The white diamond product of power series

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The purpose of these notes is to introduce a novel multiplication of power series, which we call the white diamond product. The white diamond product is obtained by deforming the Hadamard product of power series by the action of the invertible lower triangular matrix $\left(\binom{n}{k}k!\right)$. Many well-known sequences of polynomials including the Bell polynomials, the Lah polynomials, the Laguerre poynomials and the Bessel polynomials are shown to have simple expressions in terms of the white diamond product.

1 DEFORMATIONS OF THE HADAMARD PRODUCT

1.1 The Hadamard product. We recall the definition of the Hadamard product of sequences and of power series.

DEFINITIONS

D1: The Hadamard product (or the pointwise product) $a * b$ of a pair of vectors $a = (a(n))$ and $b = (b(n))$ is defined to be the vector $a * b = (a(n)b(n))$.

D2: The Hadamard product $A(x) * B(x)$ of the power series $A(x) = \sum_{n=0}^{\infty} a(n)x^n \in \mathbb{C}[[x]]$ and $B(x) = \sum_{n=0}^{\infty} b(n)x^n \in \mathbb{C}[[x]]$ is defined as the power series

$$A(x) * B(x) \quad = \quad \sum_{n=0}^{\infty} a(n)b(n)x^n\,. \tag{1}$$

There is a slight abuse of notation here in using the same symbol $*$ to denote the product of power series and the product of column vectors.

FACTS

F1: The pointwise product of vectors is clearly commutative and associative and distributes over addition of vectors.

F2: The Hadamard product of power series is clearly commutative, associative and distributes over addition of power series.

F3: The identity element for the algebra $\mathbb{C}[[x]]$ equipped with the Hadamard product is the power series $1 + x + x^2 + \cdots = \frac{1}{1-x}$.

F4: The set of monomial polynomials $\{x^n\}_{n\geq 0}$ form a complete set of mutually orthogonal idempotents in the algebra of power series equipped with the Hadamard product, that is,

$$x^i * x^j \quad = \quad \delta_{ij} x^i \quad i,j \geq 0 \quad \text{and} \sum_{i=0}^{\infty} x^i = \text{multiplicative identity.} \tag{2}$$

1.2 Deforming the Hadamard product. In what follows it will be convenient for us to represent a sequence $a(n)$ by an infinite column vector. There is an obvious bijective correspondence ϕ between formal power series and their coefficient sequences:

$$A(x) = a(0) + a(1)x + a(2)x^2 + \cdots \quad \overset{\phi}{\longleftrightarrow} \quad \begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \\ \end{pmatrix}.$$

If M is an infinite lower triangular matrix we let M act on the column vector of coefficients of a power series by matrix multiplication. We can then use the bijection ϕ to pull back this action to an action of M on the corresponding power series.

DEFINITIONS

D3: We define the action of the lower triangular matrix M on the power series $A(x) = \sum_{n\geq 0} a(n)x^n$ by

$$MA(x) \quad = \phi^{-1} \left(M \begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \\ \end{pmatrix} \right) \quad .$$

D4: Let M now be an **invertible** infinite lower triangular matrix. We define the M-Hadamard product $\underset{M}{a * b}$ of a pair of column vectors $a \equiv a(n)$ and $b \equiv b(n)$ to be the column vector

$$\underset{M}{a * b} \quad = \quad M^{-1} \left(Ma * Mb \right). \tag{3}$$

D5: Let M be an **invertible** infinite lower triangular matrix. The M-Hadamard product $\underset{M}{A(x) * B(x)}$ of the formal power series $A(x)$ and $B(x)$ is defined as the power series corresponding to the column vector $\underset{M}{a * b}$ under the bijection ϕ:

$$\underset{M}{A(x) * B(x)} \quad = \quad \phi^{-1}\left(a \underset{M}{*} b \right). \tag{4}$$

Equivalently,

$$\underset{M}{A(x) * B(x)} \quad = \quad M^{-1}\left(MA(x) \ * \ MB(x) \right). \tag{5}$$

FACTS

F5: If M is the identity matrix then the M-Hadamard product is simply the Hadamard multiplication of power series. We can therefore view the M-Hadamard product as a deformation of the Hadamard product by the matrix M.

