

# A307426 Examples

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## Abstract

Examples of Canonical Polygons with 4-fold rotational symmetry.

## 1 Introduction

### 1.1 Canonical polygon

From the description of A052436 the following rules must be true for a canonical polygon:

1. The vertices lie on a point lattice.
2. The edges consist of straight (vertical or horizontal) steps of unit length or diagonal steps (at angles which are multiples of 45 degrees with respect to the lattice axes) of length  $\sqrt{2}$ .
3. No two consecutive steps may be taken in the same direction.
4. No edge intersections are allowed.
5. No point may be a vertex of two edges.
6. It must be closed.

The number of distinct canonical polygons with  $n=3, 4, \dots, 18$  sides are 1, 3, 3, 9, 13, 52, 140, 501, 1763, 6785, 25571, 99907, 392230, 1564989, 6297892, 25601641, ... .

In the following we will use CP as a shorthand for Canonical Polygon or just call it "polygon".

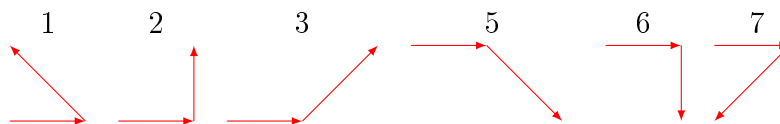
## 1.2 History

The concept Canonical Polygon was (as far as I know) invented by Roland Kyrmse and others in 1977 at the Universidade Federal de Santa Catarina, Florianópolis / SC / Brasil.

## 2 Properties of Canonical Polygons

### 2.1 The interior angles

The six different angles that are possible between consecutive edges are numbered as follows:



The angle is 45 degrees times numbers 1-7. The number 4 is not used because of rule 3, and not the number 0 because of rule 5.

### 2.2 Sum of the interior angles

The sum of the interior angles is  $180 * (n - 1)$  degrees, or  $4 * (n - 1)$  in 45 degree units.

### 2.3 Restrictions on number of straight and diagonal edges

Straight edges (those parallel to one of the axes) are  $(1, 0)$ ,  $(-1, 0)$ ,  $(0, 1)$  and  $(0, -1)$ . Call the number of such edges  $x_+$ ,  $x_-$ ,  $y_+$ ,  $y_-$  respectively. Similarly for the diagonal edges  $(1, -1)$ ,  $(1, 1)$ ,  $(-1, -1)$  and  $(-1, 1)$  and call the number of such edges  $d_{ne}$ ,  $d_{se}$ ,  $d_{nw}$ ,  $d_{sw}$  respectively.

In order for the figure to be closed, the contribution from all the edges must add to  $(0, 0)$ . This means

$$\text{In the } x \text{ direction } x_+ - x_- + d_{ne} + d_{se} - d_{nw} - d_{sw} = 0$$

$$\text{In the } y \text{ direction } y_+ - y_- - d_{ne} + d_{se} - d_{nw} + d_{sw} = 0$$

Consider the total number of straight edges:  $x_+ + x_- + y_+ + y_-$ .

Substitute  $x_-$  and  $y_-$ :

$$\begin{aligned}
&= x_+ + (x_+ + d_{ne} + d_{se} - d_{nw} - d_{sw}) + y_+ + (y_+ - d_{ne} + d_{se} - d_{nw} + d_{sw}) \\
&= 2x_+ + 2y_+ + 2d_{se} - 2d_{nw}
\end{aligned}$$

which is even, so the number of straight edges is always even.


It follows from this:

1. The number of straight edges is even for all  $n$ .
2. For odd  $n$  the number of diagonal edges is also odd, and vice versa.
3. Canonical polygons with all straight edges are only possible for even  $n$ .


### 3 Representing Canonical Polygons

#### 3.1 Angles plus start direction

A natural way to represent a polygon is the sequence of angles in the mathematical positive direction (counter-clockwise).



For example, the triangle  is represented by "211".

However, the drawing must start with straight edge (the edge in East direction).

Starting with a diagonal edge to the NE will result in  which is not a proper CP.

To mark this difference, a 0 for straight and a 1 for diagonal starting edge is appended. In the first case we then have "211.0" and in the second "211.1".

Sometimes both start directions are allowed

as in "2222.0":  and "2222.1": . To make it easier for the human reader, the angles are displayed in groups of three, for example "2.222.0".

#### 3.2 Uniqueness

The above representation is not unique. For example, the triangle can be represented by "211.0", "121.1" or "112.0".

Therefore the numerically largest of the of the possible values will be used. This means cyclically shifting the angles (equivalent to rotation), and reversing the order (equivalent to mirroring), all the time keeping track of the initial direction.

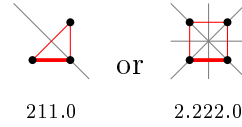
This representation then uniquely describes a CP and thus it can be used as a key to detect duplicates.

### 3.3 Drawing

For clarity, the starting edge is drawn wider than the other edges. If the key is drawn it is placed below the CP. For example



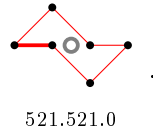
Reflection symmetries axes are shown as gray lines:



Rotational symmetries are drawn with a filled circle for 4-fold



or an unfilled circle for 2-fold



## 4 Canonical Polygons with 4-fold rotation symmetry

Polygons with this property must have the number of edges divisible by 4.

The sequence begins 2, 2, 7, 22, 79, 289, 1145, 4655, 19314, 81242, 345785, 1484065, 6414999, 27895652, ... with *offset* = 2 and  $a(n)$  is the number of polygons with  $4n$  edges.

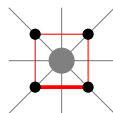
The identifying key is shown for  $n \leq 12$ .

For each  $n$ , the polygons with smallest and largest area are shown separately followed by the remaining ones. The area values given must be divided by 2 to give the true area for a lattice with 1 unit straight edges.

## 5 Polygons with 4-fold rotation symmetry

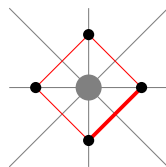
5.1 For  $n = 4$  there are 2 polygons

5.1.1 With smallest area (2) there is 1 polygon



2.222.0

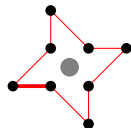
5.1.2 With largest area (4) there is 1 polygon



2.222.1

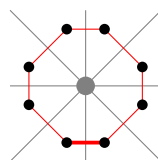
5.2 For  $n = 8$  there are 2 polygons

5.2.1 With smallest area (6) there is 1 polygon



51.515.151.0

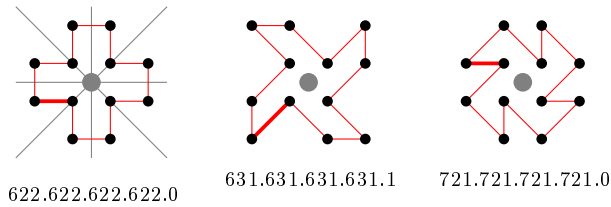
5.2.2 With largest area (14) there is 1 polygon



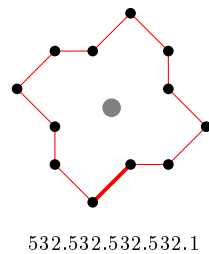
33.333.333.0

### 5.3 For $n = 12$ there are 7 polygons

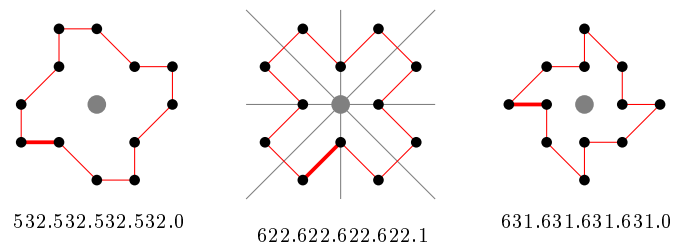
#### 5.3.1 With smallest area (10) there are 3 polygons



#### 5.3.2 With largest area (26) there is 1 polygon

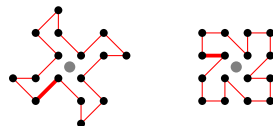


#### 5.3.3 For other area sizes there are 3 polygons

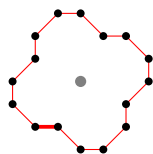


### 5.4 For $n = 16$ there are 22 polygons

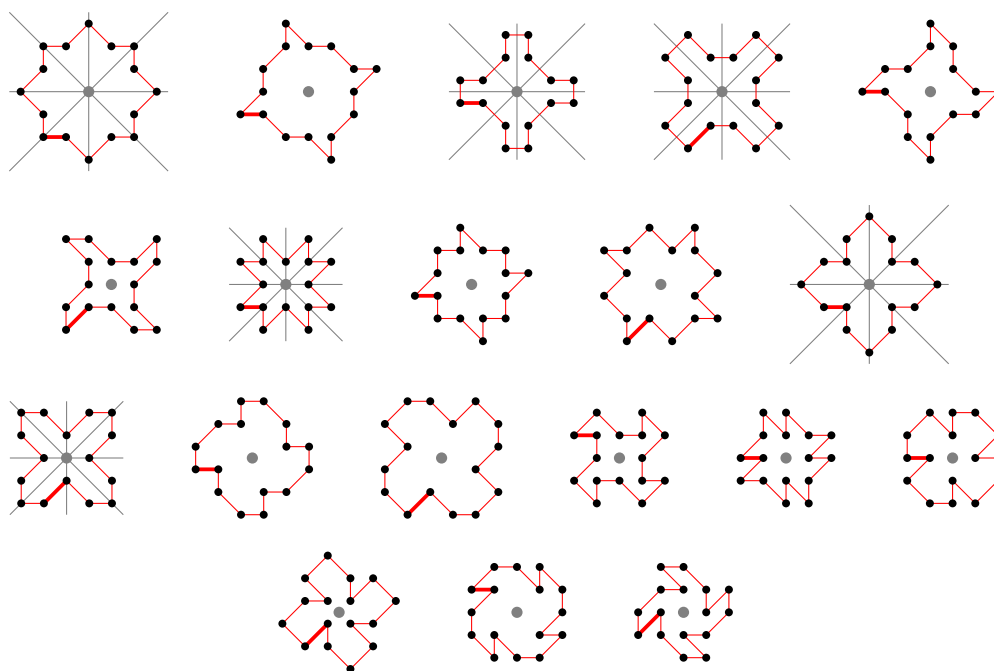
#### 5.4.1 With smallest area (14) there are 2 polygons



5.4.2 With largest area (48) there is 1 polygon

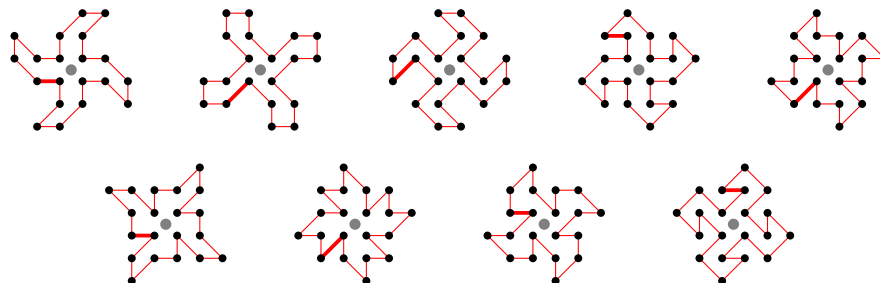


5.4.3 For other area sizes there are 19 polygons

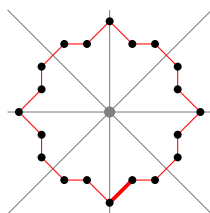


5.5 For  $n = 20$  there are 79 polygons

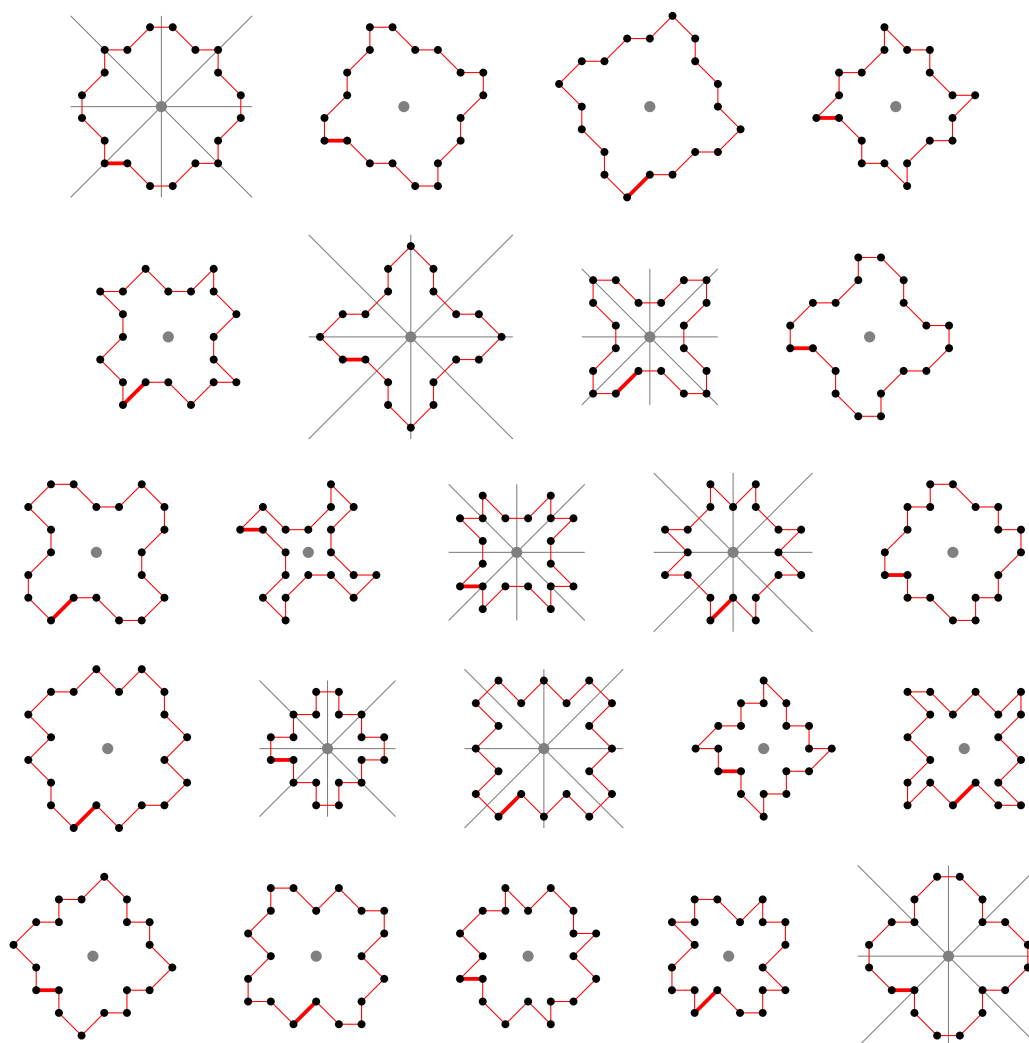
5.5.1 With smallest area (18) there are 9 polygons



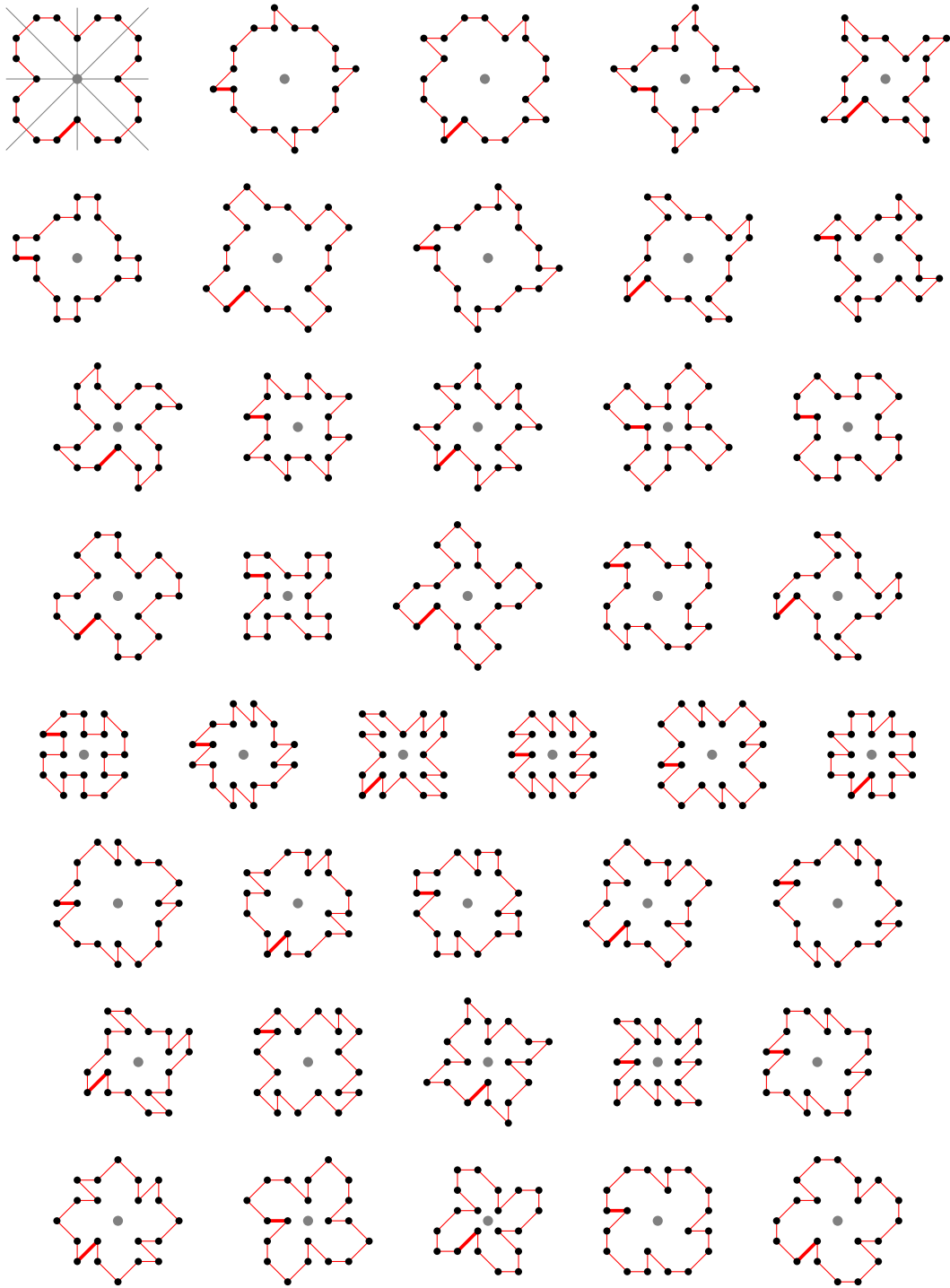
5.5.2 With largest area (76) there is 1 polygon

