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from another curve C, is called *a parallel* to C. A curve and any of its parallels have the same normals and the same evolute. The parallel curves of a non-circular ellipse are curves of the eighth degree. See, *e.g.*, Ex. 3, art. 372, of Salmon's *Conic Sections*. R. C. Yates has devised a linkage for describing curves parallel to an ellipse. See this MONTHLY [1938, 607].

Skew Ordered Sequences

E 754 [1947, 39 and 1947, 163.] *Proposed by S. T. Thompson, Tacoma, Wash.*

A finite sequence of positive integers will be said to be *skew ordered* if either each integer in an even position of the sequence is greater than or each such integer is less than its immediate neighbors. If the eight integers $1, \cdots, 8$ are placed in random order in a sequence, what is the probability that the sequence will be skew ordered?

I. *Solution by J. B. Kelly, Hampton, Va.* Let Q_n be the number of skew ordered arrangements of n distinct integers. There is no loss in generality in supposing that the integers are $1, 2, \cdots, n$. Suppose that the integer n occurs in the kth position. If we have a skew ordered arrangement, the sequences on either side of n must be skew ordered. The sequence to the left of n must have its last element less than its next to last element. Once the integers in this sequence are chosen, its elements may be arranged in $\frac{1}{2}Q_{k-1}$ different ways so as to satisfy this condition. The sequence to the right of n must have its first element less than its second element. Once the integers in this sequence are chosen, its elements may be arranged in $\frac{1}{2}Q_{n-k}$ different ways so as to satisfy this condition. There are $\binom{n-1}{k-1}$ ways of choosing the integers in the sequence to the left of n and once these integers are chosen, the integers in the sequence to the right of n are determined. Thus the number of skew ordered permutations of n distinct integers for which the greatest integer (here n) occurs in the kth position is $\frac{1}{4}\binom{n-1}{k-1}Q_{k-1}Q_{n-k}$. It follows that

$$Q_n = \frac{1}{4}\sum_{k=1}^{n}\binom{n-1}{k-1}Q_{k-1}Q_{n-k}. \tag{1}$$

twice Euler nos

In applying this formula, it is necessary to make the convention that $Q_0 = Q_1 = 2$. Let P_n be the probability that a given permutation of n distinct integers will be skew ordered. Evidently $P_n = Q_n/n!$, and relation (1) becomes

$$P_n = \frac{1}{4n}\sum_{k=1}^{n} P_{k-1}P_{n-k}, \tag{2}$$

where again we make the convention that $P_0 = P_1 = 2$. Calculating P_8 by successively calculating $P_2, P_3, \cdots, P_7$ by means of (2) we obtain $P_8 = 277/4032$.

II. *Solution by Frederick Mosteller, Harvard University.* The probability that a finite sequence of n unequal integers will be skew ordered when all permutations are equally likely can be translated directly into statistical terminology. It is the probability that the sequence will have no *run up* or *run down* of length greater than or equal to 2. This problem was treated by P. S. Olmstead, *Distribution of sample arrangements for runs up and down*. Annals of Mathematical Statistics, vol. 17 (1946), pp. 24–33.

For $n=3$ to 14 we derive from Olmstead's Table 2

n	Probability of skew ordering
3	0.66666667
4	0.41666667
5	0.26666667
6	0.16944444
7	0.10793651
8	0.06870040
9	0.04373898
10	0.02784447
11	0.01772647
12	0.01128499
13	0.00718426
14	0.00457364

Since for n even the number of arrangements with runs of length no more than one is just twice Euler's number E_n, and for n odd the number of arrangements is twice the tangent number, we might use the approximation

$$4\left(\frac{2}{\pi}\right)^{n+1}.$$

(See, *e.g.*, Milne-Thompson, *Calculus of Finite Differences*, p. 147.)

For $n=4$ this approximation agrees to 0.002, for $n=6$ to 0.0001, for $n=8$ to 0.000003, and for $n=9$ to 0.000001. The exact answer for $n=8$ is 277/4032.