F6: The power series Mx^n is the ordinary generating function for the nth column of the matrix M.

F7: If $A(x) = \sum_{i \geq 0} a(i)x^i$ and $B(x) = \sum_{j \geq 0} b(j)x^j$ then

$$\underset{M}{A(x) * B(x)} \quad = \quad \sum_{i,j \geq 0} a(i)b(j) \left(x^i \underset{M}{*} x^j \right). \tag{6}$$

F8: Both the M-Hadamard product of column vectors and the M-Hadamard product of power series are commutative, associative and distribute over addition of column vectors and power series, respectively.

F9: It follows from fact F4 that the power series $E_i(x) := M^{-1}x^i, i = 0, 1, 2, ...$ form a complete set of orthogonal idempotents in the algebra of power series $\mathbb{C}[[x]]$ equipped with the M-Hadamard product; that is,

$$E_i \underset{M}{*} E_j = \delta_{ij} E_i \quad i, j \geq 0$$

and

$$\sum_{i=0}^{\infty} E_i = M^{-1} \frac{1}{1-x} = \text{multiplicative identity.}$$

Every power series $A(x)$ has an idempotent expansion $A(x) = \sum_{n=0}^{\infty} a(n) E_n(x)$, where the coefficents $a(n)$ are determined by the power series expansion $MA(x) = \sum_{n=0}^{\infty} a(n) x^n$.

F10: If $A(x) = \sum_{n=0}^{\infty} a(n) E_n(x)$ and $B(x) = \sum_{n=0}^{\infty} b(n) E_n(x)$ are the expansions of the powers series $A(x)$ and $B(x)$ in terms of the basis of orthogonal idempotents $E_n(x)$ then

$$A(x) \underset{M}{*} B(x) = \sum_{n=0}^{\infty} a(n) b(n) E_n(x) . \tag{7}$$

It follows inductively that the k-fold product

$$\underbrace{A(x) \underset{M}{*} A(x) \underset{M}{*} \cdots \underset{M}{*} A(x)}_{k \text{ factors}} = \sum_{n=0}^{\infty} a(n)^k E_n(x) . \tag{8}$$

Dukes and White [DuWh'16], in a study of the combinatorics of web diagrams and web matrices, defined a commutative and associative binary operation on formal power series, which they called the black diamond product. They gave several examples of polynomial sequences of combinatorial interest, such as the Fubini polynomials $\mathcal{F}_n(x)$ and the shifted Legendre polynomials $P_n(2x+1)$, that have simple expressions in terms of the black diamond product. In [Ba'18] we showed the black diamond product is a particular case of the M-Hadamard product where M is equal to Pascal's triangle of binomial coefficients $\left(\binom{n}{k}\right)$. In the next section we look at an M-Hadamard product operator closely related to the black diamond product.

2 THE WHITE DIAMOND PRODUCT

In this section we work in the algebra $\mathbb{C}[[x]]$ with multiplication of power series given by the M-Hadamard product with M taken to be the lower triangular array $\left(\binom{n}{k}k!\right)$. We call this multiplication operator the *white diamond product* of power series and denote it by the symbol $\Diamond$. Thus

$$A(x) \Diamond B(x) \quad := \quad \underset{M}{A(x) * B(x)}\ , \quad M = \left(\binom{n}{k}k!\right)_{n,k\ge 0}. \tag{9}$$

The array M is A008279 in the OEIS, described as the triangle of permutation coefficients. The first few rows of M are shown in Table 1. It is not difficult to show the inverse array $M^{-1} = \left((-1)^{n-k}\binom{n}{k}\frac{1}{n!}\right)$. The first few rows of M^{-1} are shown in Table 2.