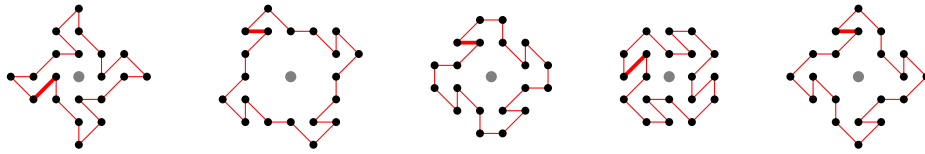


5.5.3 For other area sizes there are 69 polygons



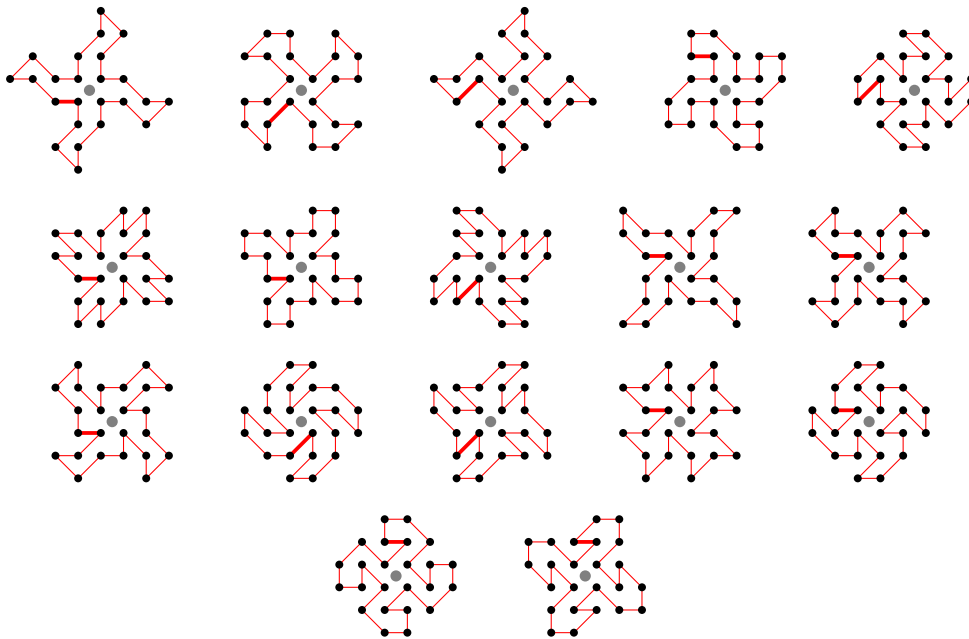




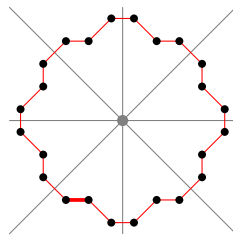


**5.6 For  $n = 24$  there are 289 polygons**

**5.6.1 With smallest area (22) there are 17 polygons**

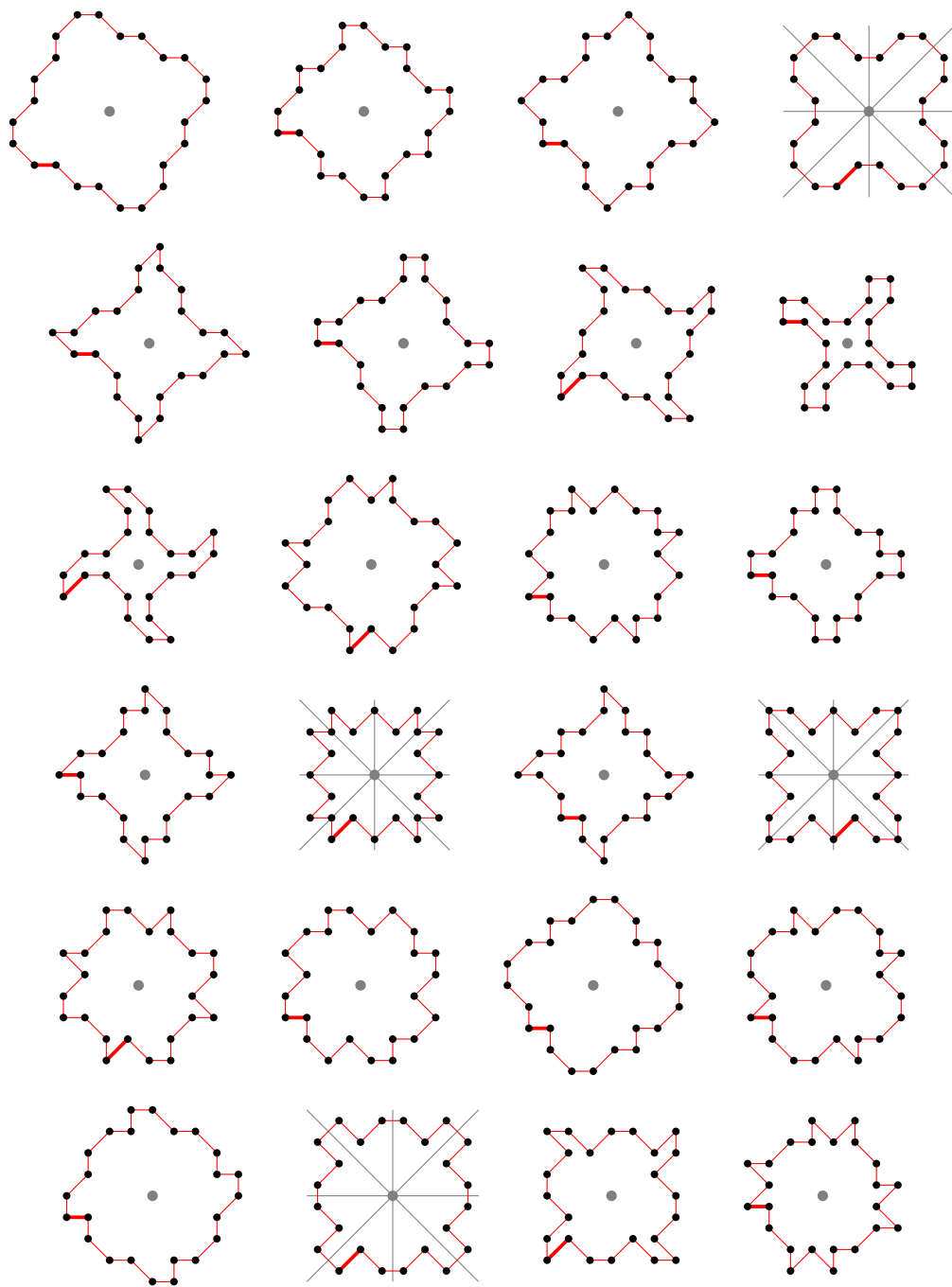


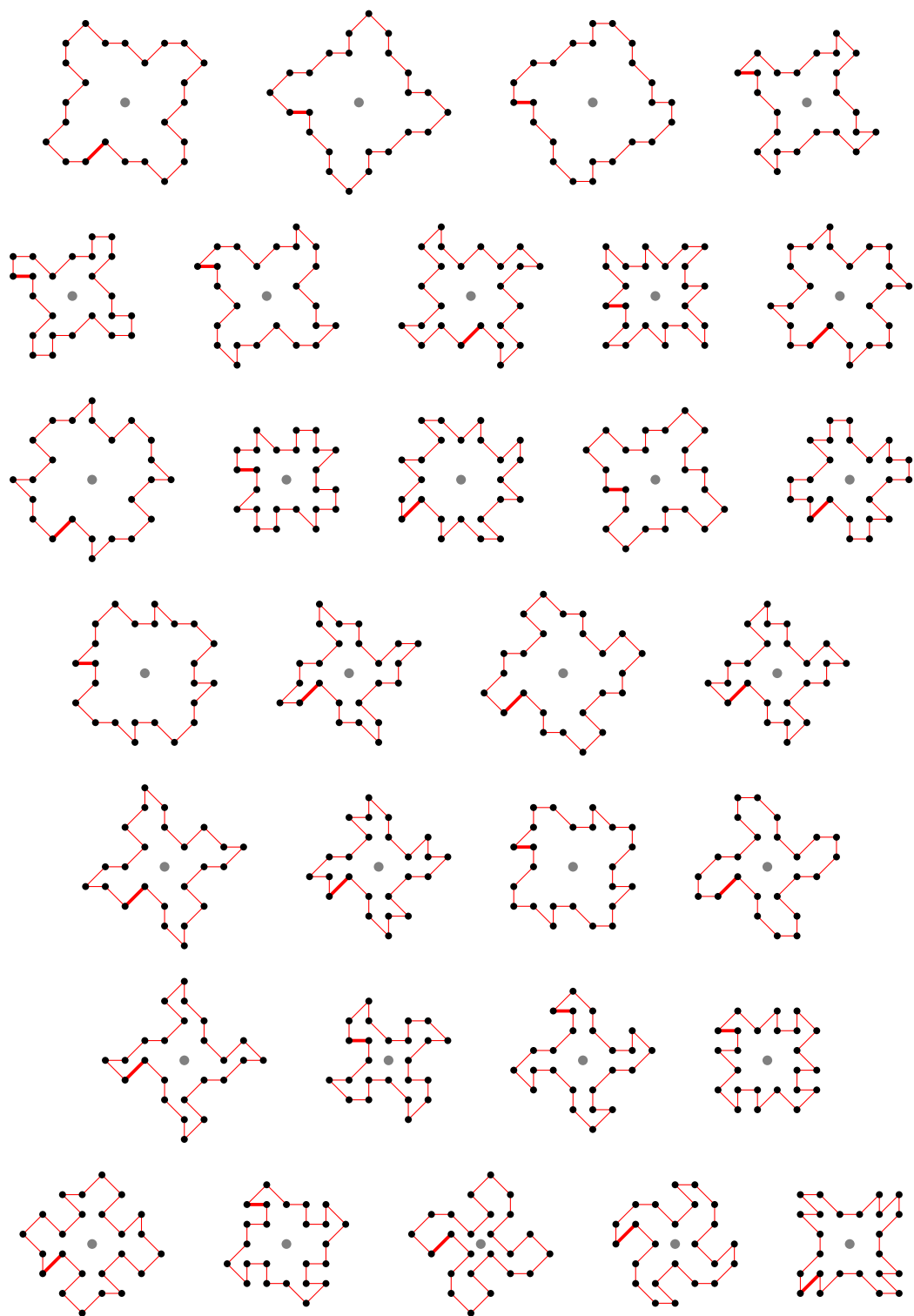
**5.6.2 With largest area (110) there is 1 polygon**

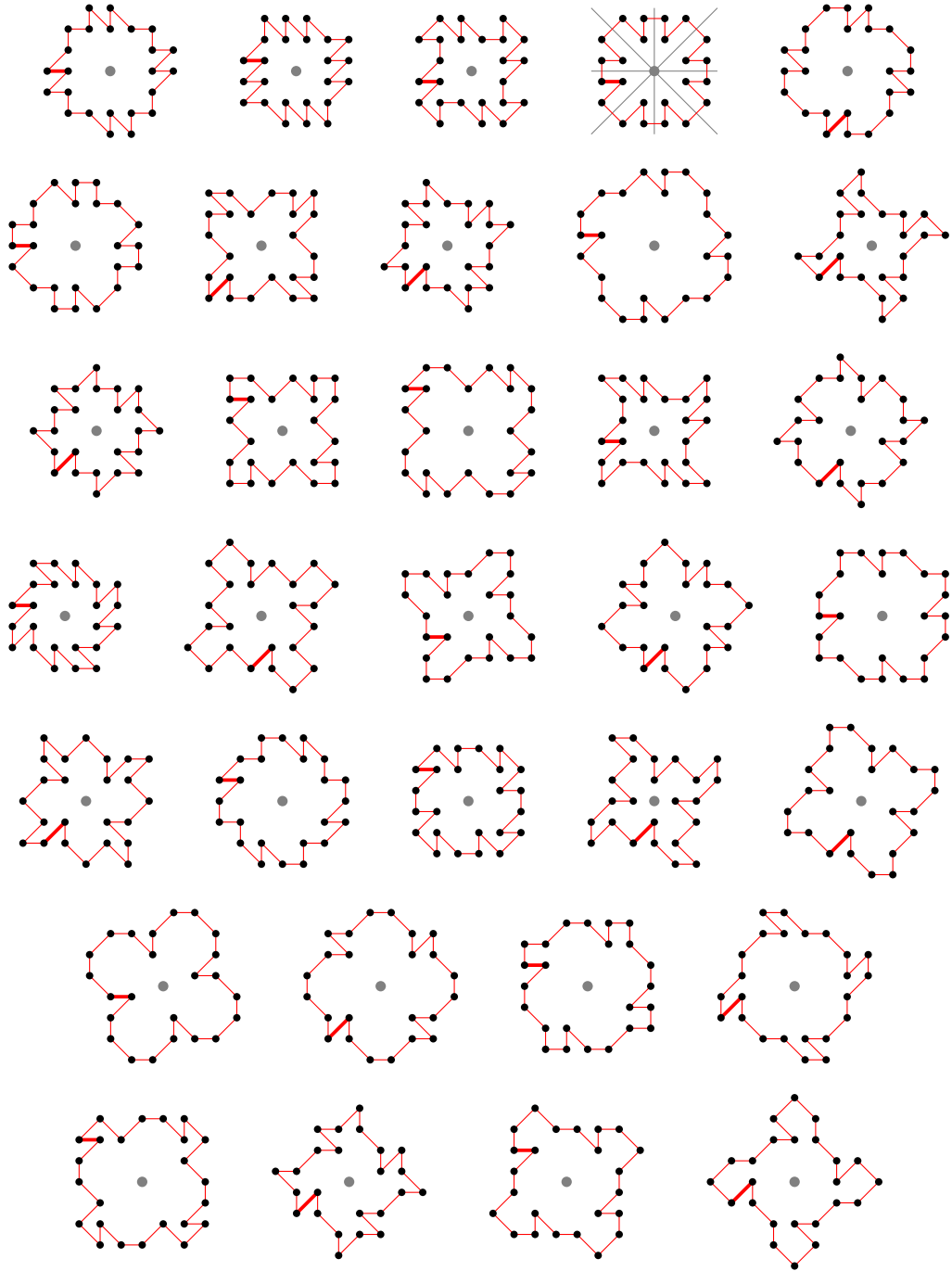


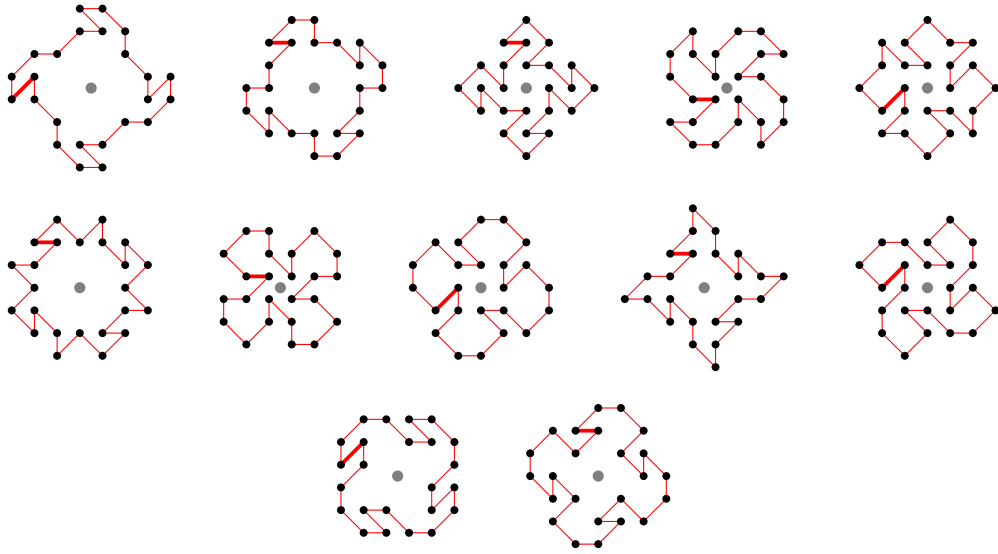
**5.6.3 For other area sizes there are 271 polygons**

(100 samples)



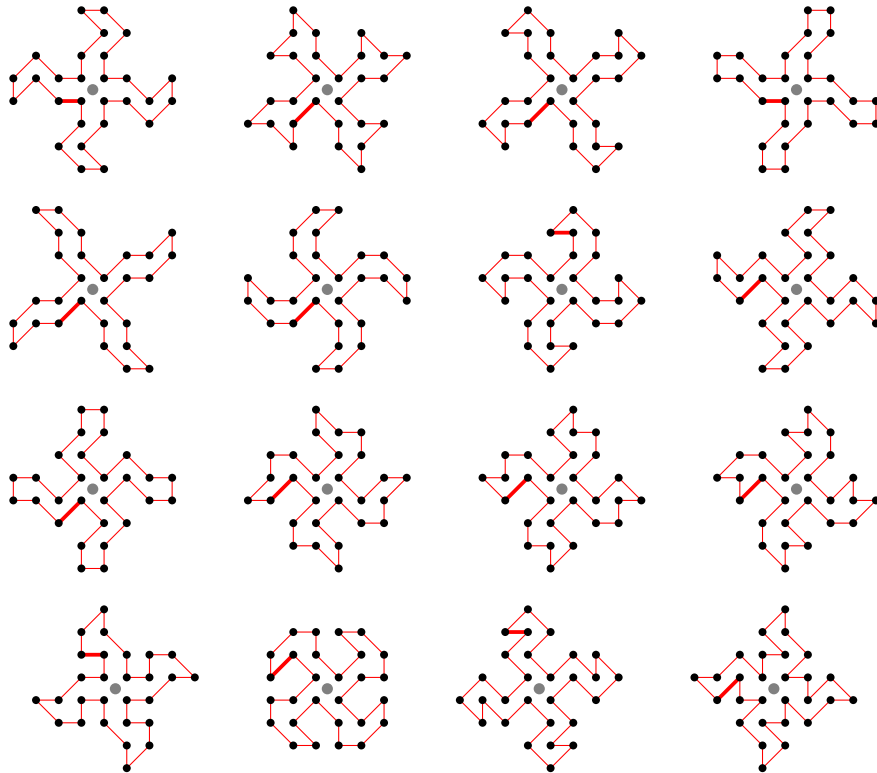


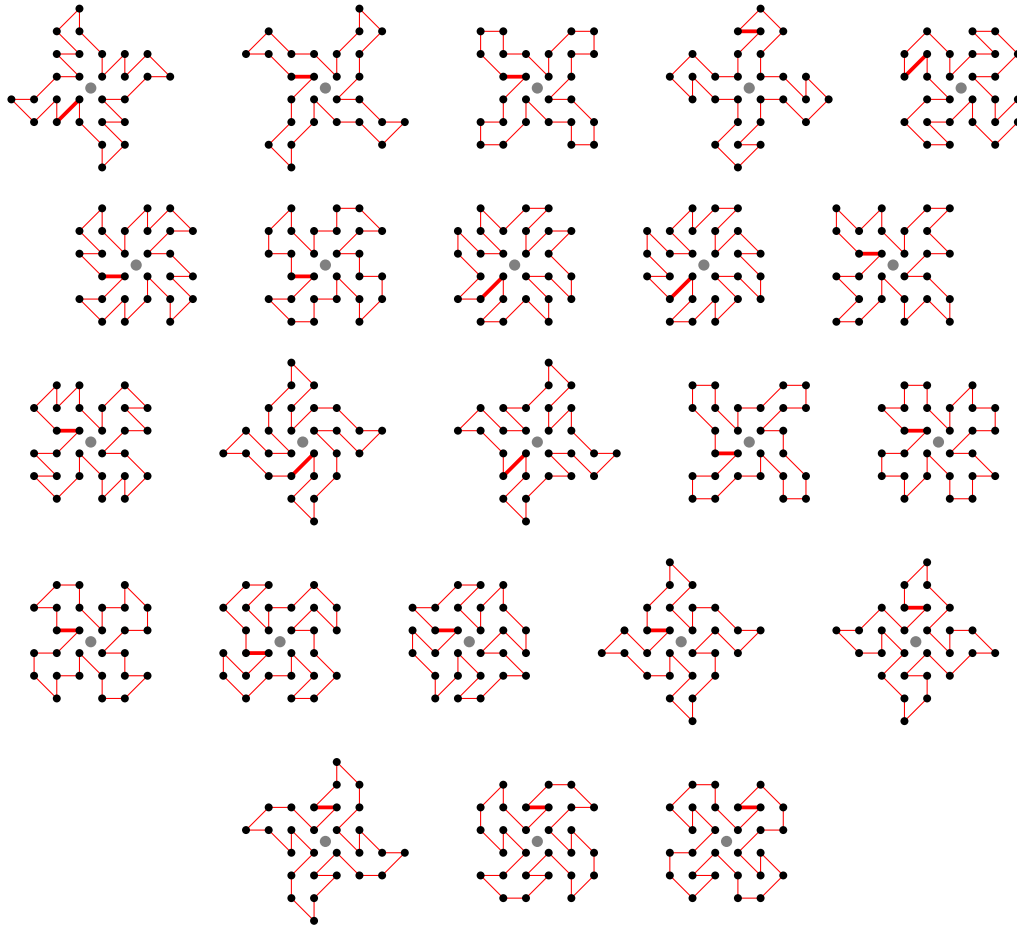




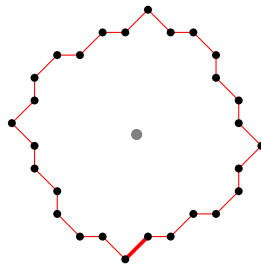
5.7 For  $n = 28$  there are 1145 polygons

