Notes on A105794

Peter Bala, July 2013

A105794 is a lower unit triangular array defined as the inverse of the triangle A105793, an array of generalized Stirling numbers of the first kind. We thus expect A105794 to generalize in some way the Stirling numbers of the second kind. We shall see in a moment in what sense this is the case.

Generating function

The exponential generating function for A105793 is

$$(1+t)^{(1+x)} = (1+t)\exp(x\log(1+t)) \tag{1}$$

Hence A105793 is a Sheffer triangle, that is to say, the rows of the triangle are the coefficients of a sequence of Sheffer polynomials for the pair $(\exp(-t)$, $\exp(t)-1)$ [2, Theorem 2.3.4]. It follows from [2, Theorem 3.5.5] that the inverse array, A105794, is also of Sheffer type, with the bivariate exponential generating function

$$\begin{aligned} A(x,t) &= \exp(-t)\exp(x(\exp(t)-1)) \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} T(n,k)x^k\right)\frac{t^n}{n!}, \end{aligned} \tag{2}$$

as given by Copeland. The generating function is a particular case of the generating function of the Actuarial polynomials [2, Chapter 4, Section 3.4, p.123].

Explicit expression for T(n,k)

If we differentiate (2) with respect to t we find

$$\frac{\partial A(x,t)}{\partial t} + A(x,t) = x\exp(x(\exp(t)-1)). \tag{3}$$

Now the function $\exp(x(\exp(t)-1))$ appearing on the rhs is the generating function for $S(n,k)$, the Stirling numbers of the second kind. Thus equating the coefficients of t^n on both sides of (3) yields the relation

$$T(n+1,k) + T(n,k) = S(n,k-1)$$

from which we get the explicit formula

$$T(n,k) = \sum_{i=0}^{n-1}(-1)^i S(n-i-1,k-1). \tag{4}$$

Combinatorial interpretation

For a graph G and a positive integer k, the graphical Stirling number of the second kind, $S(G;k)$, is the number of partitions of the vertex set of G into k non-empty independent sets (an independent set in G is a subset of the vertices, no two elements of which are adjacent). It is shown in [1, Section 3] that for $n \geq 3$, the rhs of (4) equals the graphical Stirling number $S(C_n;k)$ of the n-cycle graph C_n. This provides a combinatorial interpretation for the entries of A105794 (apart from the first two columns).

Column generating functions

The ordinary generating function for the k-th column of the Stirling numbers of the second kind is

$$\sum_{k=0}^{n} S(n,k)x^k = \frac{x^k}{(1-x)(1-2x)...(1-kx)}.$$

It follows from (4) that the ordinary generating function for the k-th column of A105794 is

$$\sum_{n=0}^{\infty} T(n,k)x^k = \frac{1}{(1+x)}\frac{x^k}{(1-x)(1-2x)...(1-(k-1)x)}. \tag{5}$$

Recurrence equation

It is easy to verify that the generating function $A(x,t)$ for A105794 given in (2) satisifes the partial differential equation

$$x\frac{\partial A(x,t)}{\partial x} - \frac{\partial A(x,t)}{\partial t} = (1-x)A(x,t). \tag{6}$$

Equating coefficients of $x^k t^n$ on both sides leads to the recurrence equation

$$T(n+1,k) = (k-1)T(n,k) + T(n,k-1). \tag{7}$$

REFERENCES

[1] D. Galvin and D. T. Thanh *Stirling numbers of forests and cycles*, Electr. J. Comb., Vol. 20(1): P73 (2013)

[2] S. Roman *The umbral calculus*, Pure and Applied Mathematics 111, Academic Press Inc., New York, 1984. Reprinted by Dover in 2005.