Table 1. Array $M = \left(\binom{n}{k}k!\right), 0 \le k \le n \le 4$.

	k =0	1	2	3	4
n =0	1				
1	1	1			
2	1	2	2		
3	1	3	6	6	
4	1	4	12	24	24

Table 2. Array $M^{-1} = \left((-1)^{(n-k)}\binom{n}{k}\frac{1}{n!}\right), 0 \le k \le n \le 4$.

	k =0	1	2	3	4
n =0	1				
1	-1	1			
2	$\frac{1}{2}$	-1	$\frac{1}{2}$		
3	-$\frac{1}{6}$	$\frac{1}{2}$	-$\frac{1}{2}$	$\frac{1}{6}$	
4	$\frac{1}{24}$	-$\frac{1}{6}$	$\frac{1}{4}$	-$\frac{1}{6}$	$\frac{1}{24}$

If $a(n)$ is a column vector and $b(n) := Ma(n)$ then it is easy to see from the definition of M that the exponential generating function for the sequence $b(n)$ and the ordinary generating function for the sequence $a(n)$ are related by

$$\sum_{n=0}^{\infty} b(n)\frac{x^n}{n!} \quad = \quad e^x \sum_{n=0}^{\infty} a(n)x^n.$$

This observation leads to the following characterisation of the white diamond product of power series.

Let $A(x), B(x) \in \mathbb{C}[[x]]$. Associate to this pair of power series a pair of sequences $\alpha(n)$ and $\beta(n)$ defined by the equations

$$\begin{aligned}\sum_{n=0}^{\infty} \alpha(n) \frac{x^n}{n!} &= A(x) e^x \\ \sum_{n=0}^{\infty} \beta(n) \frac{x^n}{n!} &= B(x) e^x .\end{aligned}$$

Then the white diamond product $C(x) = A(x) \lozenge B(x)$ of the series $A(x)$ and $B(x)$ is the power series defined by the equation

$$\sum_{n=0}^{\infty} \alpha(n)\beta(n) \frac{x^n}{n!} = C(x) e^x \, . \tag{10}$$

We see from fact F7 that in order to calculate the $\lozenge$ product of two power series we need to know the $\lozenge$ product of monomial polynomials. This is given by the following result.

Proposition 1. The white diamond product of two monomial polynomials is given by

$$x^m \lozenge x^n = \sum_{k=0}^{m} \frac{m!n!}{(n+k)!} \binom{n+k}{k} \binom{n}{m-k} x^{n+k} \, . \tag{11}$$

Proof.

The proof is exactly similar to the proof of the corresponding result for the black diamond product [Ba'18, Proposition 2]. The deformation matrix $M = \left(\binom{n}{k} k!\right)$. Its inverse $M^{-1} = \left((-1)^{n-k} \binom{n}{k} \frac{1}{n!}\right)$. By Fact F6 the action of these matrices on monomial polynomials is given by

$$M x^j = \sum_{i \geq 0} \binom{i}{j} j! x^i, \;\; M^{-1} x^j = \sum_{i \geq 0} (-1)^{i-j} \binom{i}{j} \frac{1}{i!} x^i .$$

Therefore, by the definition D5 of the deformed Hadamard product we have

$$\begin{aligned} x^m \lozenge x^n &= M^{-1} \left(M x^m * M x^n \right) \\ &= M^{-1} \left(\sum_{i \geq 0} \binom{i}{m} \binom{i}{n} m!n! x^i \right) \\ &= \sum_{i \geq 0} \binom{i}{m} \binom{i}{n} m!n! M^{-1} x^i \end{aligned}$$

$$= \sum_{i\geq 0} \binom{i}{m}\binom{i}{n} m!n! \sum_{N\geq 0} (-1)^{N-i}\binom{N}{i}\frac{1}{N!}x^N \tag{12}$$

$$= \sum_{N\geq 0} \frac{m!n!}{N!} \sum_{i=0}^{N} (-1)^{N-i}\binom{N}{i}\binom{i}{m}\binom{i}{n} x^N$$