5.7.1 With smallest area (26) there are 39 polygons



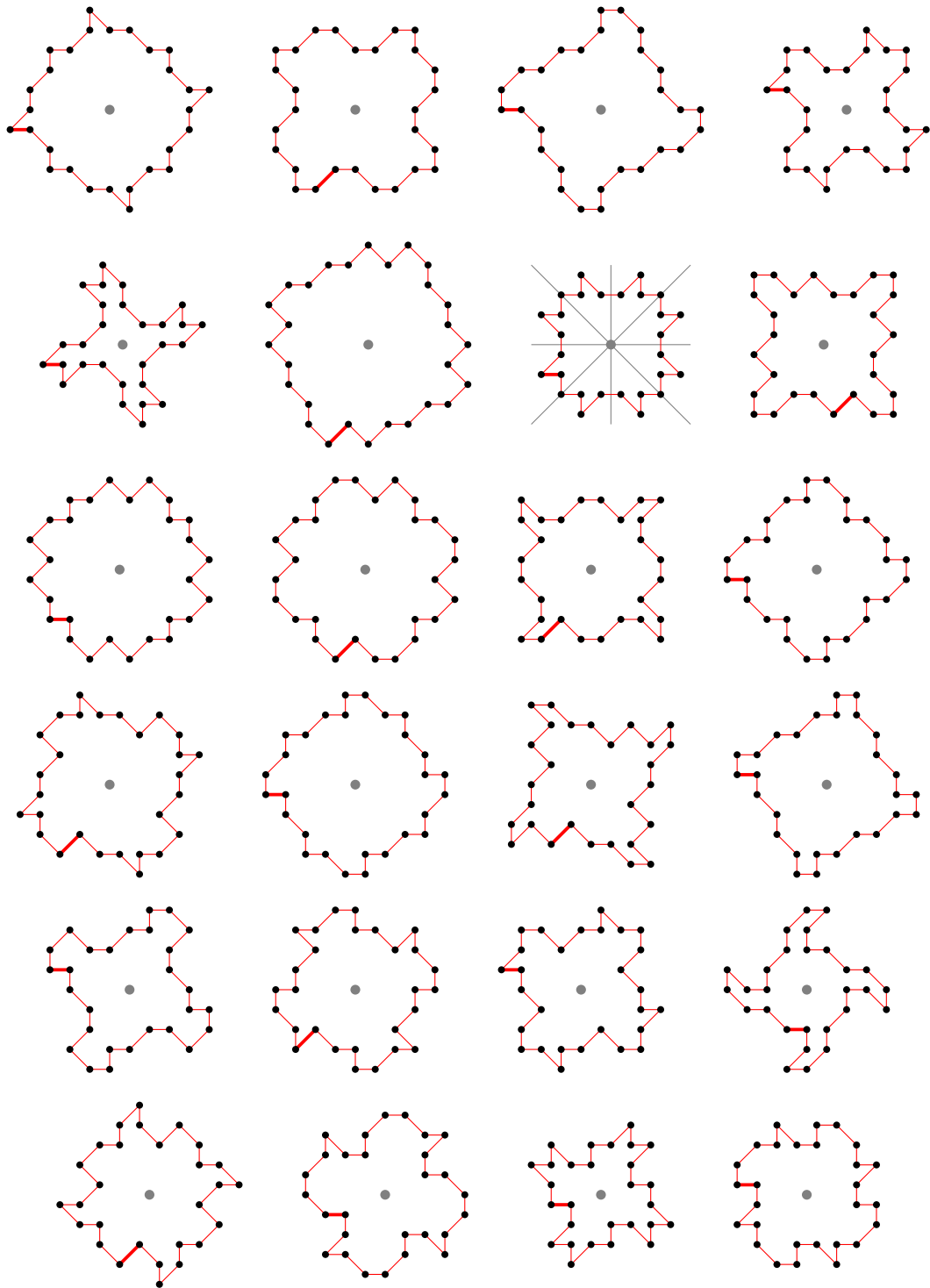


5.7.2 With largest area (146) there is 1 polygon

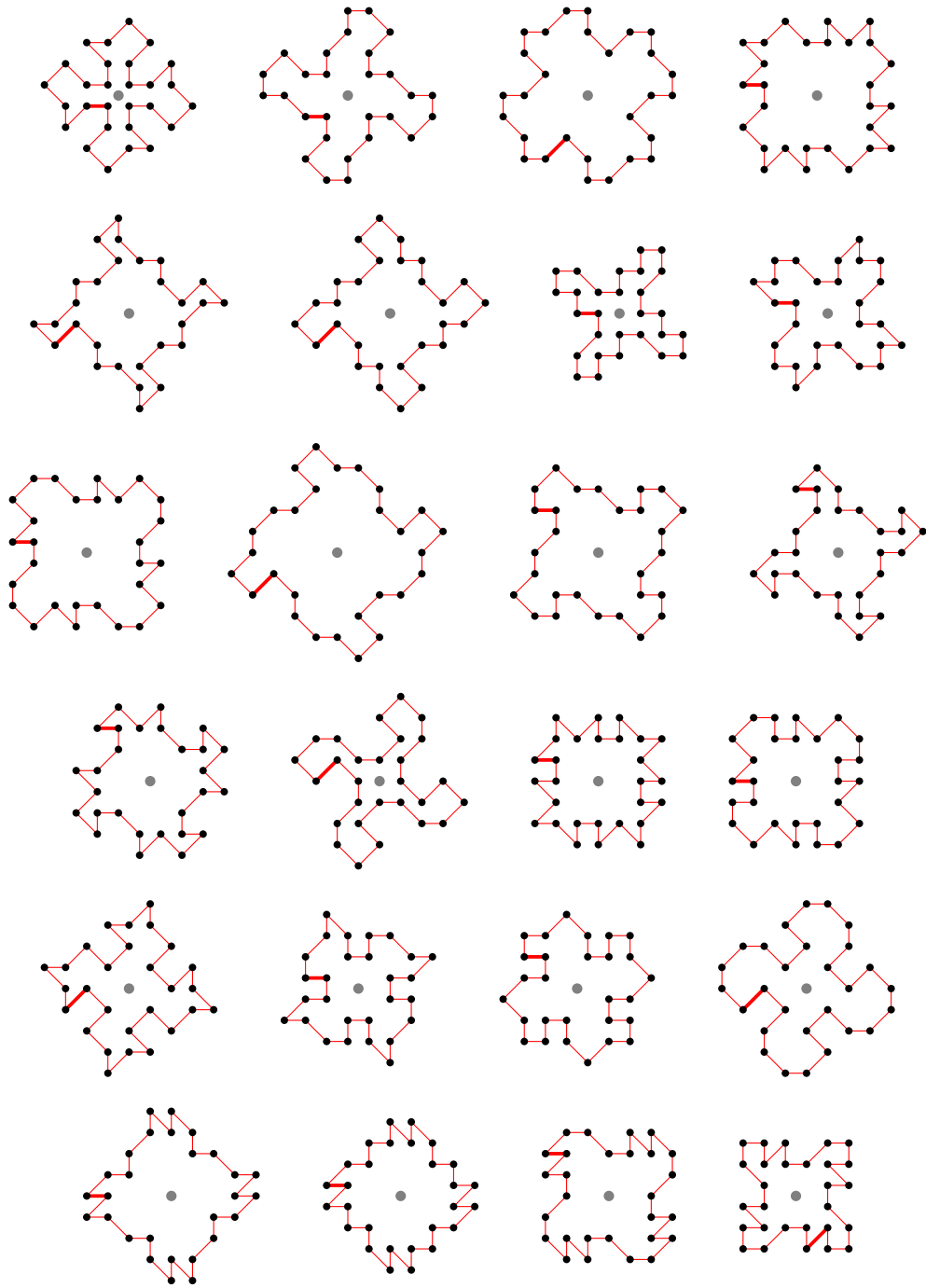


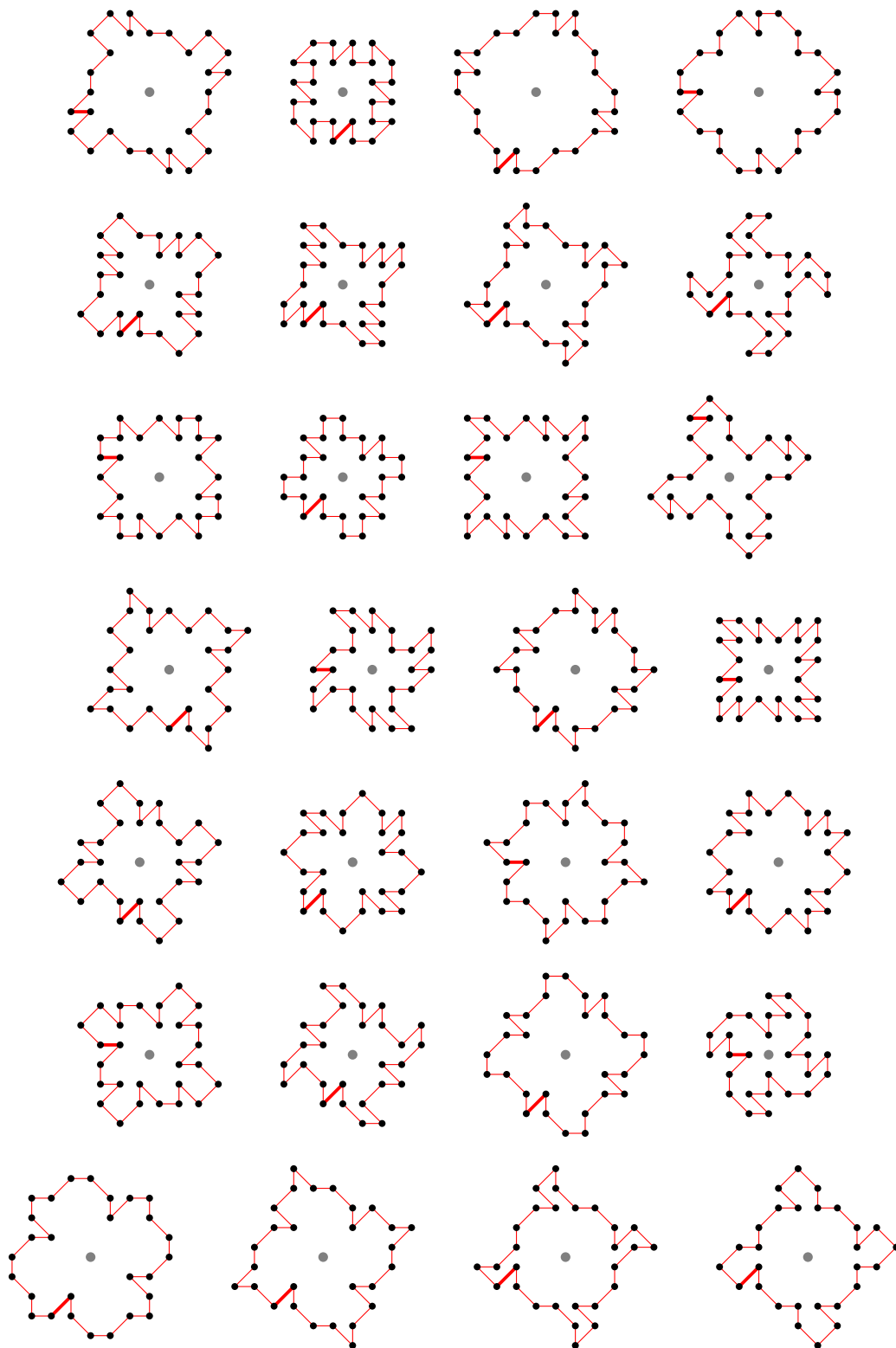
5.7.3 For other area sizes there are 1105 polygons

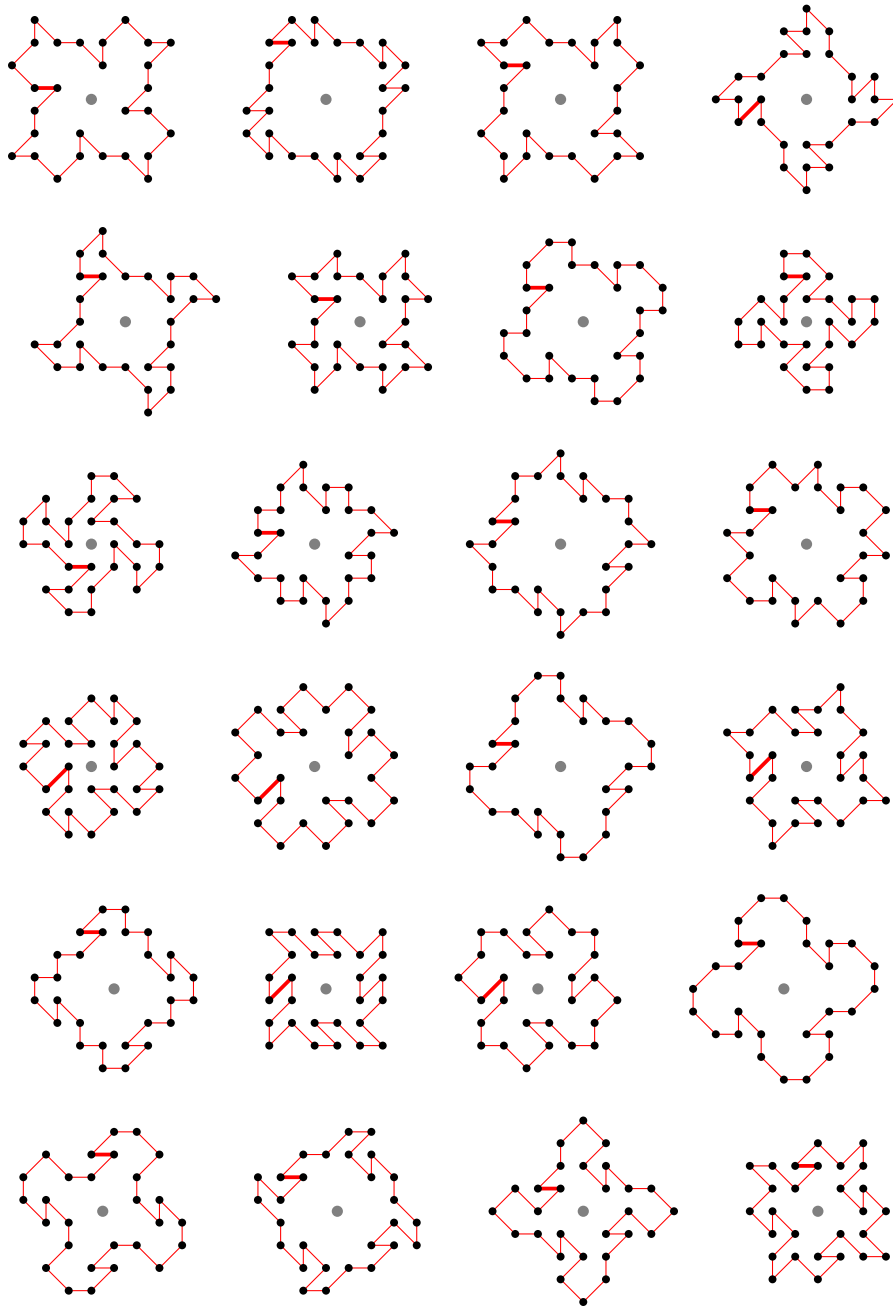
(100 samples)







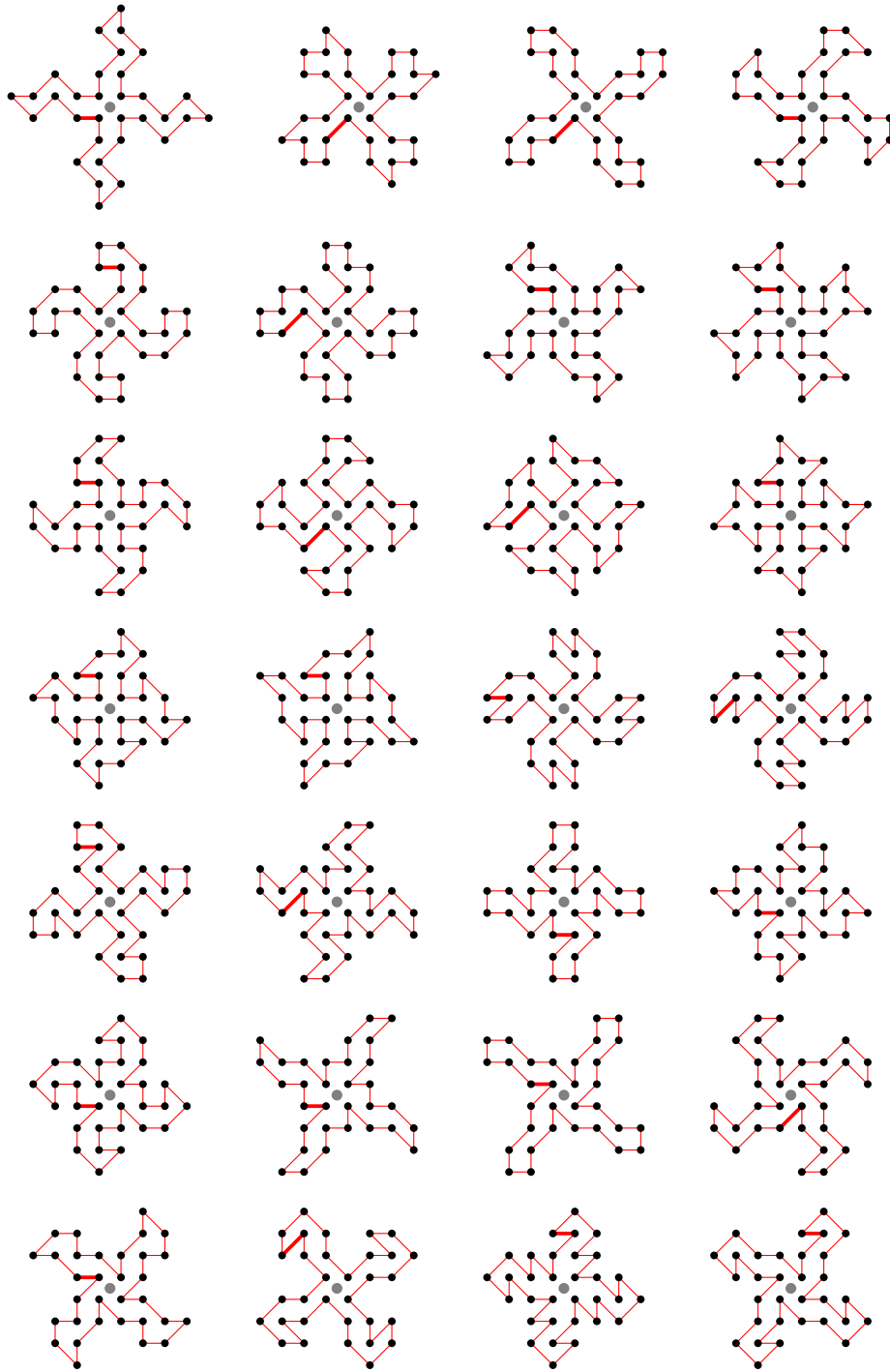


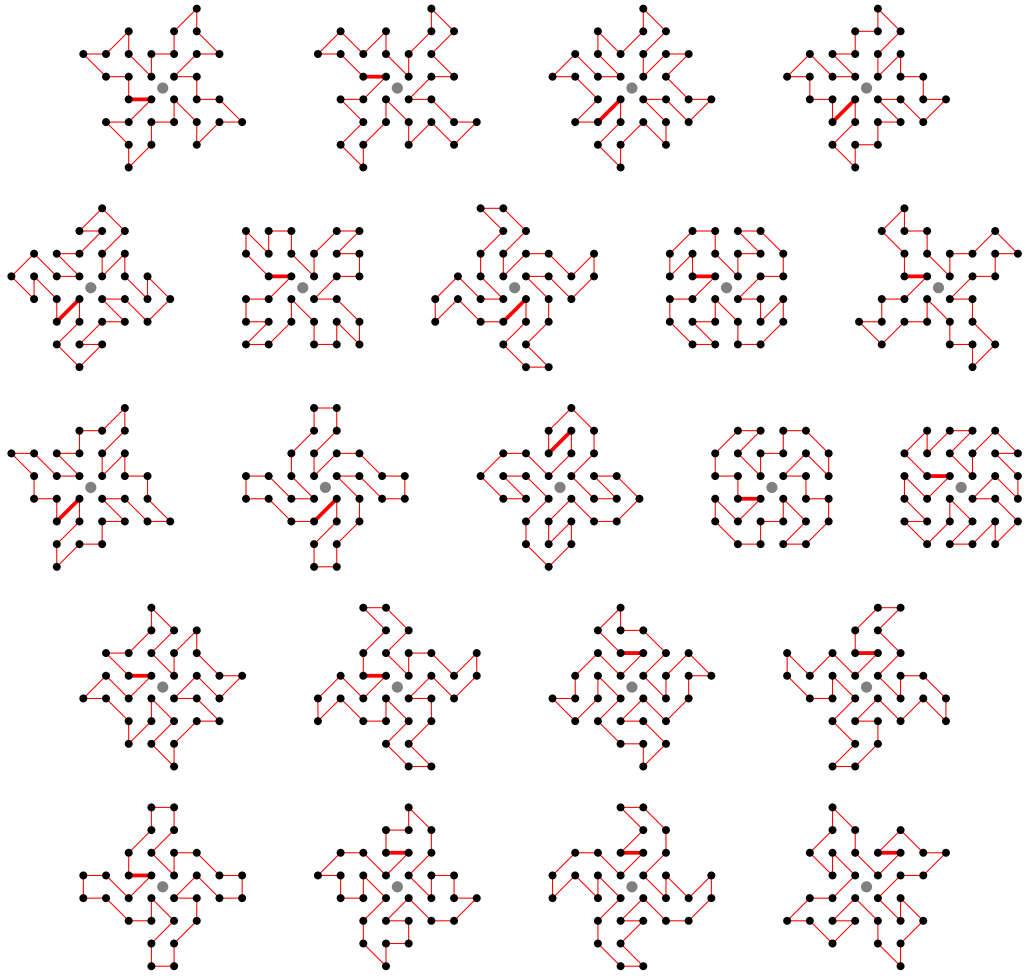


5.8 For  $n = 32$  there are 4655 polygons

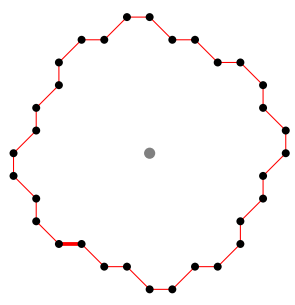
5.8.1 With smallest area (30) there are 102 polygons

(50 samples)



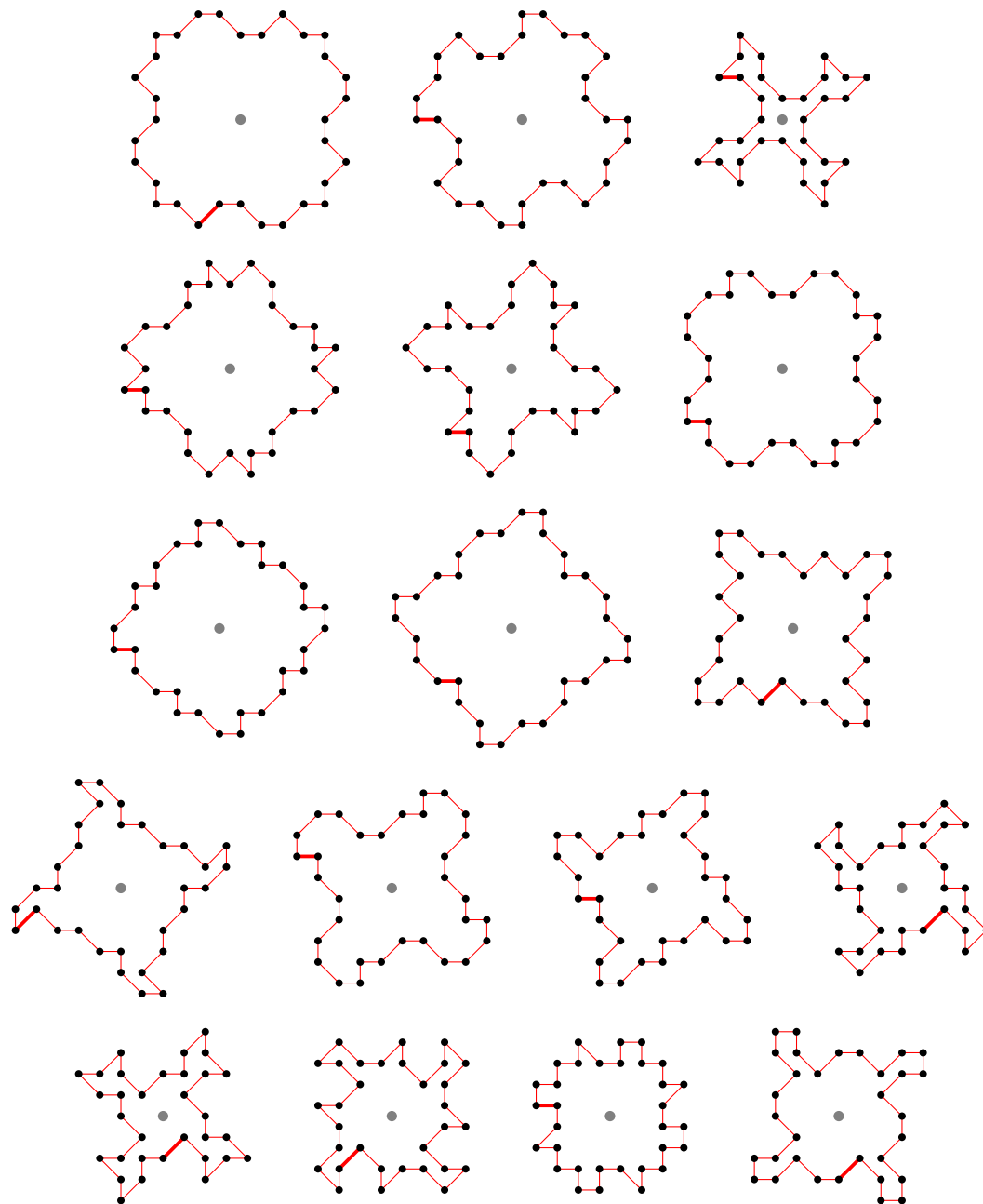


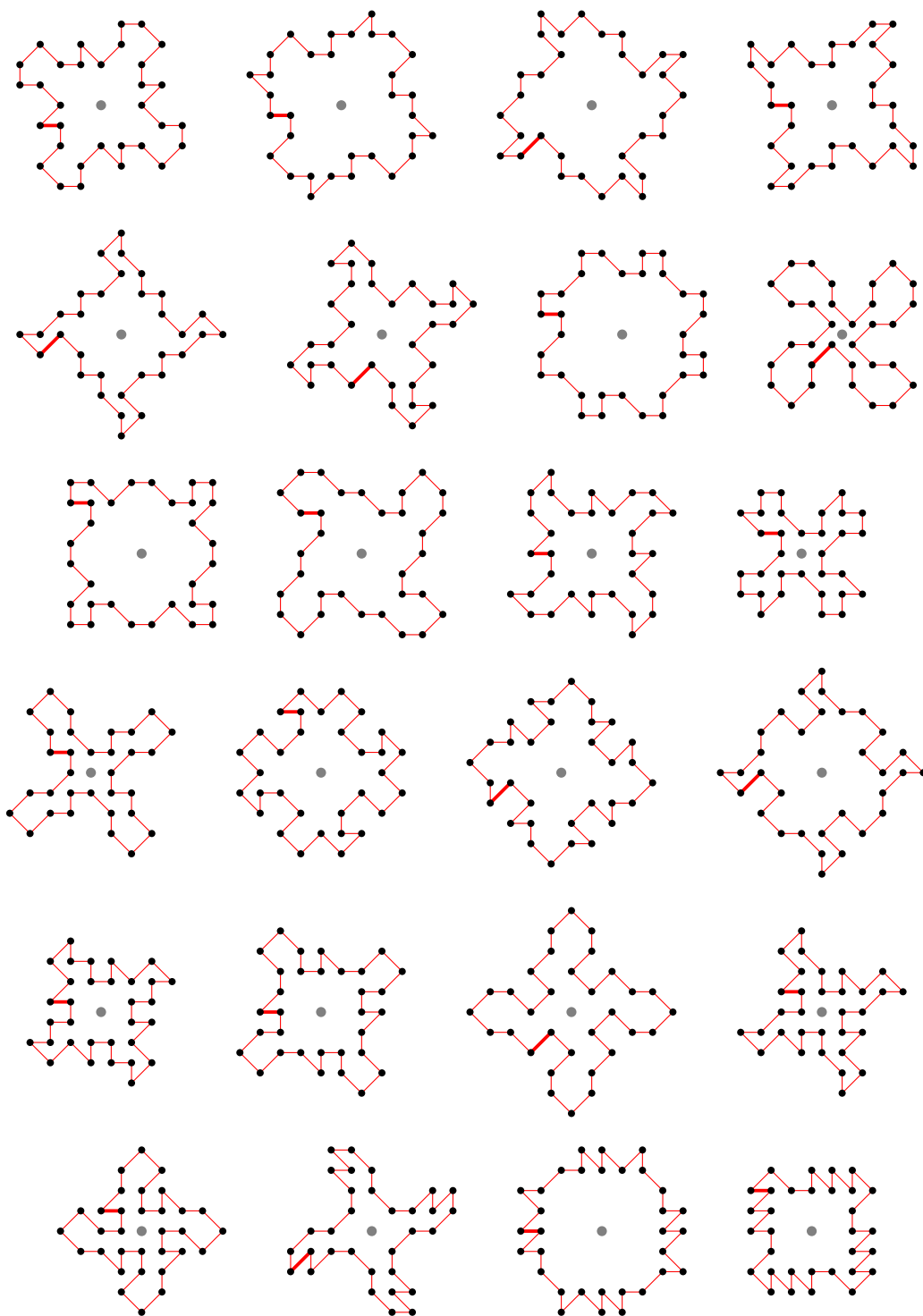
5.8.2 With largest area (192) there is 1 polygon

