positive solution than X_0 and Y_0 of eq. (4); but these correspond to the smallest positive solution of $(x_0,\,y_0)$ of eq. (3). Therefore, one will automatically find the smallest positive solution of eq. (1),

ii) Let $(R_0,\,S_0)$ be the smallest positive solution of eq. (1), and put $d\,=\,C\,B$ with positive B and C, $C\,\geq\,2$. Then d is not a square as shown above after eq. (1). Define $X_0\,:=\,B\,R_0^2$ and $Y_0\,:=\,R_0\,S_0$. Then eq. (3) follows from eq. (1) for R_0 and S_0 because B and R_0 are non-vanishing. The solution $(x_0\,:=\,2\,X_0+1,\,y_0\,:=\,2\,Y_0)$ of eq. (3) will then be the positive fundamental solution, because otherwise there would be smaller positive R_0 and S_0 values but they have been chosen minimal. □

Note: If we take *Conway*'s theorem then the above proof of the 1-happy couple product numbers A007969, together with the square d numbersA000290, lead to the statement that the 2-happy couple product numbers A007970 are those d values for which the *Pell* eq. (3) has positive fundamental solutions $(x_0,\, y_0)$ with odd y_0. This should also be proved independently of the theorem.

References

[1] J. H. Conway, On Happy Factorizations, https://cs.uwaterloo.ca/journals/JIS/happy.html, Journal of Integer Sequences, Vol. 1 (1998), Article 98.1.1.

[2] T. Nagell, Introduction to Number Theory, 1964, Chelsea Publishing Company, New York.

[3] The On-Line Encyclopedia of Integer Sequences, https://oeis.org/.

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Concerned with OEIS sequences A000290, A007969, A007970, A000217, A002378, A191854, A191855, A007913, A000188, A261250, A262024 and A262025, A263006 and A263007.

TAB. : d, X_0, Y_0, C, B, S_0, R_0

d	X_0	Y_0	C	B	S_0	R_0
2	1	1	2	1	1	1
5	4	2	5	1	1	2
6	2	1	3	2	1	1
10	9	3	10	1	1	3
12	3	1	4	3	1	1
13	324	90	13	1	5	18
14	7	2	2	7	2	1
17	16	4	17	1	1	4
18	8	2	9	2	1	2
20	4	1	5	4	1	1
21	27	6	7	3	2	3
22	98	21	11	2	3	7
26	25	5	26	1	1	5
28	63	12	4	7	4	3
29	4900	910	29	1	13	70
30	5	1	6	5	1	1
33	11	2	3	11	2	1
34	17	3	2	17	3	1
37	36	6	37	1	1	6
38	18	3	19	2	1	3
39	12	2	13	3	1	2
41	1024	160	41	1	5	3 2
42	6	1	7	6	1	1
44	99	15	4	11	5	3
45	80	12	9	5	3	4
46	12167	1794	2	23	78	23
50	49	7	50	1	1	7
52	324	45	13	4	5	9
53	33124	4550	53	1	25	182
54	242	33	27	2	3	11
55	44	6	5	11	3	2
56	7	1	8	7	1	1
57	75	10	19	3	2	5
58	9801	1287	58	1	13	99
60	15	2	4	15	2	1
61	883159524	113076990	61	1	3805	29718
62	31	4	2	31	4	1
65	64	8	65	1	1	8
66	32	4	33	2	1	4
68	16	2	17	4	1	2
69	3887	468	3	23	36	13
70	125	15	14	5	3	5
72	8	1	9	8	1	1
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