$$= \sum_{N\geq 0} \frac{m!n!}{N!} s(N) x^N, \tag{13}$$

where, as in [Ba'18, Proposition 2], we define

$$s(N) = \sum_{i=0}^{N} (-1)^{N-i}\binom{N}{i}\binom{i}{m}\binom{i}{n},$$

a sum dependent on the parameters m and n. With the aid of Maple's sumtools package we proved in [Ba'18] that $s(N)$ has the closed-form expression

$$s(N) \quad = \quad \binom{N}{N-n}\binom{n}{m+n-N}. \tag{14}$$

Therefore

$$x^m \Diamond x^n \quad = \quad \sum_{N} \frac{m!n!}{N!}\binom{N}{N-n}\binom{n}{m+n-N} x^N. \tag{15}$$

The coefficient of x^N in the series on the right-hand side of (15) is zero if N lies outside the closed interval $[n, m+n]$. If we write $N = n+k$, (15) becomes

$$x^m \Diamond x^n \quad = \quad \sum_{k=0}^{m} \frac{m!n!}{(n+k)!}\binom{n+k}{k}\binom{n}{m-k} x^{n+k}$$

completing the proof of the proposition. □

EXAMPLES

E1: $x \Diamond x^n = nx^n + x^{n+1}$.

E2: $x^2 \Diamond x^n = 2!\binom{n}{2}x^n + 2 \times 1!\binom{n}{1}x^{n+1} + \binom{n}{0}x^{n+2}$.

E3: $x^3 \Diamond x^n = 3!\binom{n}{3}x^n + 3 \times 2!\binom{n}{2}x^{n+1} + 3 \times 1!\binom{n}{1}x^{n+2} + \binom{n}{0}x^{n+3}$.

It follows from Proposition 1 that if $A(x)$ and $B(x)$ are integral polynomials (resp. integral power series) then the series $A(x) \Diamond B(x)$ is an integral polynomial (resp. integral power series).

We give several examples of well-known sequences of polynomials of combinatorial interest that have simple expressions in terms of the $\Diamond$ product. As a matter of notation, we abbreviate the n-fold product $A(x) \Diamond \cdots \Diamond A(x)$ to $A(x)^{\Diamond n}$ with the convention that $A(x)^{\Diamond 0} = 1$.

EXAMPLES

E4: Using example E1 we calculate succesively

$$\begin{aligned} x \Diamond x &= x + x^2, \quad x \Diamond x \Diamond x = x + 3x^2 + x^3, \\ x \Diamond x \Diamond x \Diamond x &= x + 7x^2 + 6x^3 + x^4. \end{aligned}$$

More generally, the n-fold product

$$x^{\Diamond n} = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k = B_n(x), \tag{16}$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ denotes the Stirling number of the second kind and where $B_n(x)$ denotes the nth Bell (or exponential) polynomial. The Bell polynomials are the row polynomials of the triangular array of Stirling numbers of the second kind A048993. The proof of (16) is by a straightforward induction argument, making use of Example E1 and the recurrence equation

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\}$$

satisfied by the Stirling numbers of the second kind.

E5: There is an inverse relation to Example E4 involving the Stirling numbers of the first kind $s(n,k)$ (see A008275): there holds

$$\begin{aligned} x^n &= x \Diamond (x-1) \Diamond (x-2) \Diamond \cdots \Diamond (x-n+1) \\ &= \sum_{k=1}^{n} s(n,k) x^{\Diamond k}. \end{aligned} \tag{17}$$

A simple inductive proof of this identity can be given using Example E1 and the recurrence for the Stirling numbers of the first kind

$$s(n+1,k) = s(n,k-1) - ns(n,k).$$

E6: We also note the following shifted version of Example E4, again easily proved by induction:

$$\begin{aligned} (1+x)^{\Diamond n} &= \sum_{k=0}^{n} \left\{ \begin{array}{c} n+1 \\ k+1 \end{array} \right\} x^k \\ &= \frac{B_{n+1}(x)}{x}. \end{aligned} \tag{18}$$