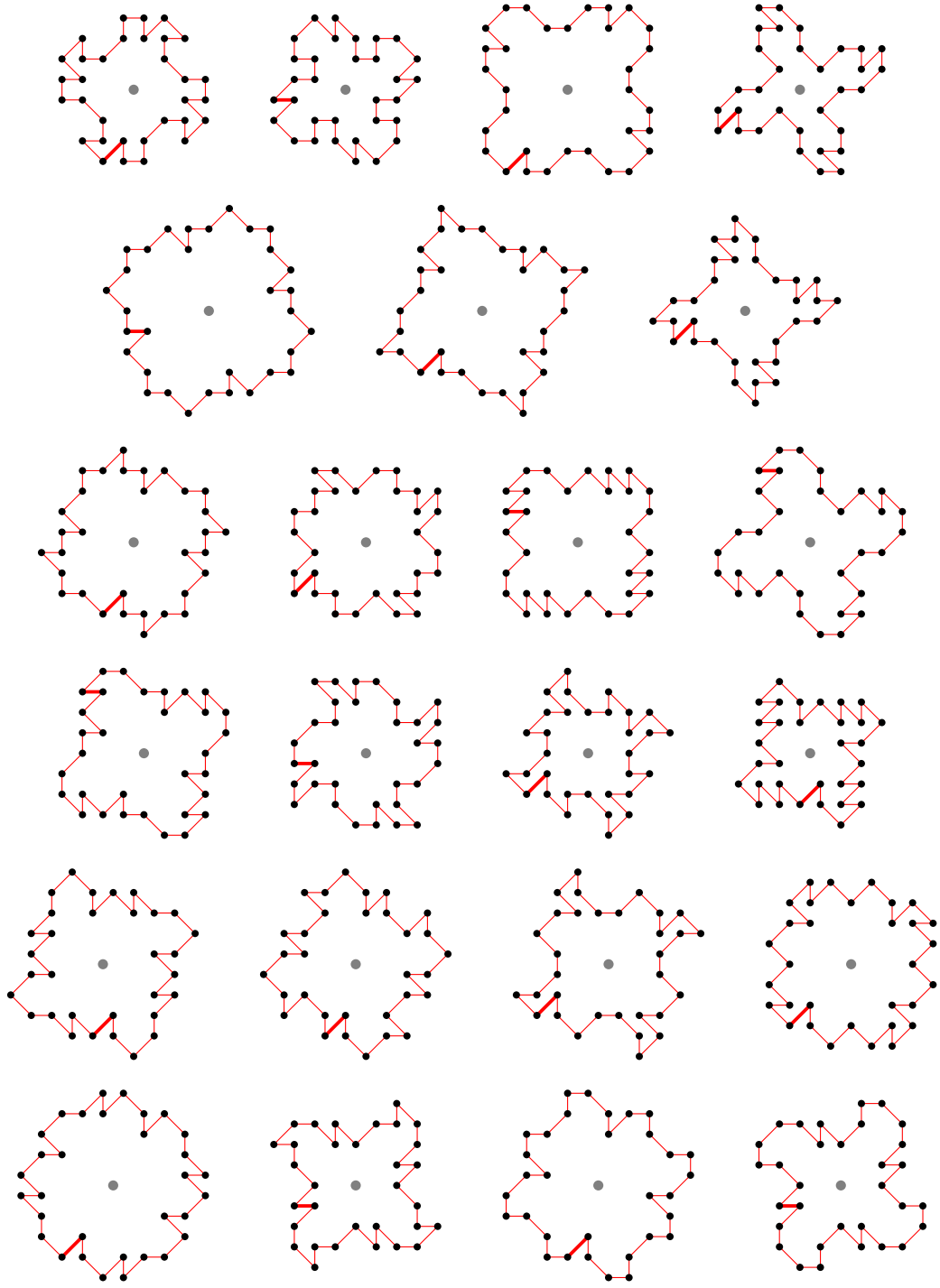


### 5.8.3 For other area sizes there are 4552 polygons

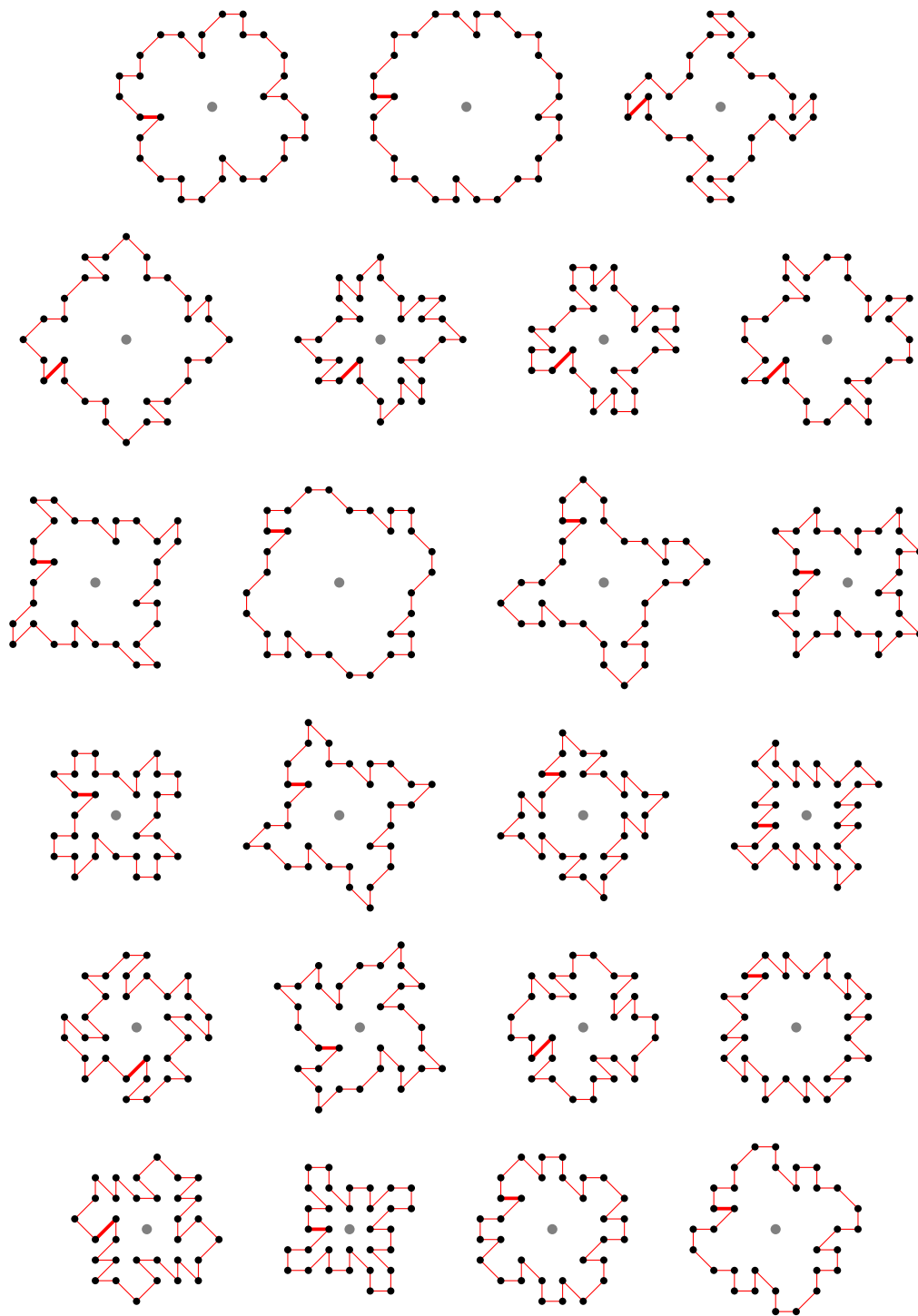
(100 samples)

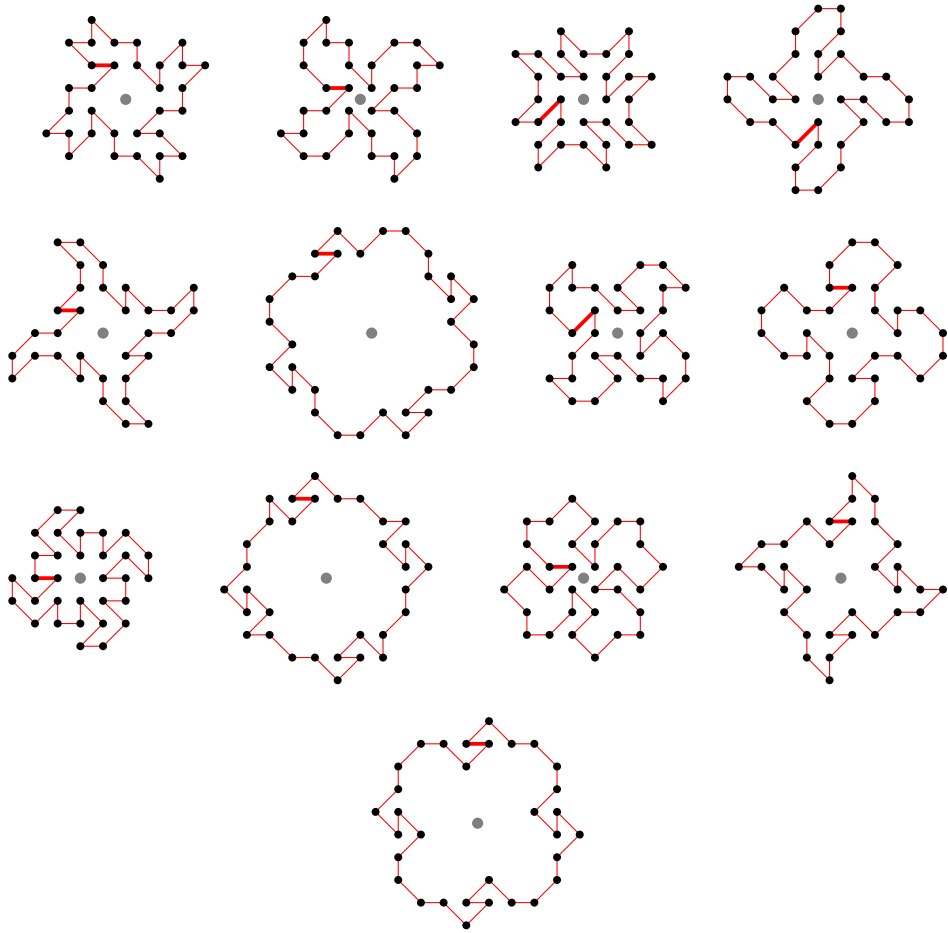








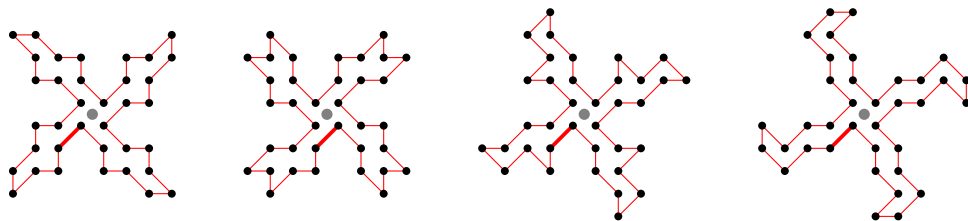


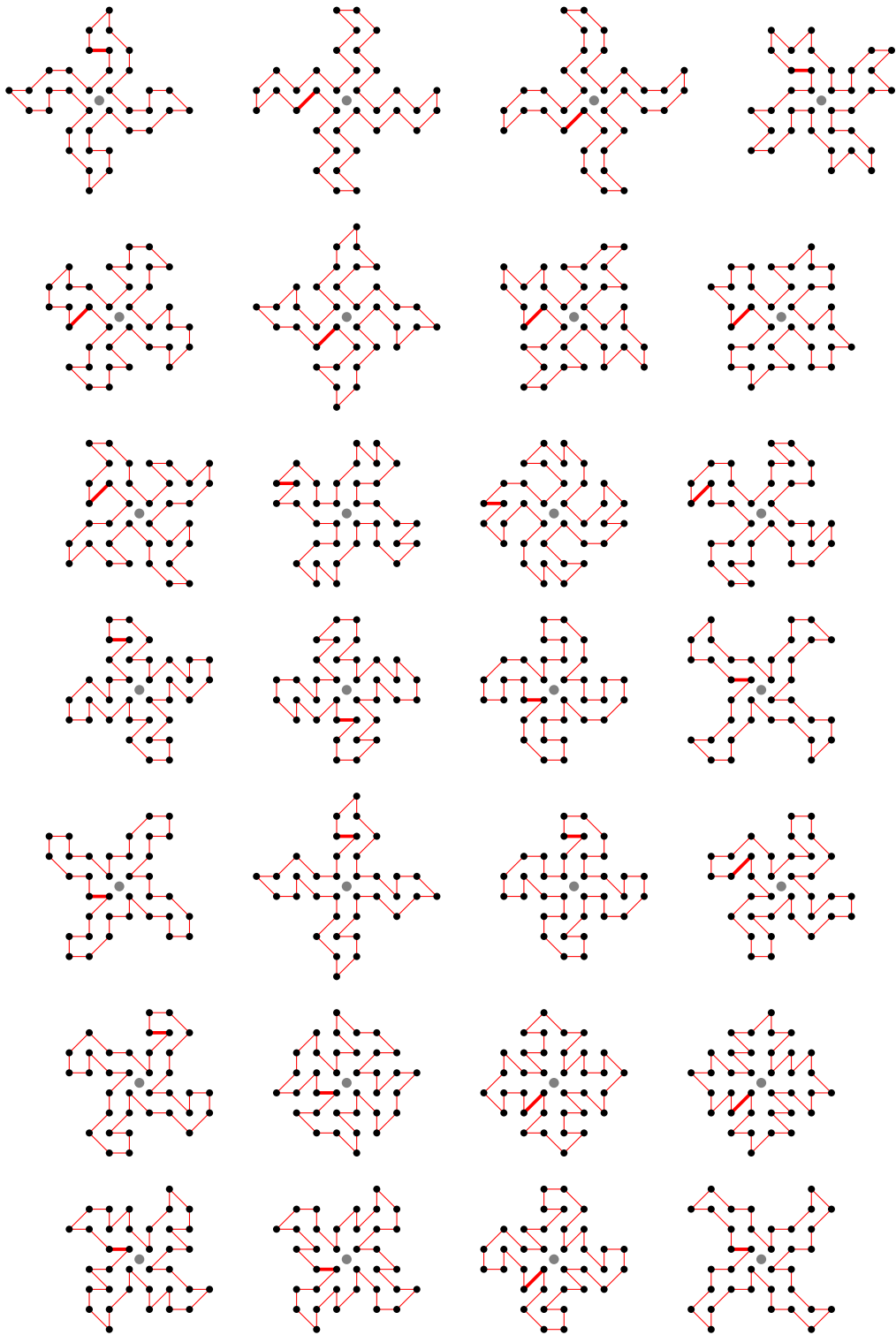


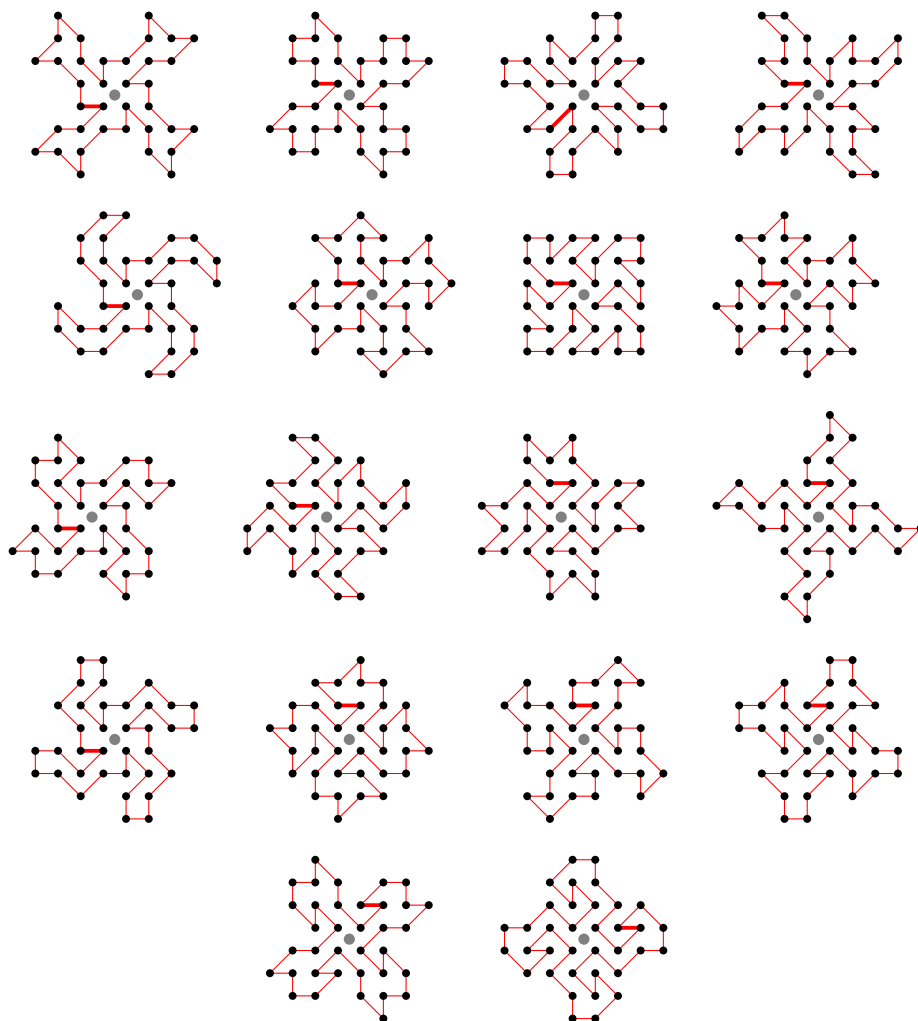
**5.9 For  $n = 36$  there are 19314 polygons**

**5.9.1 With smallest area (34) there are 285 polygons**

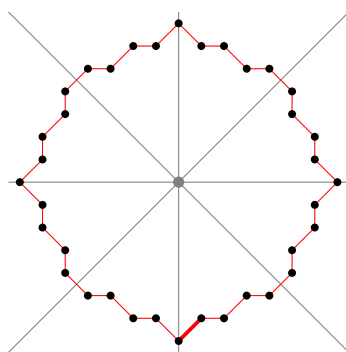
(50 samples)





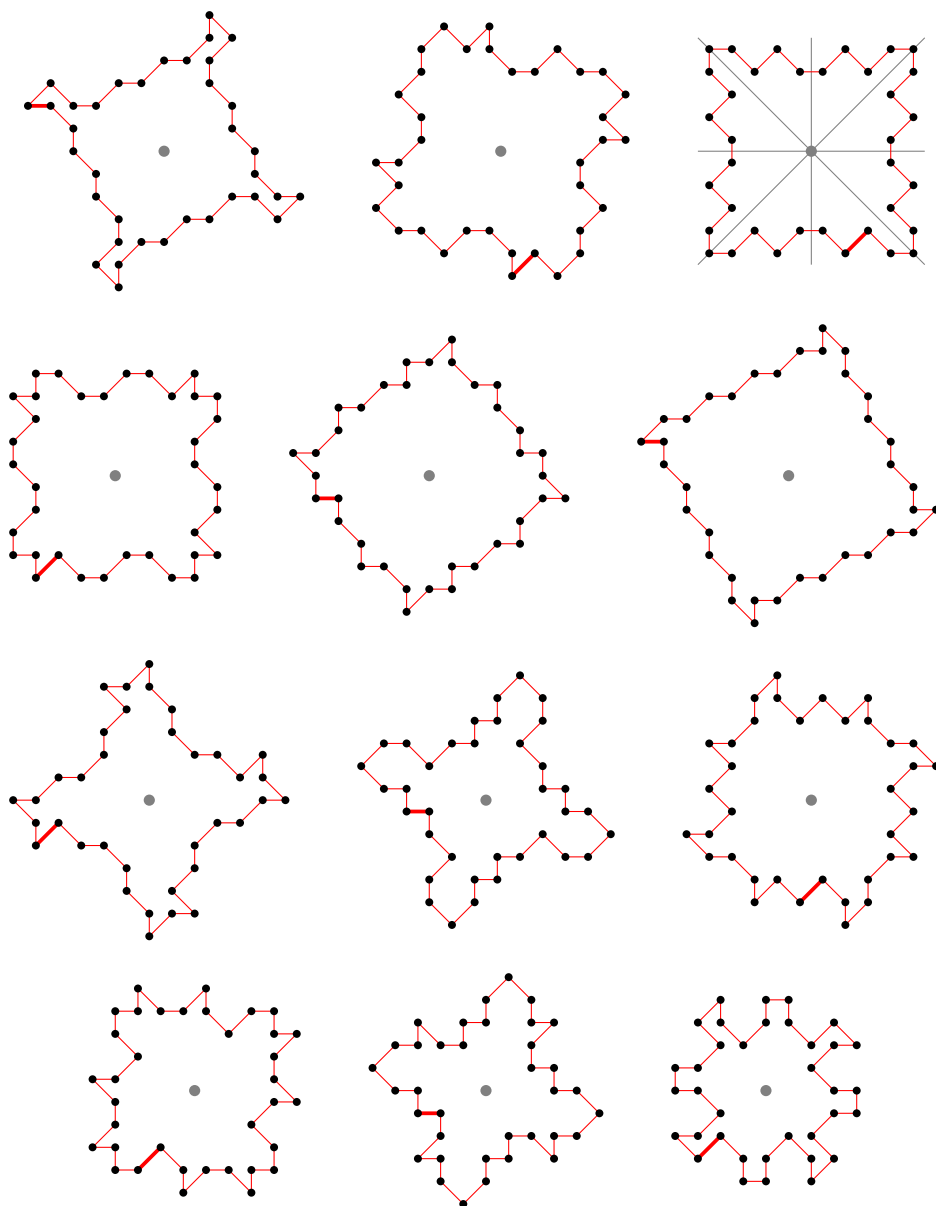


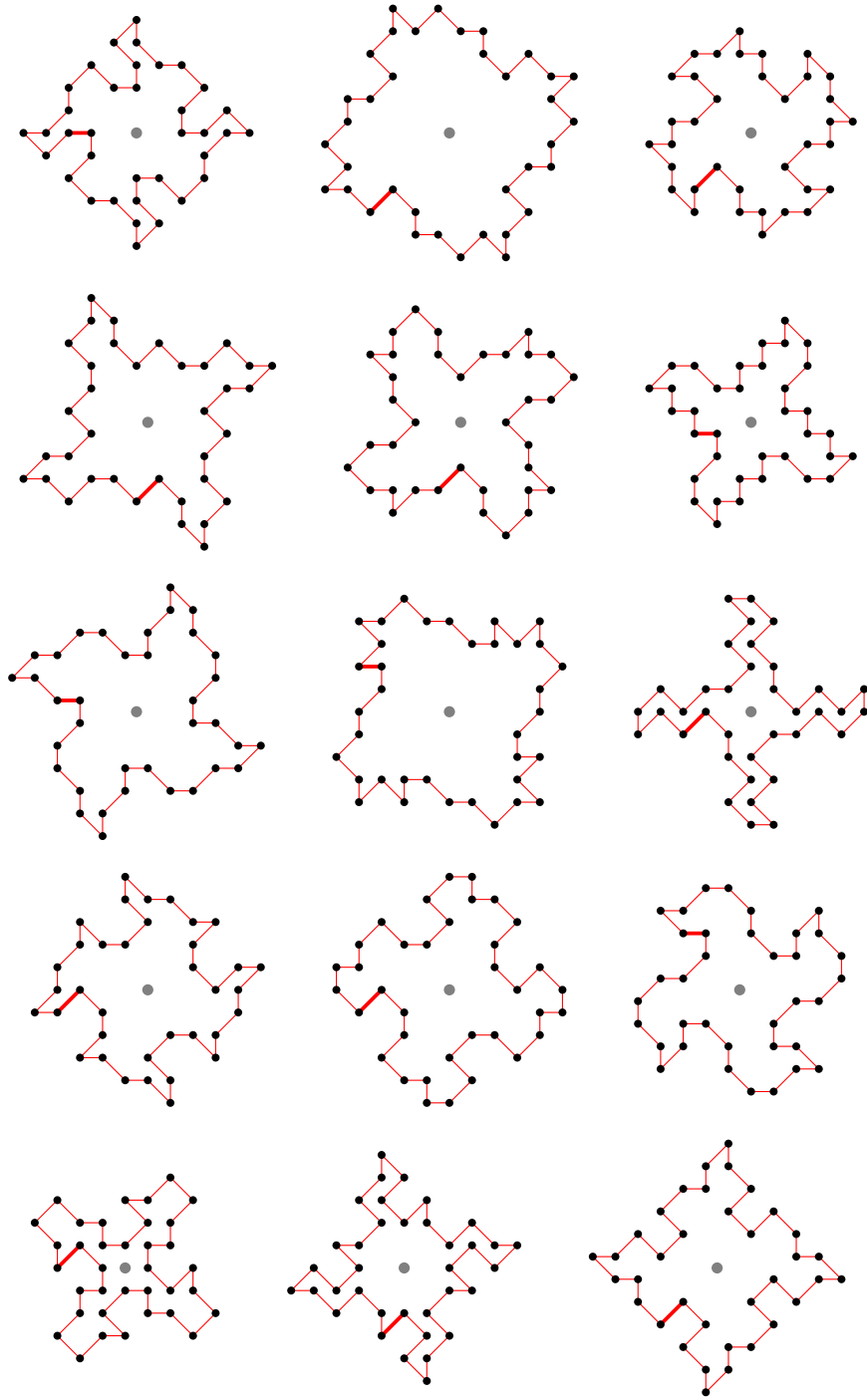
5.9.2 With largest area (244) there is 1 polygon

