Joe Roberts, the Lure of the Integers

~~7138~~

3 Figurate Numbers

A27568

Sierpiński, in his book *A Selection of Problems in the Theory of Numbers*, p 181. observed that Escott and Sendacka have shown that the only numbers less than 10^9 which are both triangular, of the form $\binom{n}{2}$, and tetrahedral, of the form $\binom{m}{3}$, are the five numbers

$$1, 10, 120, 1540, 7140$$ a finite seq. A27568

The pairs of values for n, m in these five numbers are $(2,3)$, $(5,5)$, $(16,10)$, $(56,22)$, $(120,36)$. $P_1 = T_1$ $P_3 = T_4$ $P_8 = T_{15}$ $P_{20} = T_{55}$ $P_{34} = T_{119}$

Avanesov proved that these five do indeed exhaust all possible numbers which are both triangular and tetrahedral.

Singmaster showed that three binomial coefficients $\binom{n}{k}$, $\binom{n}{k+1}$, $\binom{n}{k+2}$ can be in the ratio $1:2:3$ precisely for $n = 14$, $k = 4$ and this triple of binomial coefficients form the following portion of Pascal's triangle.

$$\begin{array}{c} 1001 \; 2002 \; 3003 \\ 3003 \; 5005 \\ 8008 \end{array}$$

Tzanakis and de Weger showed that the equation $y^2 = x^3 - 4x + 1$ has the following 22 solutions

$$(x, \pm y) = (-2,1),\ (-1,2),\ (0,1),\ (2,1),\ (3,4),\ (4,7),\ (10,31),$$

$$(12,41),\ (20,89),\ (114,1217),\ (1274,45473).$$

From this they conclude that the only triangular numbers which are products of three consecutive integers are:

$$T_3,\ T_{15},\ T_{20},\ T_{44},\ T_{608},\ T_{22736}.$$

Mohanty had announced this exact same result the previous year. However, as was pointed out in a letter to the editor by A. Bremner, there were mistakes in Mohanty's arguments. Despite the fact that the main methods of the paper were not at all elementary, it is interesting to note that elementary errors did creep in. One of the errors, quoted

Scan A27568

R G Wilson, ✓

Letter to

NJAS

25 August 1983

one page

one seq