It follows from (16) and (18) that the white diamond binomial theorem

$$(1+x)^{\diamondsuit n} \quad = \quad \sum_{k=0}^{n} \binom{n}{k} x^{\diamondsuit k}$$

is equivalent to the well-known recurrence for the Bell polynomials

$$B_{n+1}(x) \quad = \quad x \sum_{k=0}^{n} \binom{n}{k} B_k(x).$$

E7: For a natural number r, the r-Stirling number $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r$ is defined as the number of set partitions of $\{1, 2, ..., n\}$ into k blocks subject to the restriction that the numbers $1, 2, ..., r$ belong to different blocks. In particular, $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_0 = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_1 = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$. The r-Stirling numbers of the second kind satisfy the same recurrence equation as the Stirling numbers of the second kind (but with different boundary conditions).

An induction argument using Example E1 leads to the expression

$$(r+x)^{\diamondsuit n} \quad = \quad \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r x^k \, , \tag{19}$$

The polynomial on the right side of (19) is the nth row polynomial of the triangle of r-Stirling numbers (but with a factor of x^r removed). For cases see A143494, A143495, A143496 and A193685.

2.1 Idempotent expansions. According to Fact F9, the power series $E_i(x) = M^{-1}x^i, i = 0, 1, 2, ...$ are a complete set of mutually orthogonal idempotents in the algebra of power series equipped with the white diamond product. By Fact F6, the orthogonal idempotent $E_i(x)$ is the generating function of the i-th column vector of the array $M^{-1} = \left((-1)^{n-k}\binom{n}{k}\frac{1}{n!}\right)$. A simple calculation gives

$$\begin{aligned} E_i(x) \quad &= \quad M^{-1}x^i \\ &= \quad \frac{x^i}{i!}e^{-x} \, . \end{aligned} \tag{20}$$

The multiplicative identity element of the algebra is the constant power series 1, with the idempotent expansion

$$1 \quad = \quad \sum_{i=0}^{\infty} E_i(x).$$

It is easily seen that

$$x = \sum_{i=1}^{\infty} iE_i(x)$$

and hence for constants r and s we have the idempotent expansion

$$r + sx = \sum_{i=0}^{\infty} (r + si)E_i(x). \tag{21}$$

It follows that the n-fold product

$$\begin{aligned} (r + sx)^{\Diamond n} &= \sum_{i=0}^{\infty} (r + si)^n E_i(x) \\ &= e^{-x} \sum_{i=0}^{\infty} (r + si)^n \frac{x^i}{i!}\,. \end{aligned} \tag{22}$$

Using (16), the case $r = 0, s = 1$ of (22) reads

$$B_n(x) = e^{-x} \sum_{i=0}^{n} i^n \frac{x^i}{i!}\,.$$

This is the well-known Dobinski formula for the Bell polynomials. Thus (22) can be viewed as a generalised Dobinski formula.

In Table 3 below we list arrays in the OEIS whose row polynomials are of the form $(r + sx)^{\Diamond n}$ for particular values of the constants r and s (modulo some differences of offset to those used in the OEIS). The proofs are by induction using Example E1 and the known recurrences for the elements of the various arrays listed. In Table 4 we list several other triangular arrays in the OEIS whose row polynomials have simple expressions in terms of the white diamond product.