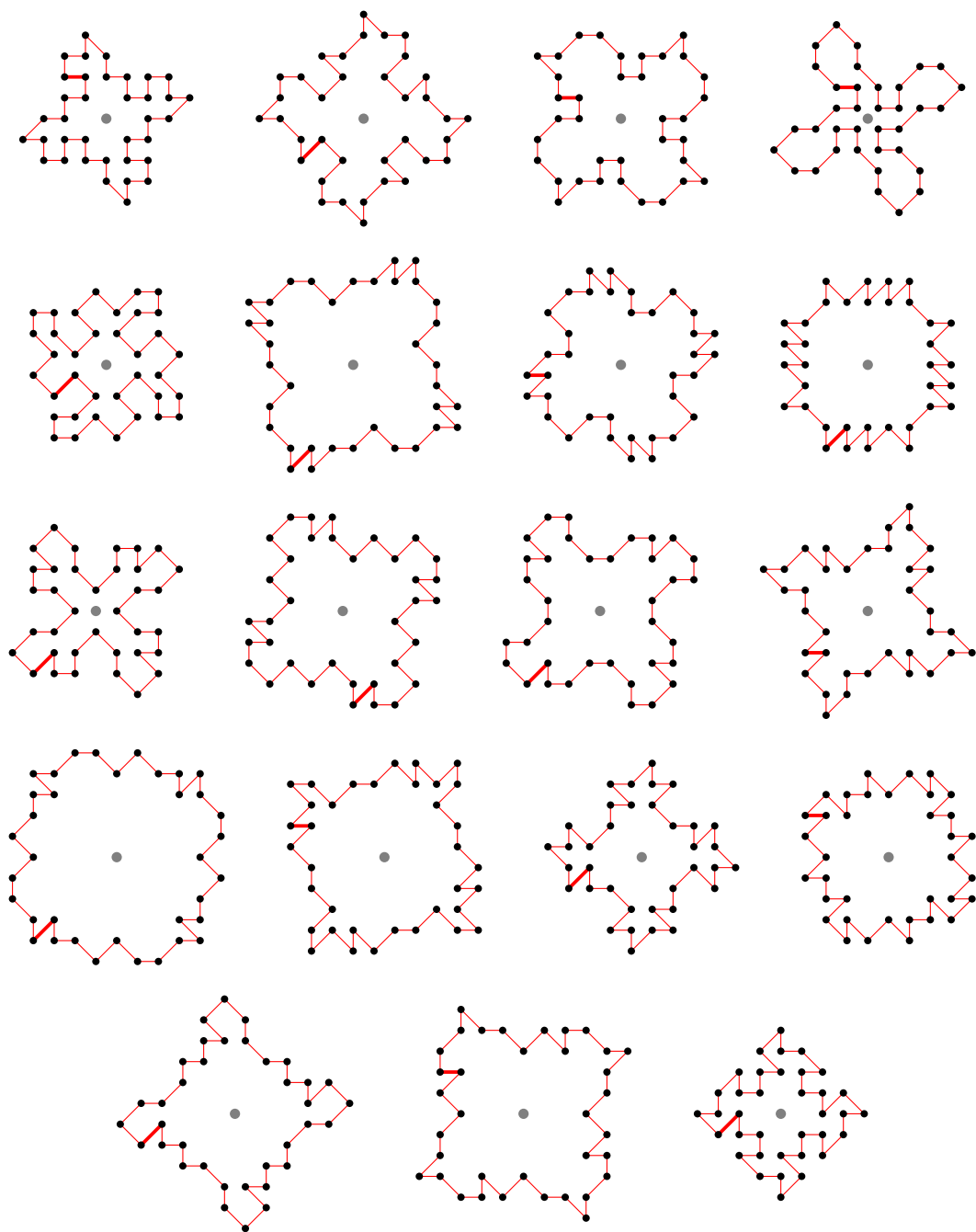


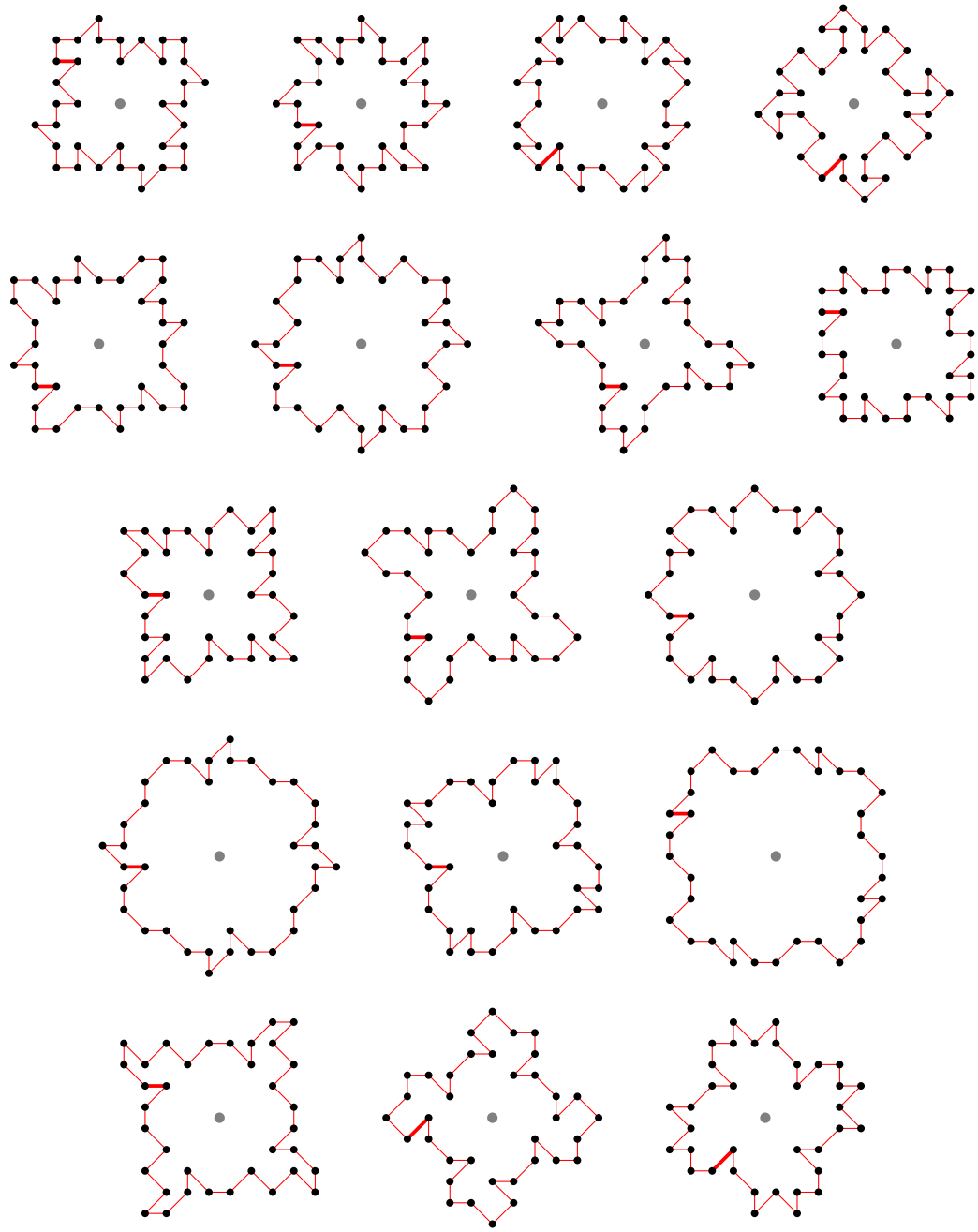
### 5.9.3 For other area sizes there are 19028 polygons

(100 samples)

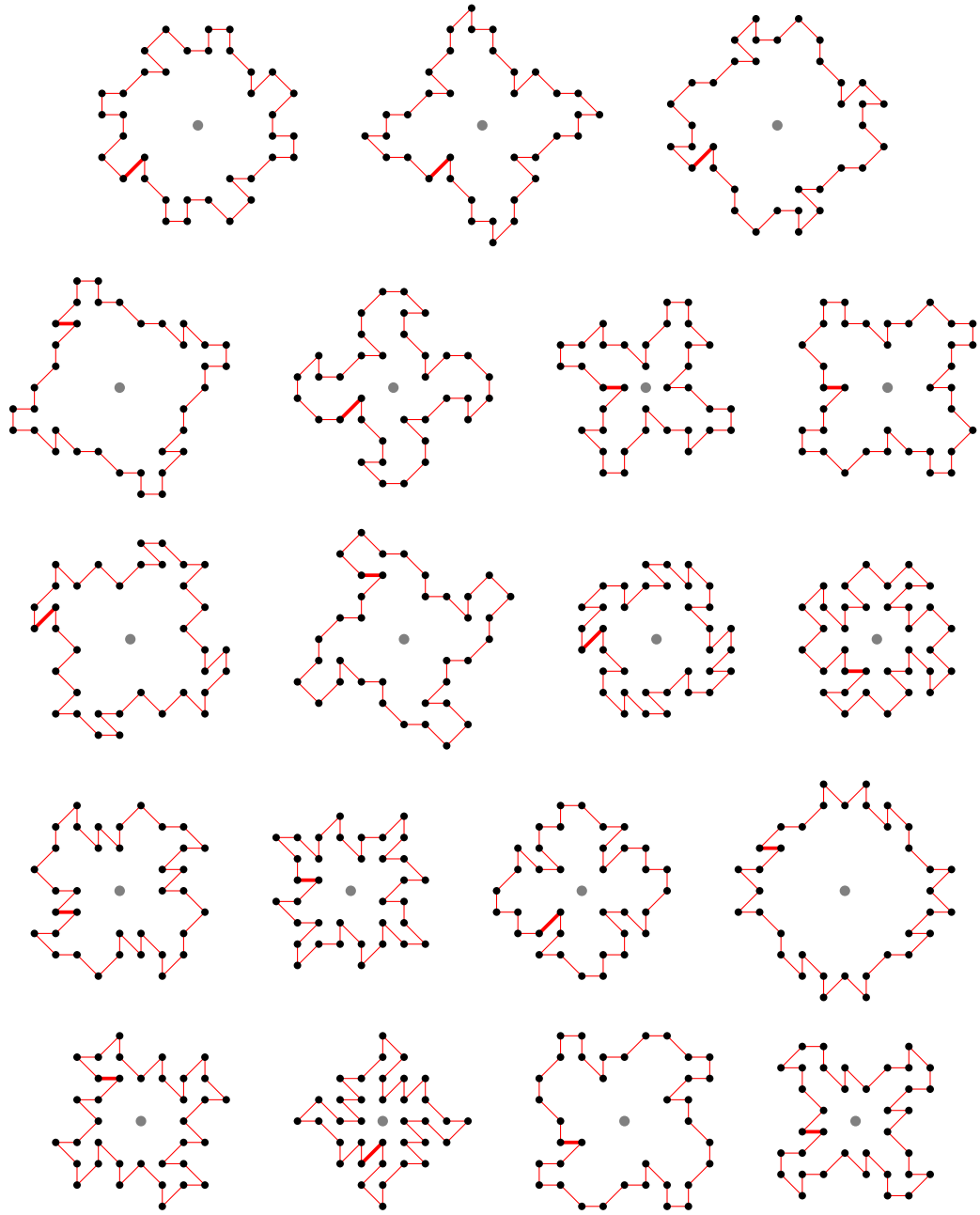


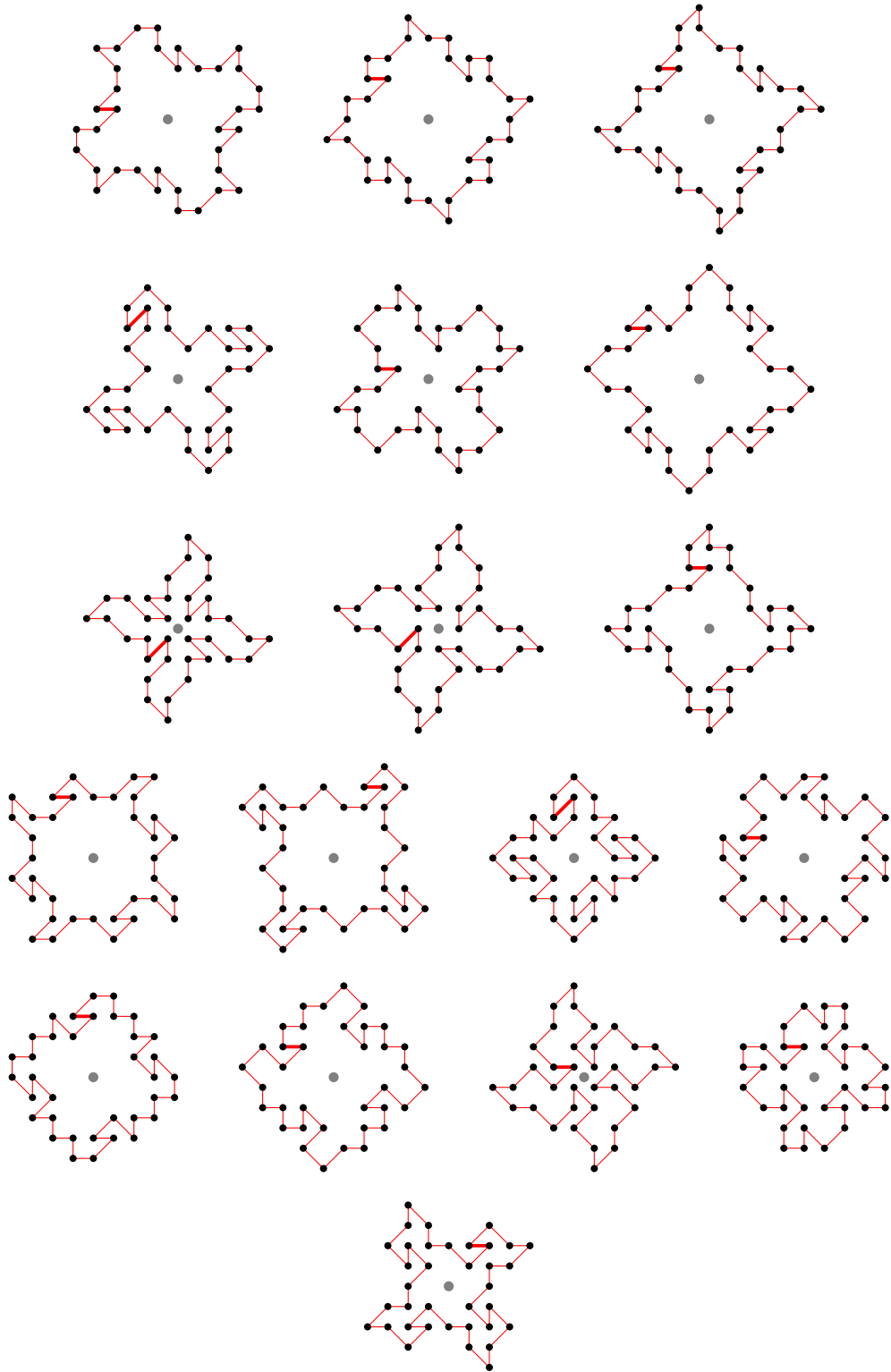








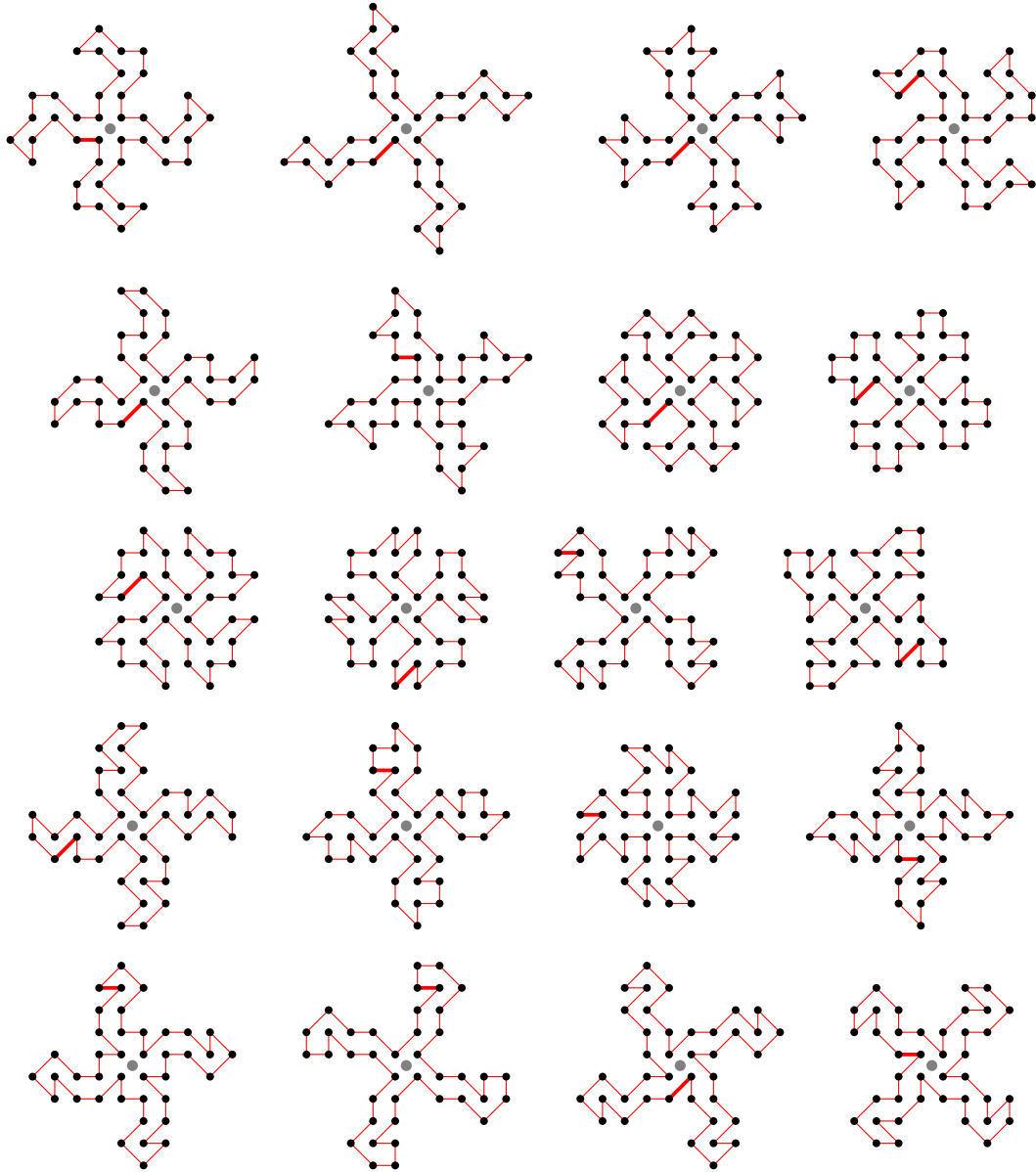


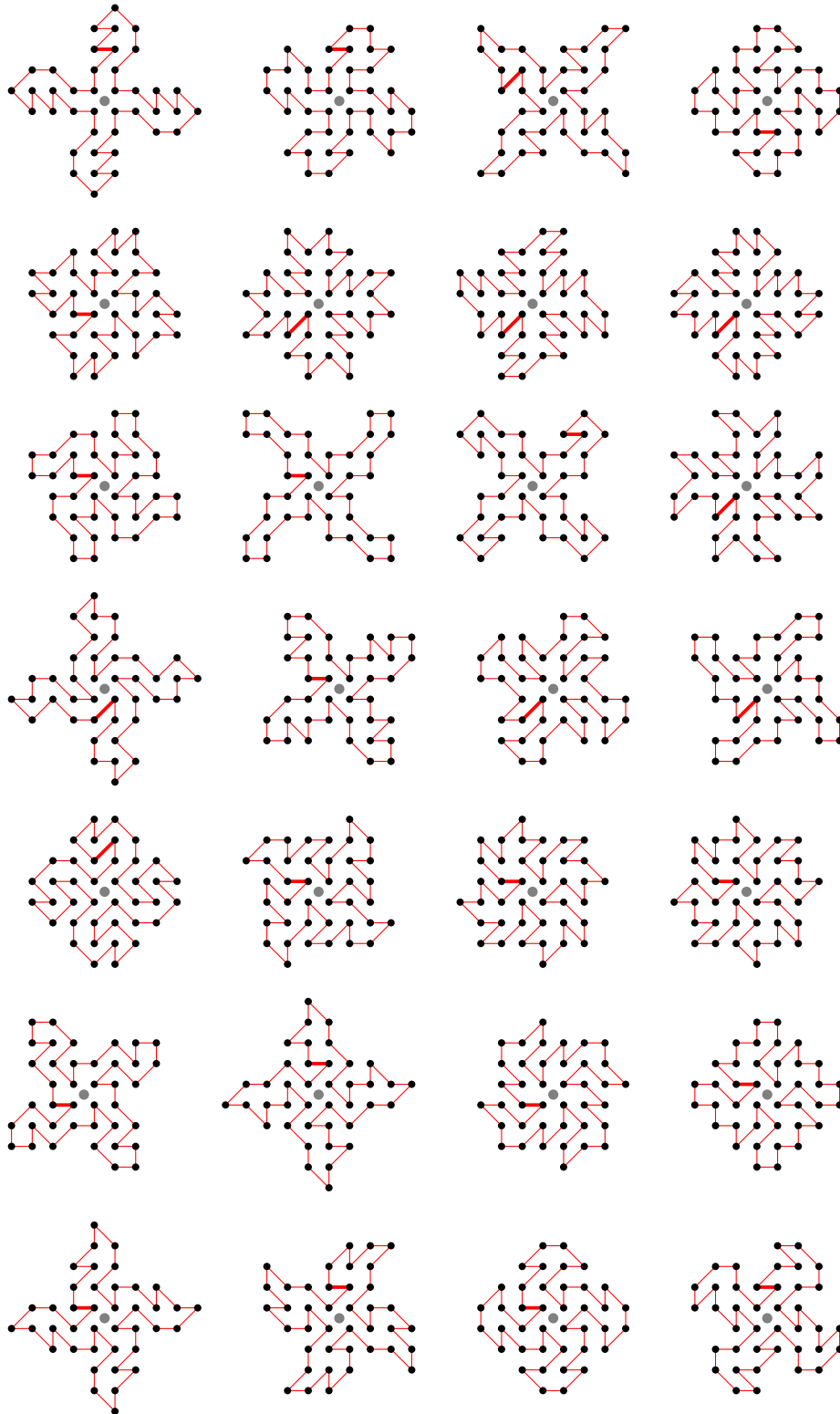


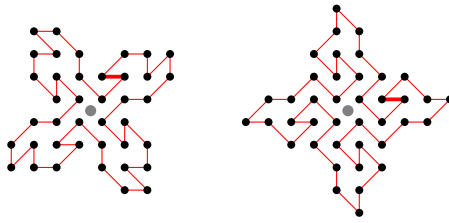
## 5.10 For $n = 40$ there are 81242 polygons

### 5.10.1 With smallest area (38) there are 772 polygons

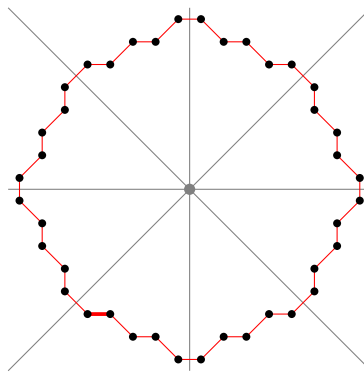
(50 samples)





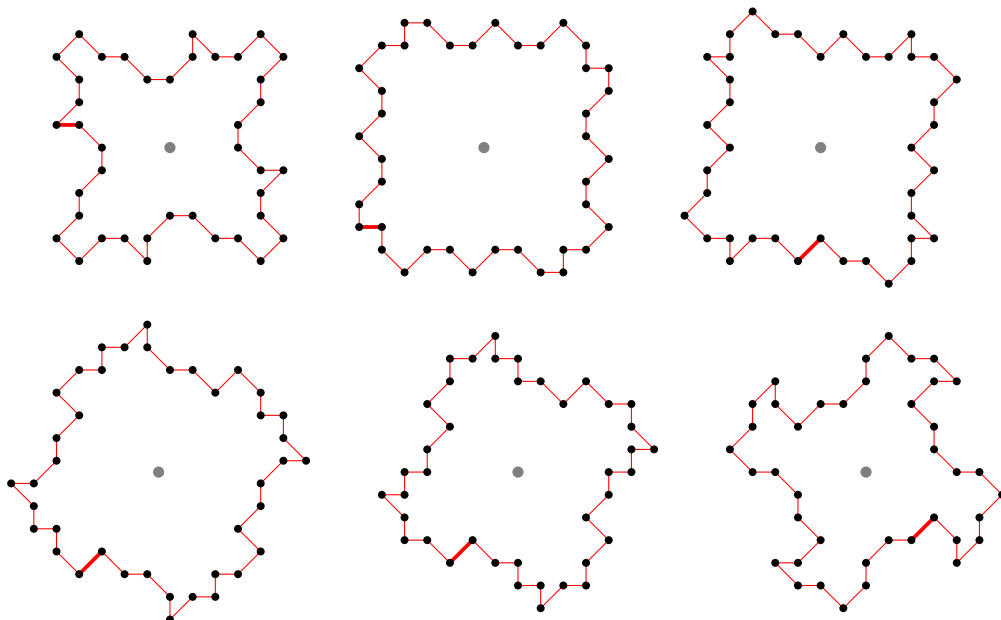


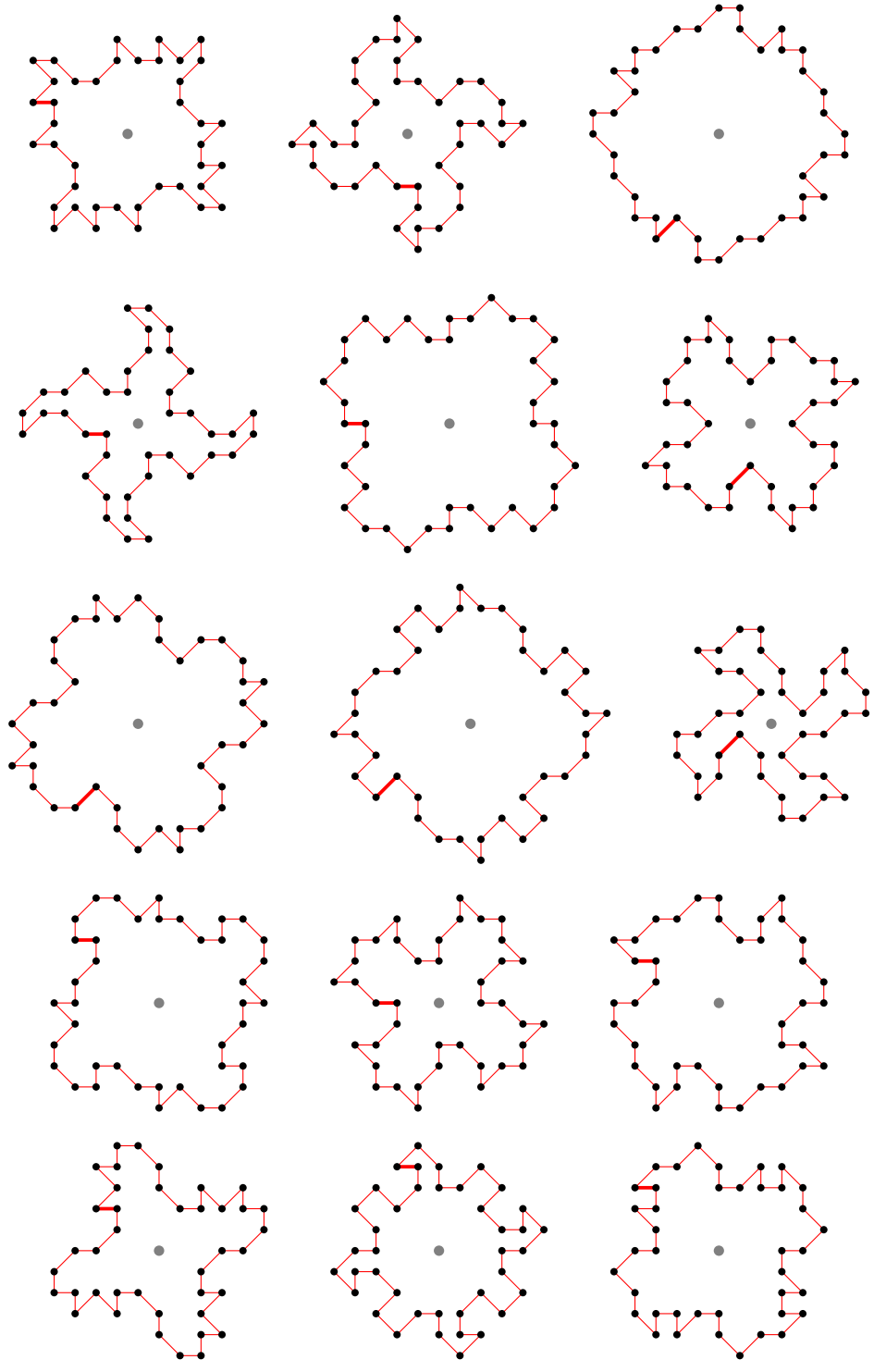
5.10.2 With largest area (302) there is 1 polygon

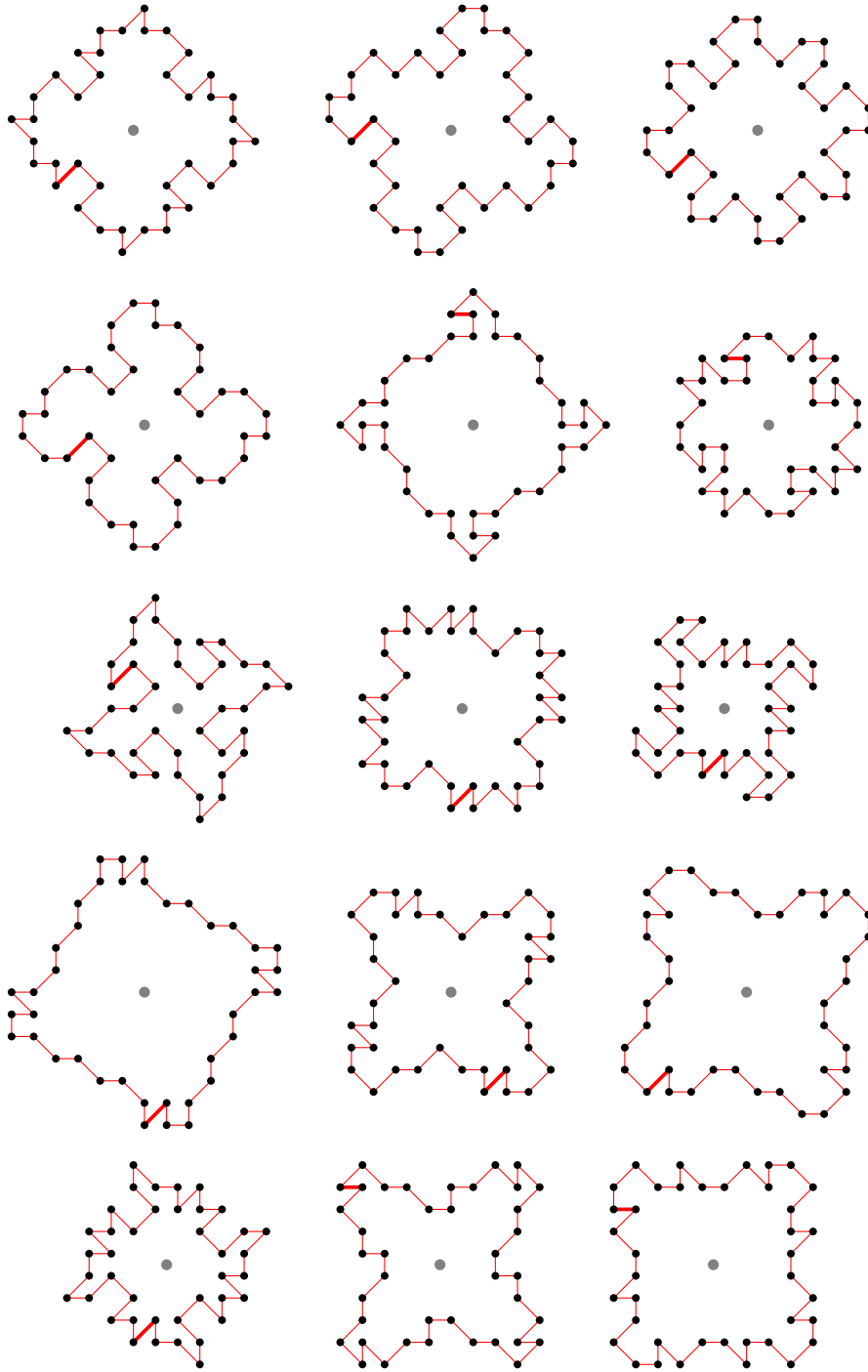


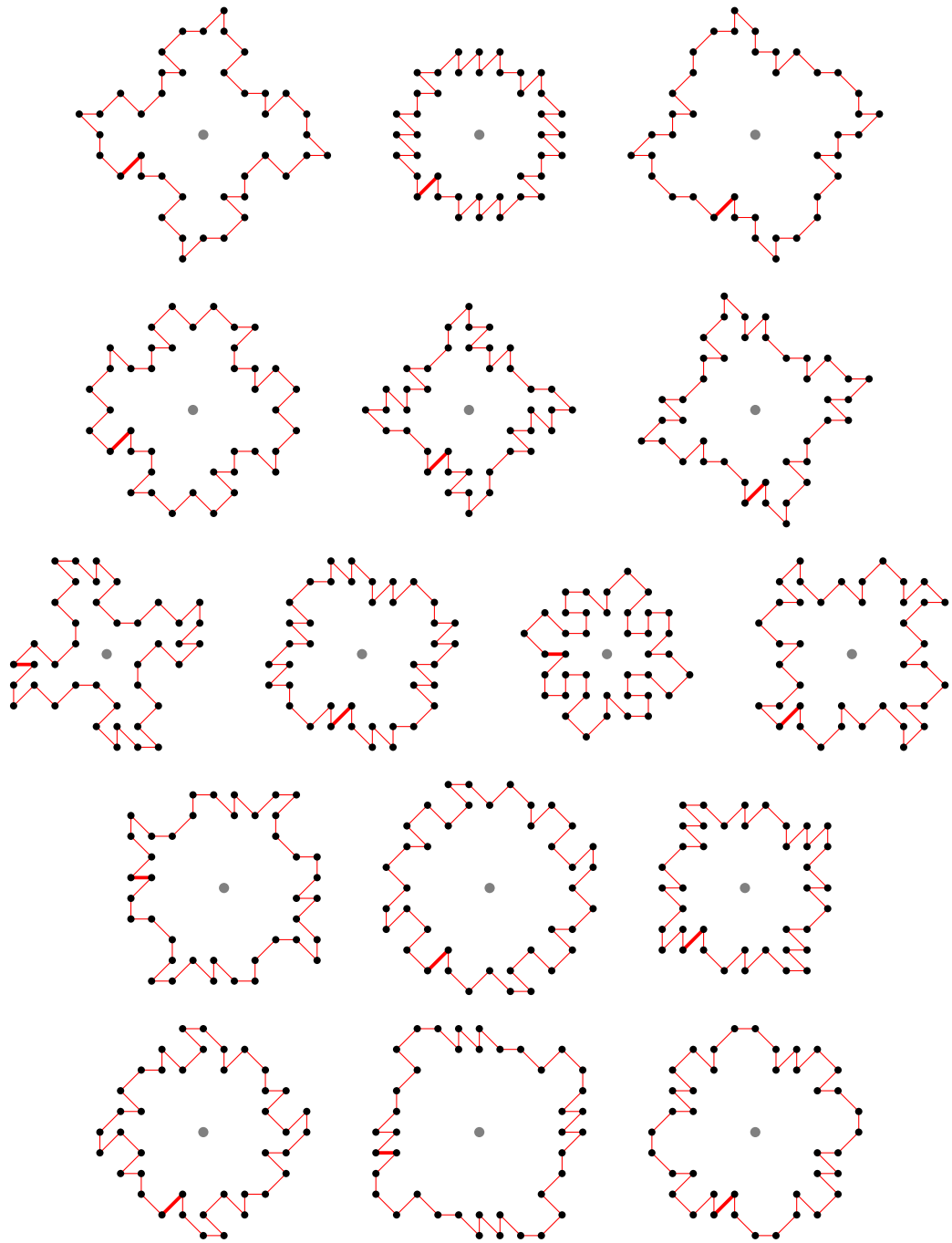
5.10.3 For other area sizes there are 80469 polygons

(100 samples)