Table 3. White diamond polynomials $(r+sx)^{\diamondsuit n}$ in the OEIS

Array	Row polynomials	In terms of Bell polynomials	Idempotent expansion (Dobinski-type formula)
A048993	$x^{\diamondsuit n}$	$B_n(x)$	$e^{-x}\sum_{i=0}^{\infty} i^n \frac{x^i}{i!}$
A008277	$(1+x)^{\diamondsuit n}$	$\frac{B_{n+1}(x)}{x}$	$e^{-x}\sum_{i=0}^{\infty}(1+i)^n \frac{x^i}{i!}$
A143494	$(2+x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}2^{(n-k)}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(2+i)^n \frac{x^i}{i!}$
A143495	$(3+x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}3^{(n-k)}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(3+i)^n \frac{x^i}{i!}$
A143496	$(4+x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}4^{(n-k)}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(4+i)^n \frac{x^i}{i!}$
A193685	$(5+x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}5^{(n-k)}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(5+i)^n \frac{x^i}{i!}$
A154537	$(1+2x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}2^{k}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(2i+1)^n \frac{x^i}{i!}$
A282629	$(1+3x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}3^{k}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(3i+1)^n \frac{x^i}{i!}$
A225466	$(2+3x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}2^{(n-k)}3^{k}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(3i+2)^n \frac{x^i}{i!}$
A285061	$(1+4x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}4^{k}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(4i+1)^n \frac{x^i}{i!}$
A225467	$(3+4x)^{\diamondsuit n}$	$\sum_{k=0}^{n}\binom{n}{k}3^{n-k}4^{k}B_k(x)$	$e^{-x}\sum_{i=0}^{\infty}(4i+3)^n \frac{x^i}{i!}$

Table 4. Other white diamond polynomials in the OEIS

Array	Row polynomials	Description
A105278	$x \lozenge (x+1) \lozenge (x+2) \lozenge \cdots \lozenge (x+n)$	Lah polynomial
A023531	$x \lozenge (x-1) \lozenge (x-2) \lozenge \cdots \lozenge (x-n)$	Monomial x^n
A035342	$x \lozenge (x+2) \lozenge (x+4) \lozenge \cdots \lozenge (x+2n)$	
A122850	$x \lozenge (x-2) \lozenge (x-4) \lozenge \cdots \lozenge (x-2n)$	Shifted reverse Bessel polynomials $(-1)^n x\theta_n(-x)$
A035469	$x \lozenge (x+3) \lozenge (x+6) \lozenge \cdots \lozenge (x+3n)$	
A004747 (signed)	$x \lozenge (x-3) \lozenge (x-6) \lozenge \cdots \lozenge (x-3n)$	
A049029	$x \lozenge (x+4) \lozenge (x+8) \lozenge \cdots \lozenge (x+4n)$	
A021009 (signed)	$(x+1) \lozenge (x+2) \lozenge \cdots \lozenge (x+n)$	Laguerre polynomial $n!L_n(-x)$
A094587 (signed)	$(x-1) \lozenge (x-2) \lozenge \cdots \lozenge (x-n)$	
A265649	$(x+1) \lozenge (x+3) \lozenge \cdots \lozenge (x+2n-1)$	
A122850	$(x-1) \lozenge (x-3) \lozenge \cdots \lozenge (x-2n+1)$	Reverse Bessel polynomial $(-1)^n\theta_n(-x)$
A048993	$x \lozenge x \lozenge \cdots \lozenge x$ (n factors)	Bell polynomial or Stirling polynomial for n copies of the complete graph K_1
A078739	$x^2 \lozenge x^2 \lozenge \cdots \lozenge x^2$ (n factors)	Stirling polynomial for n copies of the complete graph K_2
A078741	$x^3 \lozenge x^3 \lozenge \cdots \lozenge x^3$ (n factors)	Stirling polynomial for n copies of the complete graph K_3
A090214	$x^4 \lozenge x^4 \lozenge \cdots \lozenge x^4$ (n factors)	Stirling polynomial for n copies of the complete graph K_4

3 RELATED M-HADAMARD PRODUCTS OF POWER SERIES

We briefly consider two other M-Hadamard products related to the white diamond product.

3.1. Type B white diamond product

Let now $M = \left(\binom{n}{k}k!2^k\right)$. We denote the associated M-Hadamard product of power series by the symbol $\triangledown$: thus

$$A(x) \triangledown B(x) \quad := \quad \underset{M}{A(x) * B(x)}\ , \quad M = \left(\binom{n}{k}k!2^k\right)_{n,k\ge 0}. \tag{23}$$

Adapting the proof of Proposition 1 we find the $\triangledown$ product of a pair of monomials is given by

$$x^m \triangledown x^n \quad = \quad \sum_{k=0}^{m} \frac{m!n!}{(n+k)!} 2^{m-k} \binom{n+k}{m} \binom{m}{k} x^{n+k}. \tag{24}$$

One particular case is

$$x^n \triangledown x^n \quad = \quad x^n R_n(x)$$

where $R_n(x) = \sum_{k=0}^{n} 2^{n-k} \binom{n}{k} \frac{n!}{k!} x^k, n = 0, 1, 2, ...$ are the row polynomials of the array A286724.