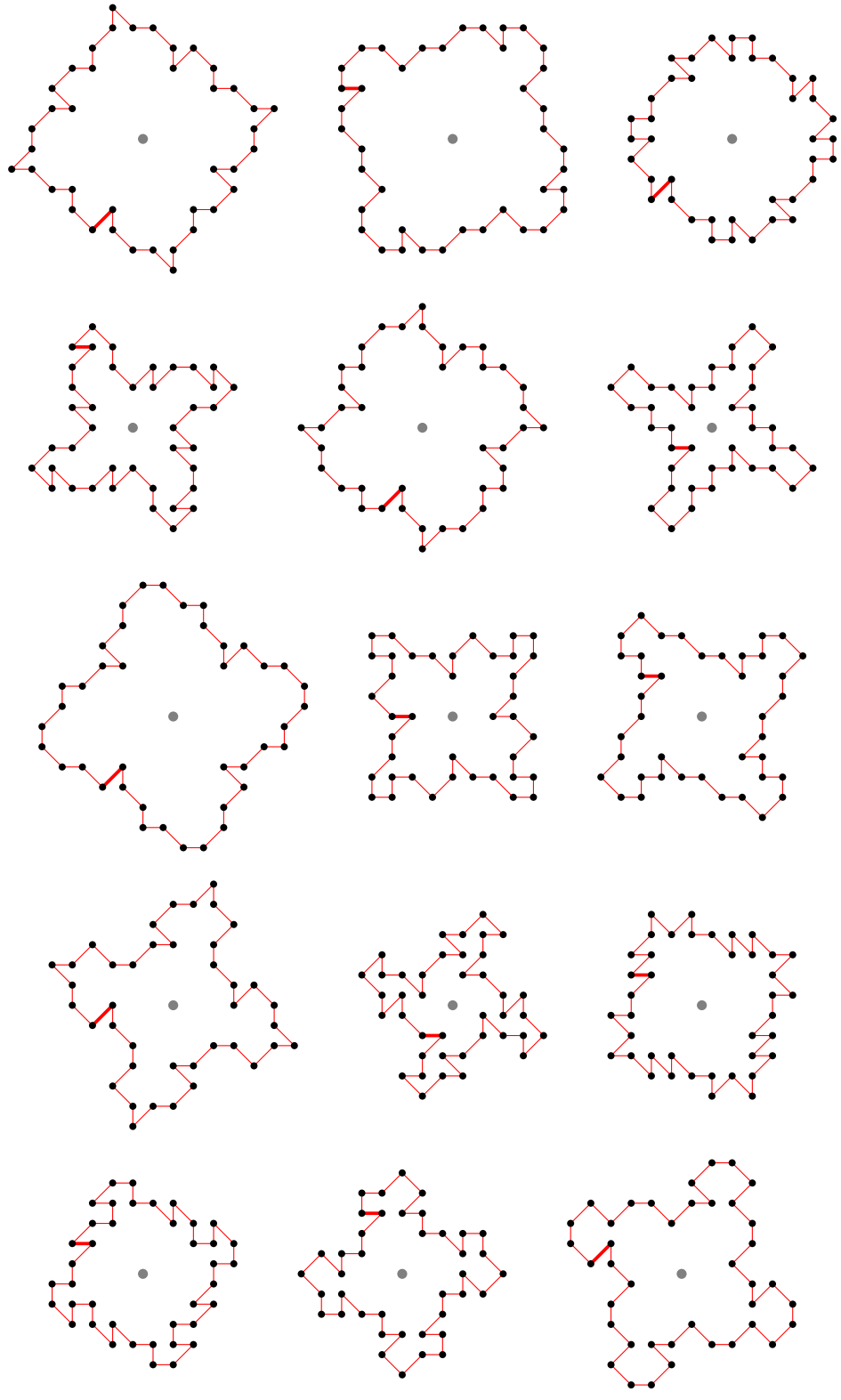


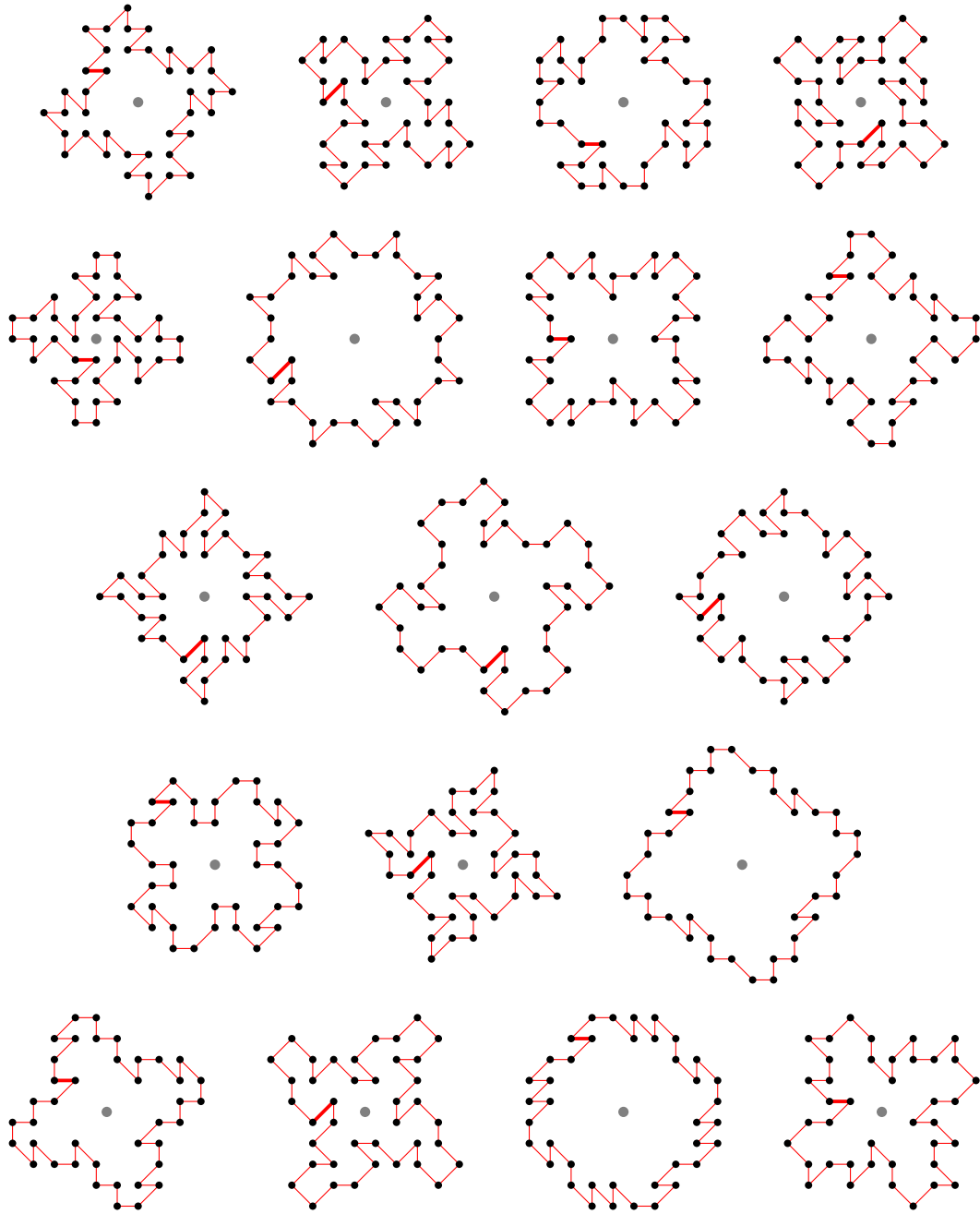


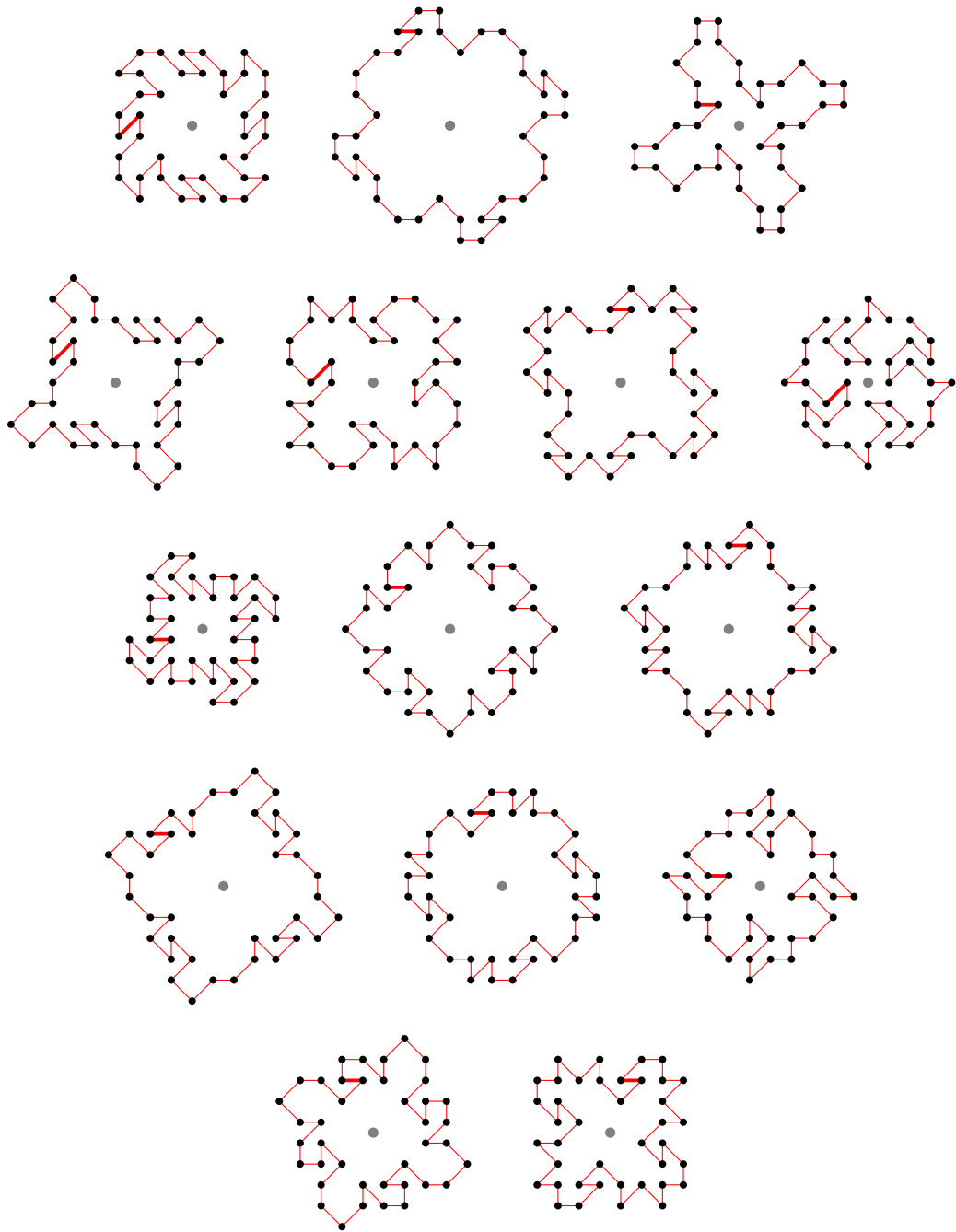








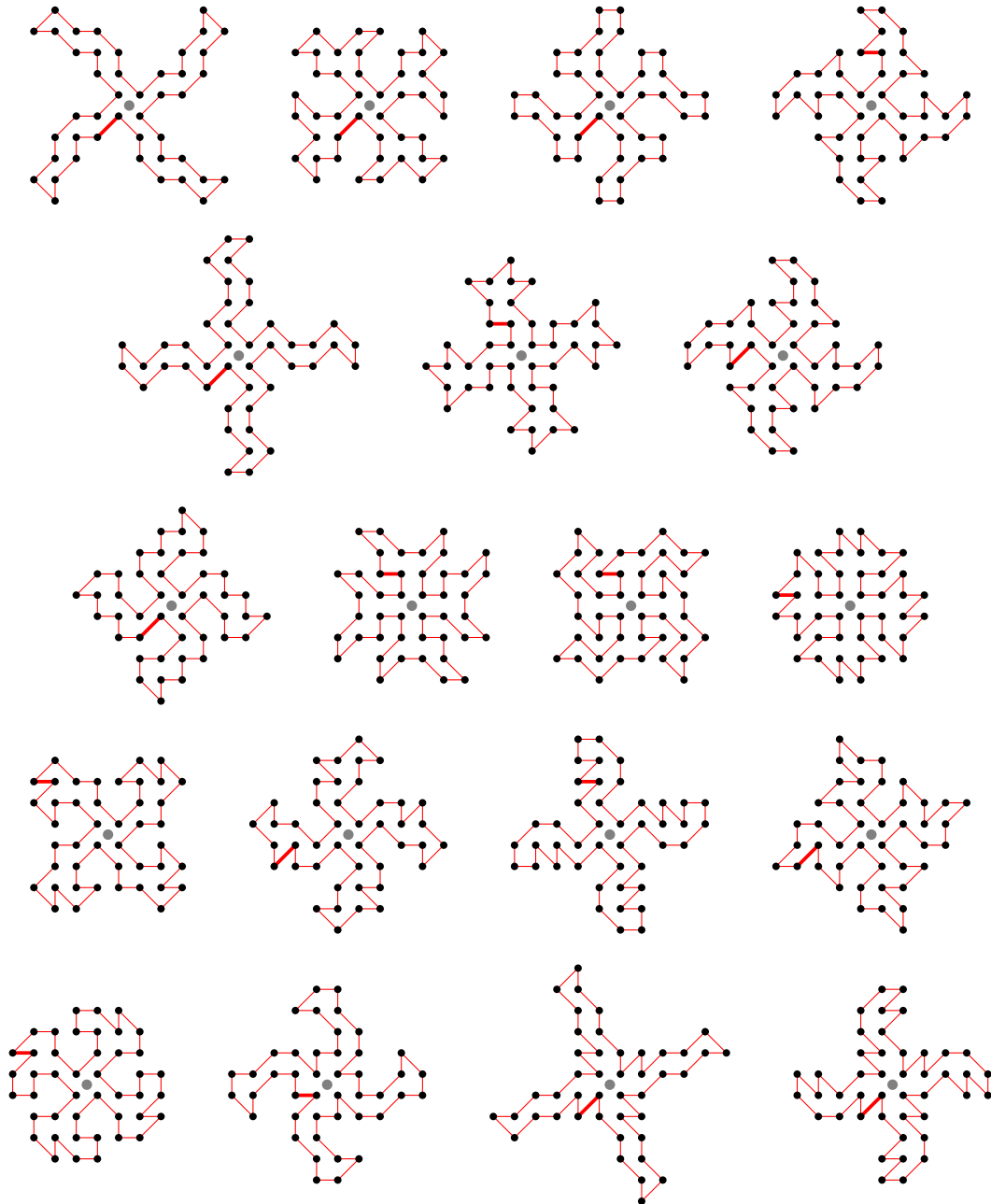




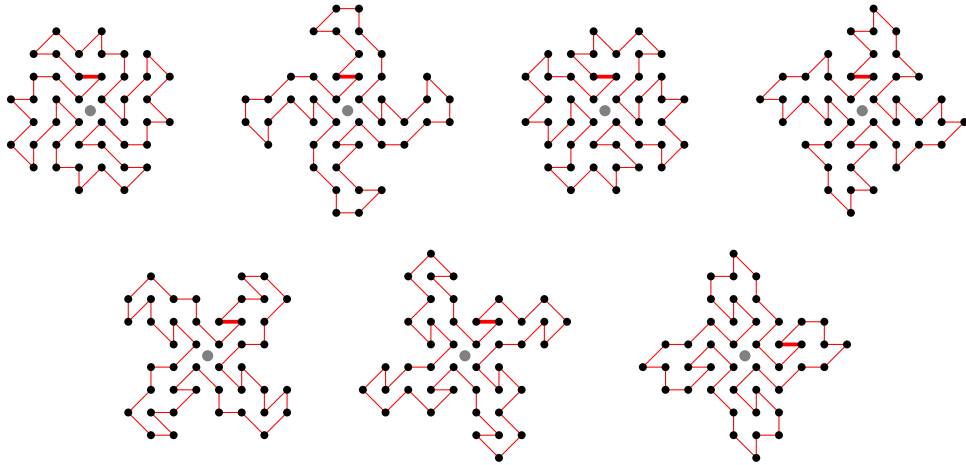
5.11 For  $n = 44$  there are 345785 polygons

5.11.1 With smallest area (42) there are 2102 polygons

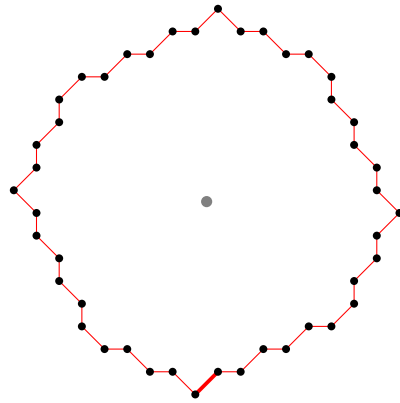
(50 samples)





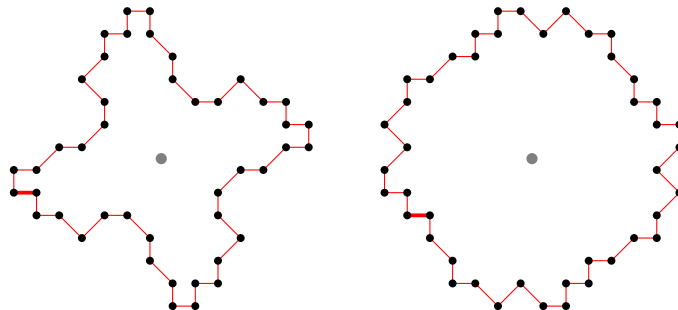


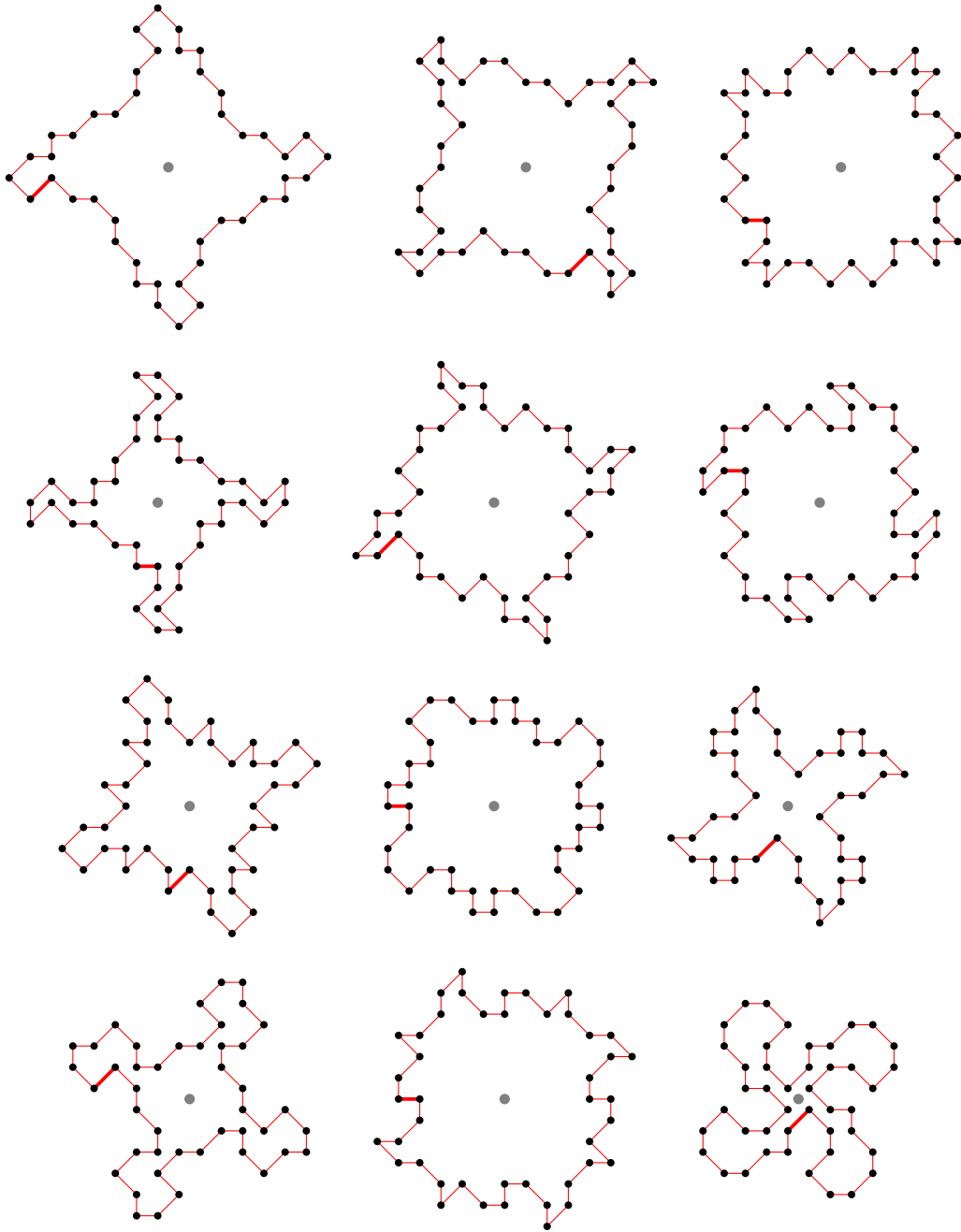
5.11.2 With largest area (362) there is 1 polygon

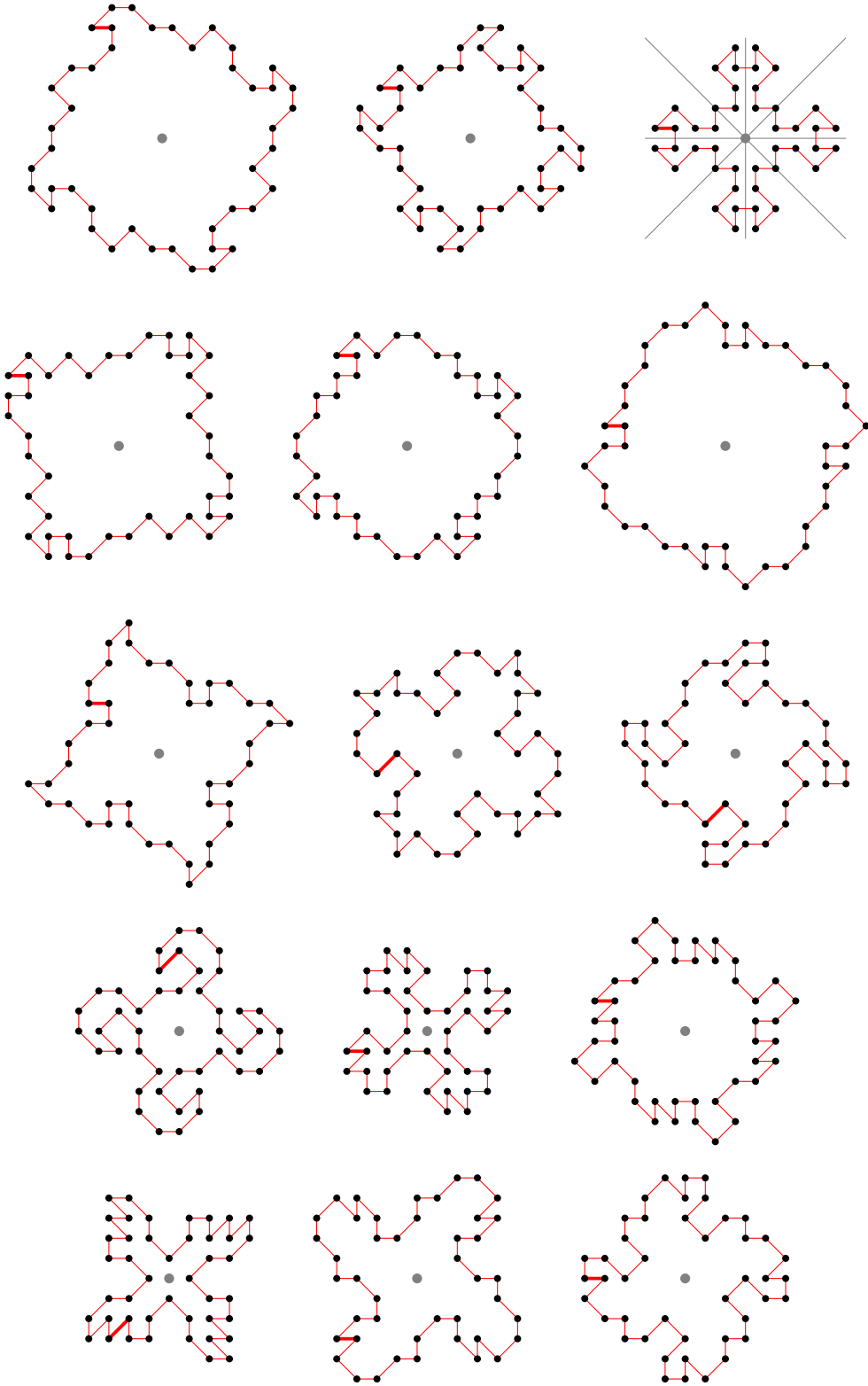


5.11.3 For other area sizes there are 343682 polygons

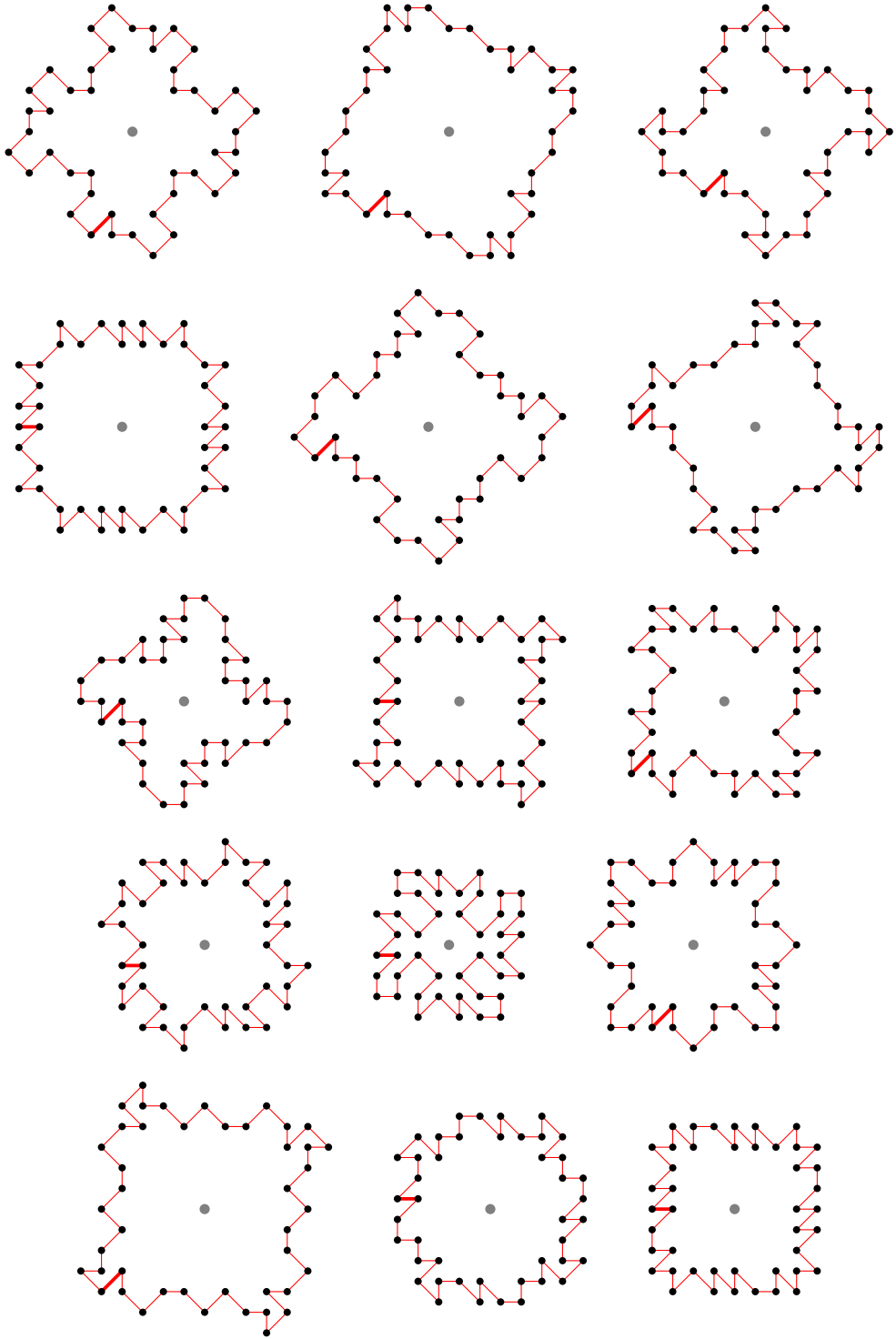
(100 samples)