Setting $m = 1$ in (24) gives

$$x \triangledown x^n \quad = \quad 2nx^n + x^{n+1}. \tag{25}$$

Using this relation, a simple induction argument shows that the n-fold product $(1 + x) \triangledown ... \triangledown (1 + x)$ is the nth row polynomial of A039755, the triangle of B-analogues of Stirling numbers of the second kind. Comparing this result with (18), it seems reasonable to refer to the $\triangledown$ product as the type B analogue of the white diamond product $\lozenge$.

In this case, the mutually orthogonal idempotent power series $E_i(x)$ are given by

$$\begin{aligned} E_i(x) &= M^{-1} x^i \\ &= \frac{x^i}{2^i i!} e^{-\frac{x}{2}} \qquad i = 0, 1, 2, ... \end{aligned} \tag{26}$$

Then the n-fold product $(r + sx)^{\triangledown n}$ has the idempotent expansion

$$\begin{aligned} (r + sx)^{\triangledown n} &= \sum_{i=0}^{\infty} (r + 2si)^n E_i(x) \\ &= e^{-\frac{x}{2}} \sum_{i=0}^{\infty} (r + 2si)^n \frac{x^i}{2^i i!}\,. \end{aligned} \tag{27}$$

The row polynomials of several triangular arrays in the OEIS have simple expressions as $\triangledown$ products. We give some examples in Table 5.

Table 5.

Array	Row polynomial	Description
A039755	$\underbrace{(1+x)\triangledown(1+x)\triangledown\ldots\triangledown(1+x)}_{n\text{factors}}$	Type B Stirling numbers of the second kind
A075497	$\underbrace{x\triangledown\ldots\triangledown x}_{n\text{factors}}$	Stirling numbers of the second kind scaled with powers of 2
A075497 (offset 0)	$\underbrace{(2+x)\triangledown(2+x)\triangledown\ldots\triangledown(2+x)}_{n\text{factors}}$	Stirling numbers of the second kind scaled with powers of 2
A046089	$\underbrace{x\triangledown(x+1)\triangledown(x+2)\triangledown\ldots\triangledown(x+n)}_{n\text{factors}}$	Exponential Riordan array$[f(x),\int_0^x f(t)dt]$, $f(x)=1/(1-x)^3$
A079621	$\underbrace{x\triangledown(x+2)\triangledown(x+4)\triangledown\ldots\triangledown(x+2n)}_{n\text{factors}}$	Exponential Riordan array$[f(x),\int_0^x f(t)dt]$, $f(x)=1/(1-2x)$
A176230	$\underbrace{(x+1)\triangledown(x+3)\triangledown\ldots\triangledown(x+2n-1)}_{n\text{factors}}$	Exponential Riordan array $\left[\frac{1}{\sqrt{1-2x}},\frac{x}{1-2x}\right]$.

3.2 q-analogue of the white diamond product

Consider the M-Hadamard product where the deformation matrix is defined as $M=\left(\binom{n}{k}_q[k]_q!\right)$. Here $\binom{n}{k}_q$ is the q-binomial coefficient and $[k]_q!$ is the q-factorial. Denote the resulting multiplication operator on power series by $\Diamond_q$. We can view this operator as a q-analogue of the white diamond product and investigate q-analogues of the results of Section 2. As an example, the q-analogue of (16) is

$$\underbrace{x\Diamond_q\ldots\Diamond_q x}_{n\text{factors}} = \sum_{k=0}^{n}\left\{\begin{matrix} n \\ k \end{matrix}\right\}_q x^k \qquad n=0,1,2,\ldots,$$

where $\left\{\begin{matrix} n \\ k \end{matrix}\right\}_q$ is a q-Stirling number of the second kind.

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