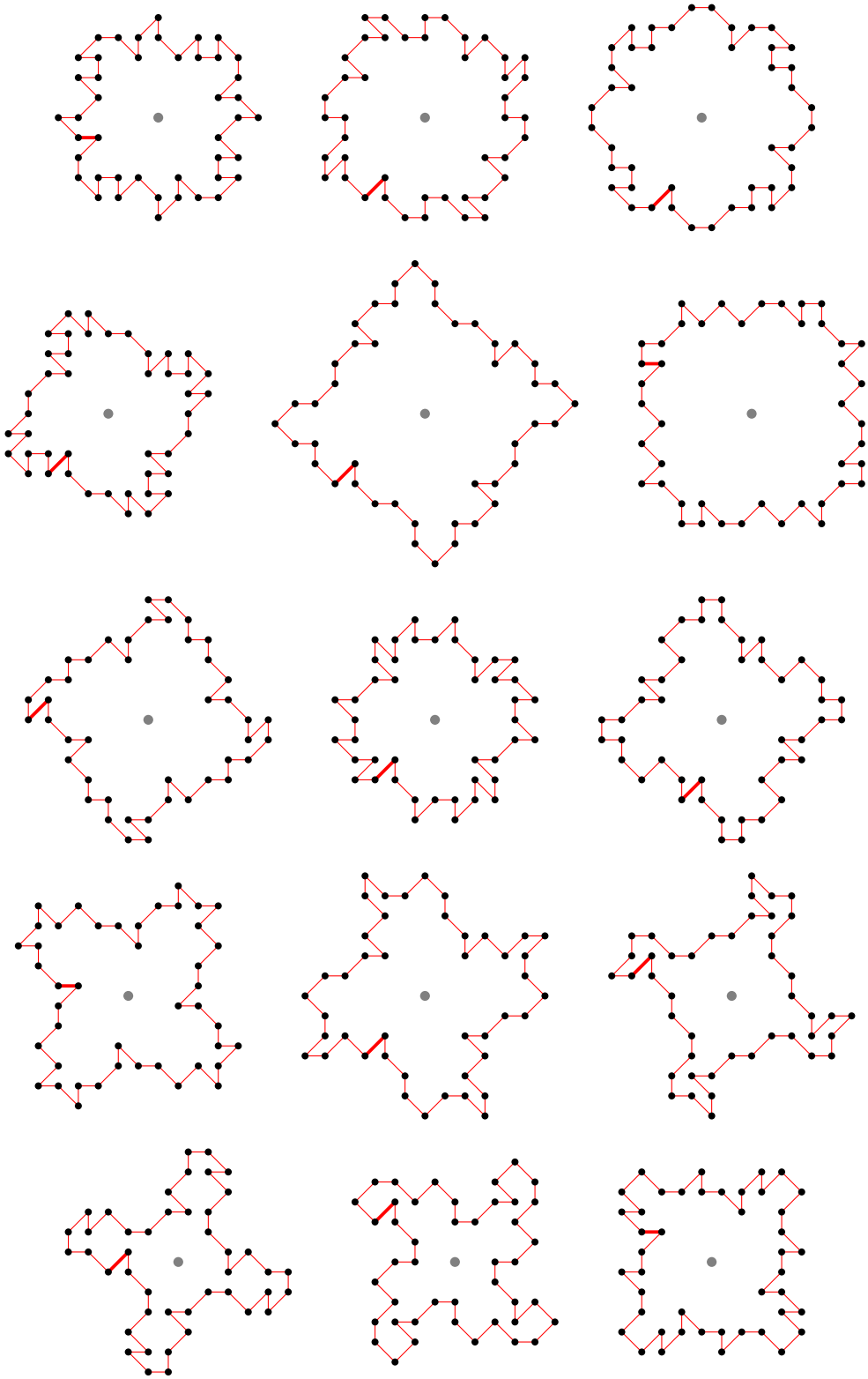


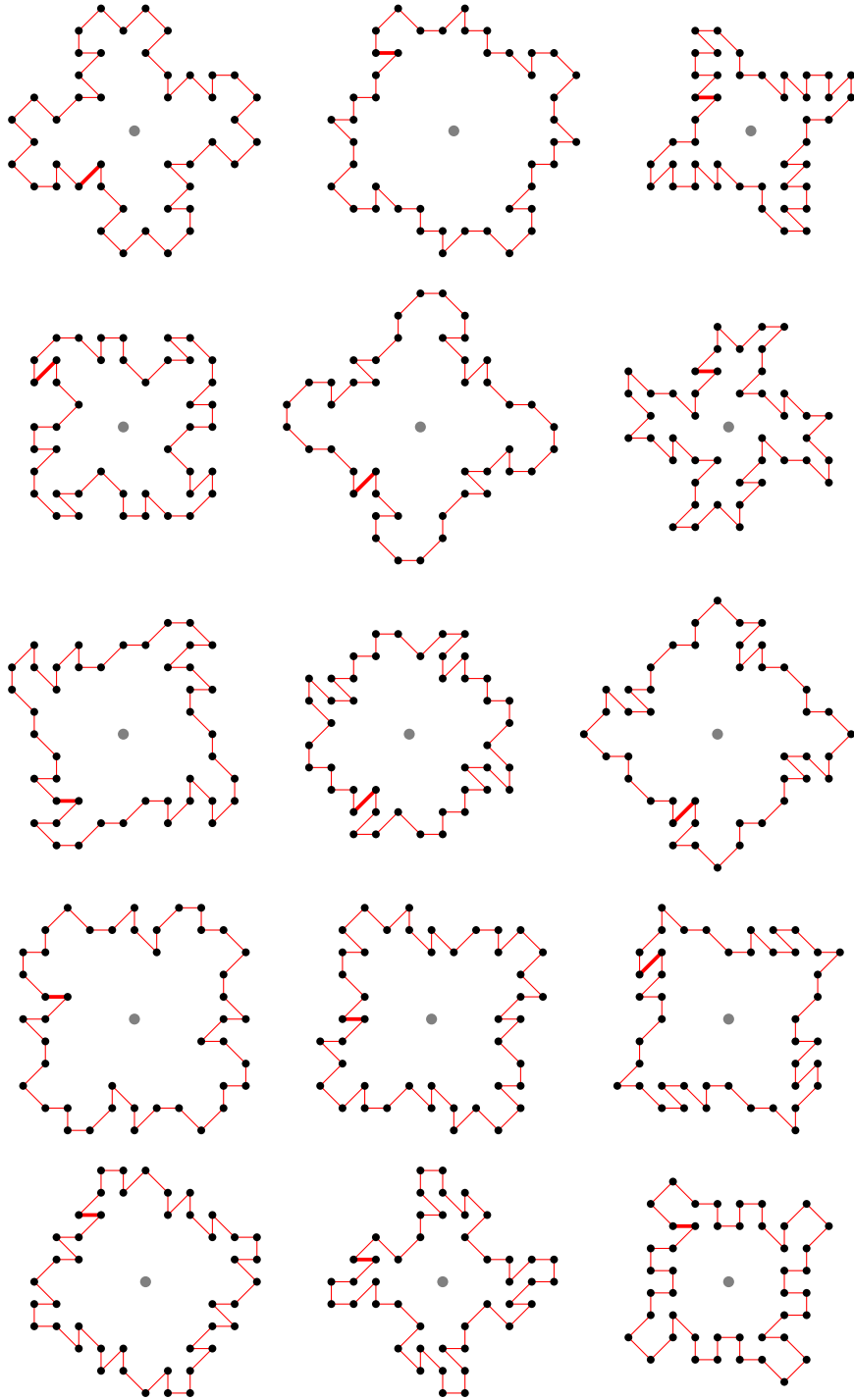


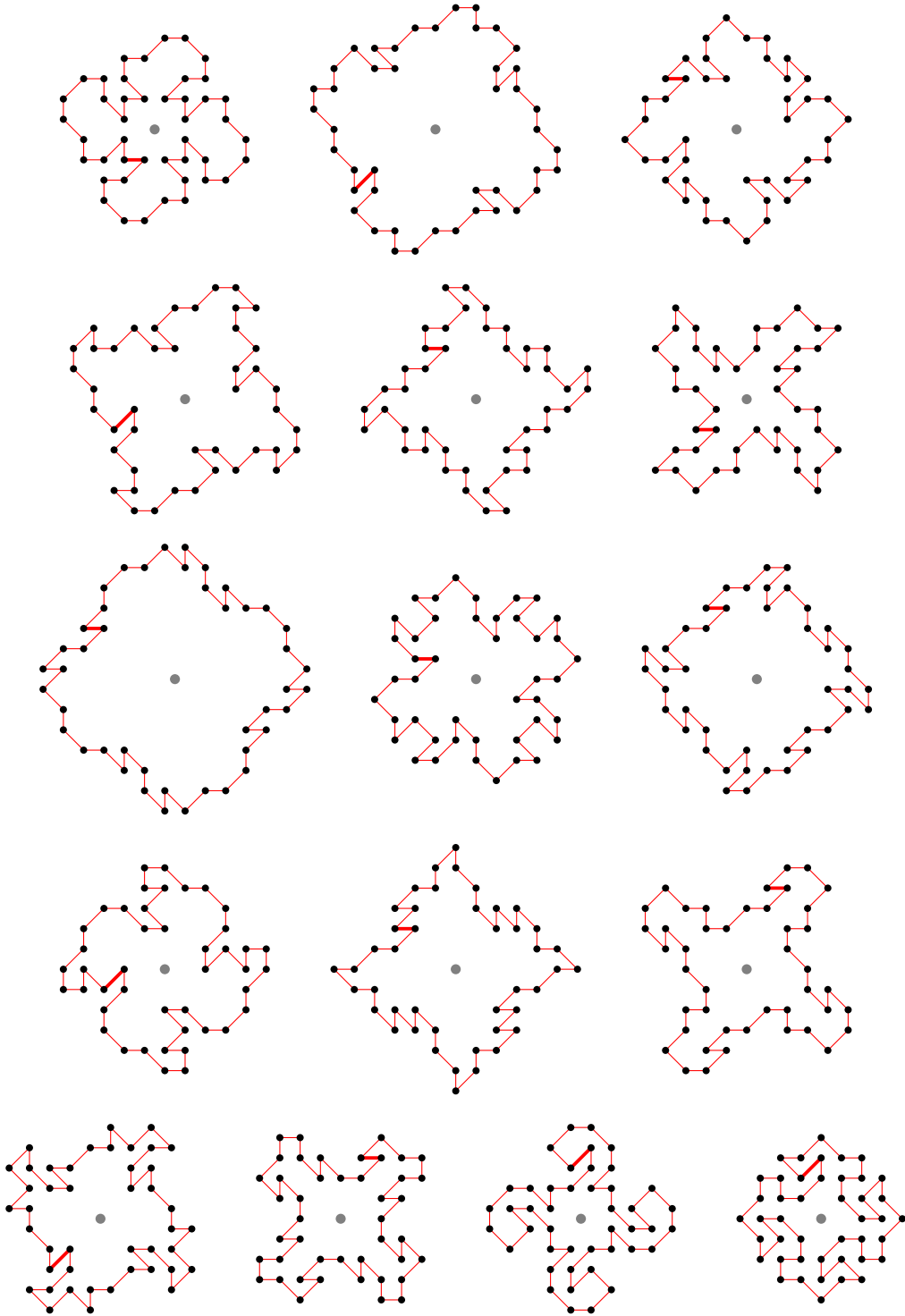


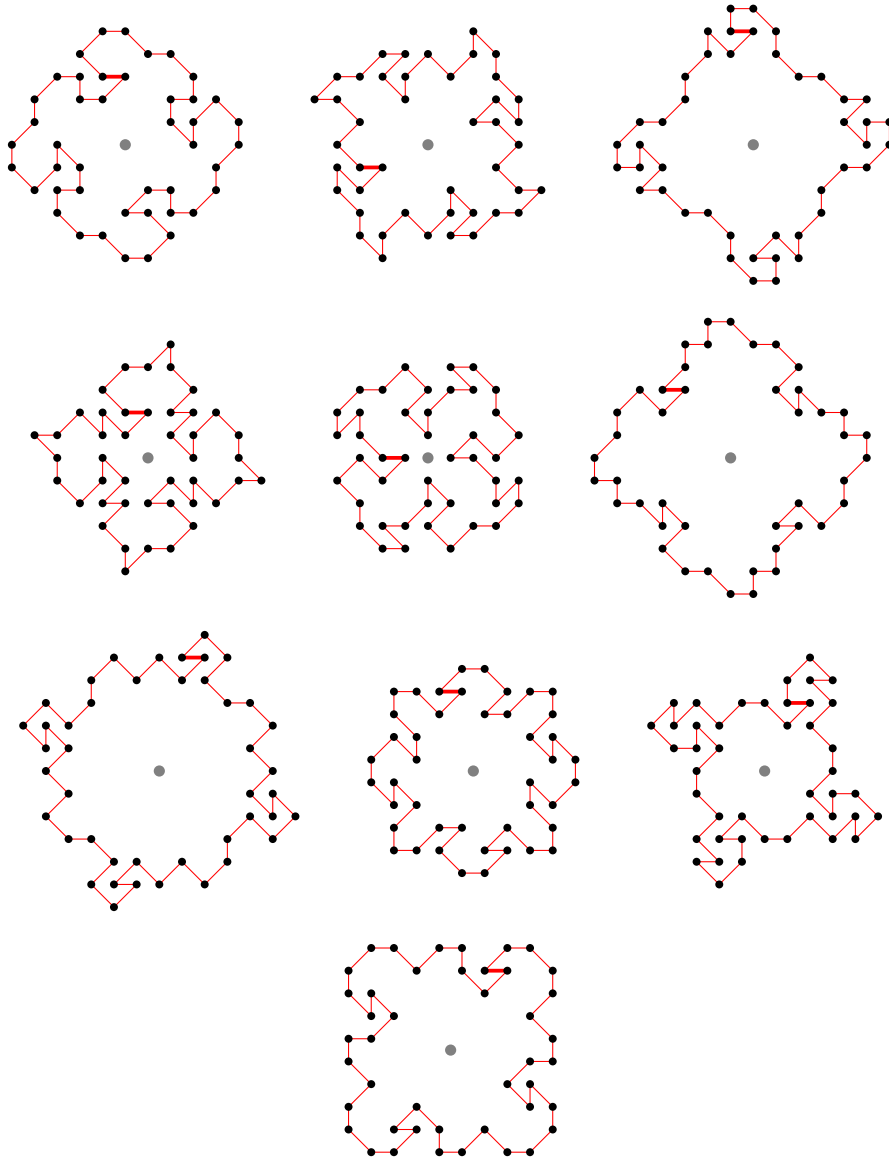








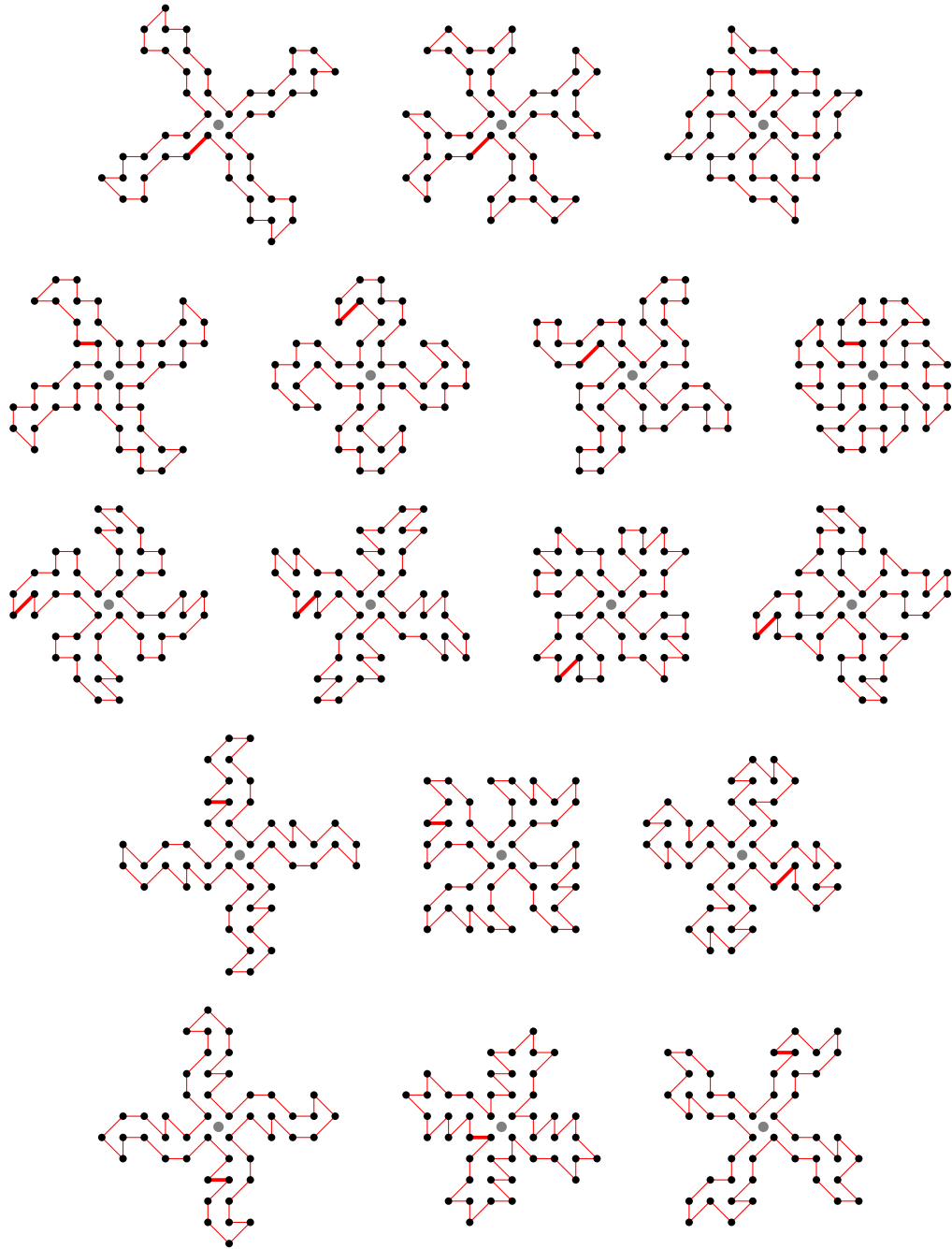


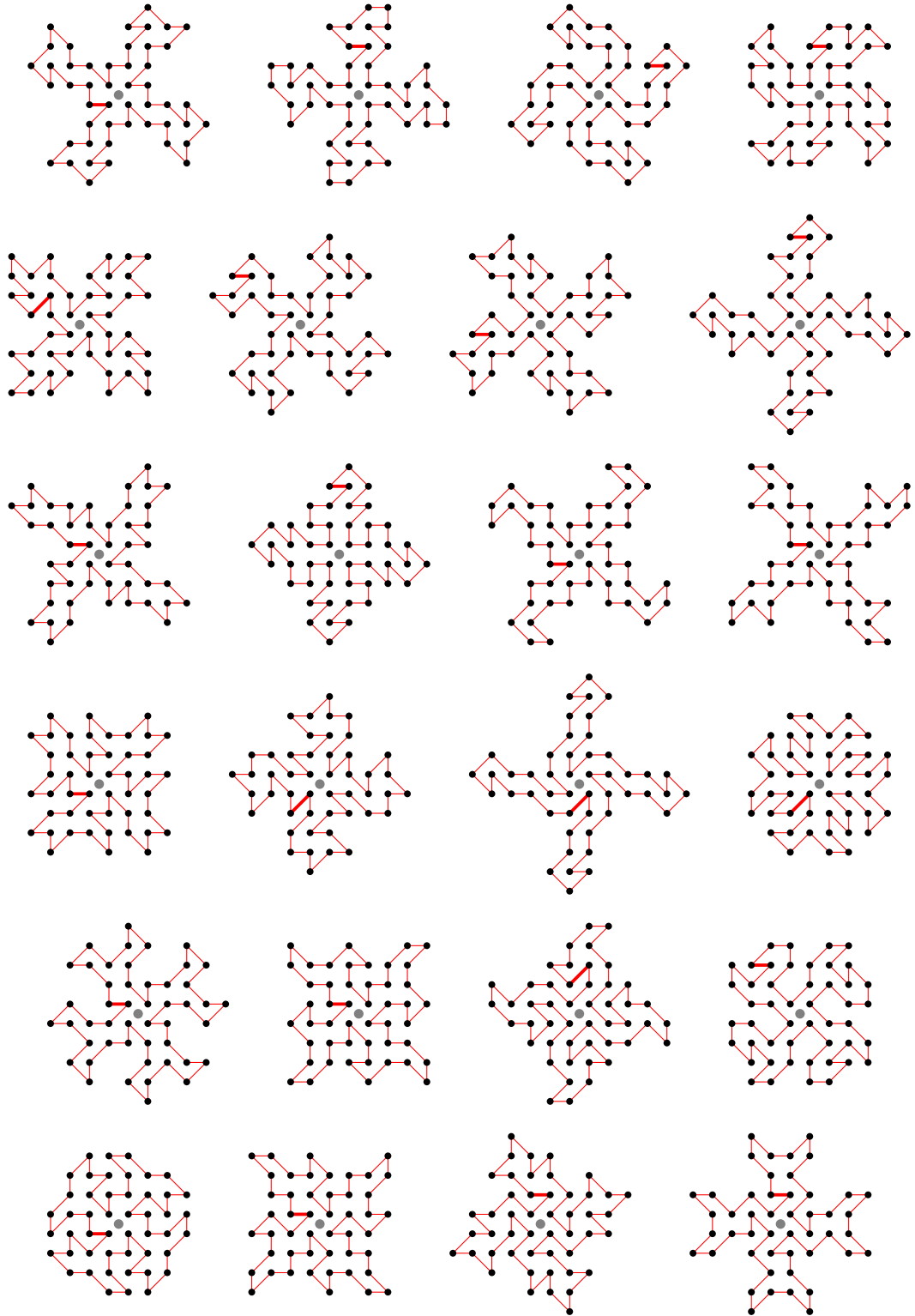


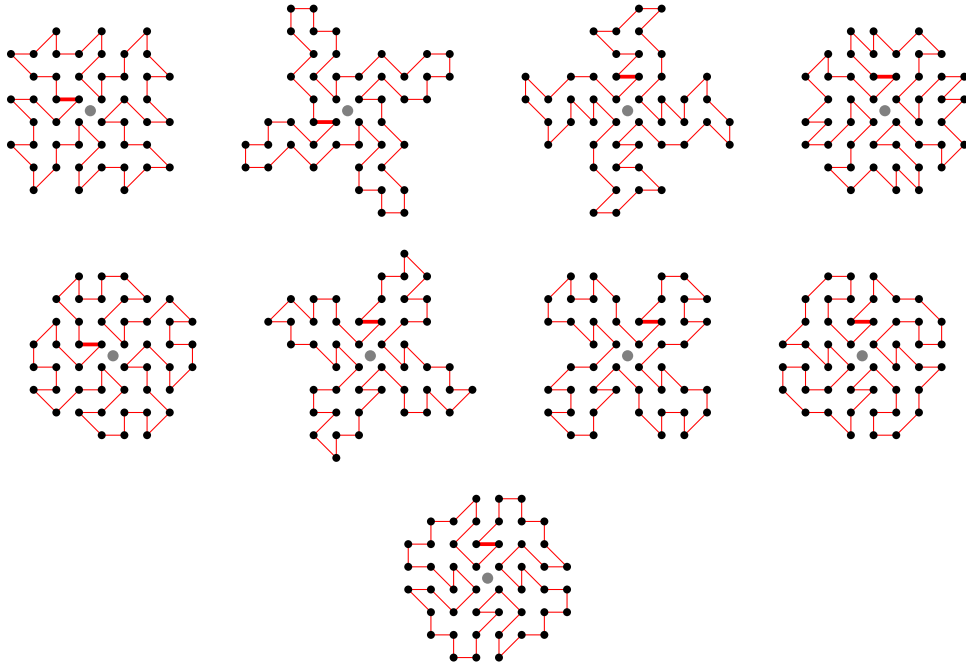
**5.12 For  $n = 48$  there are 1484065 polygons**

**5.12.1 With smallest area (46) there are 6013 polygons**

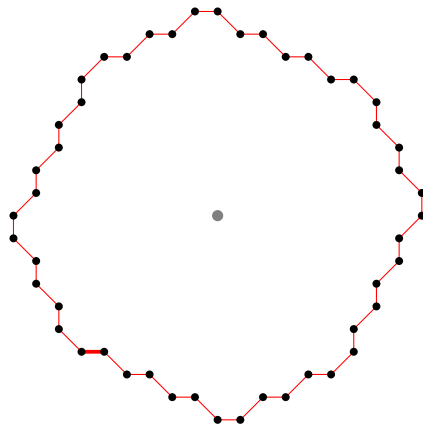
(50 samples)







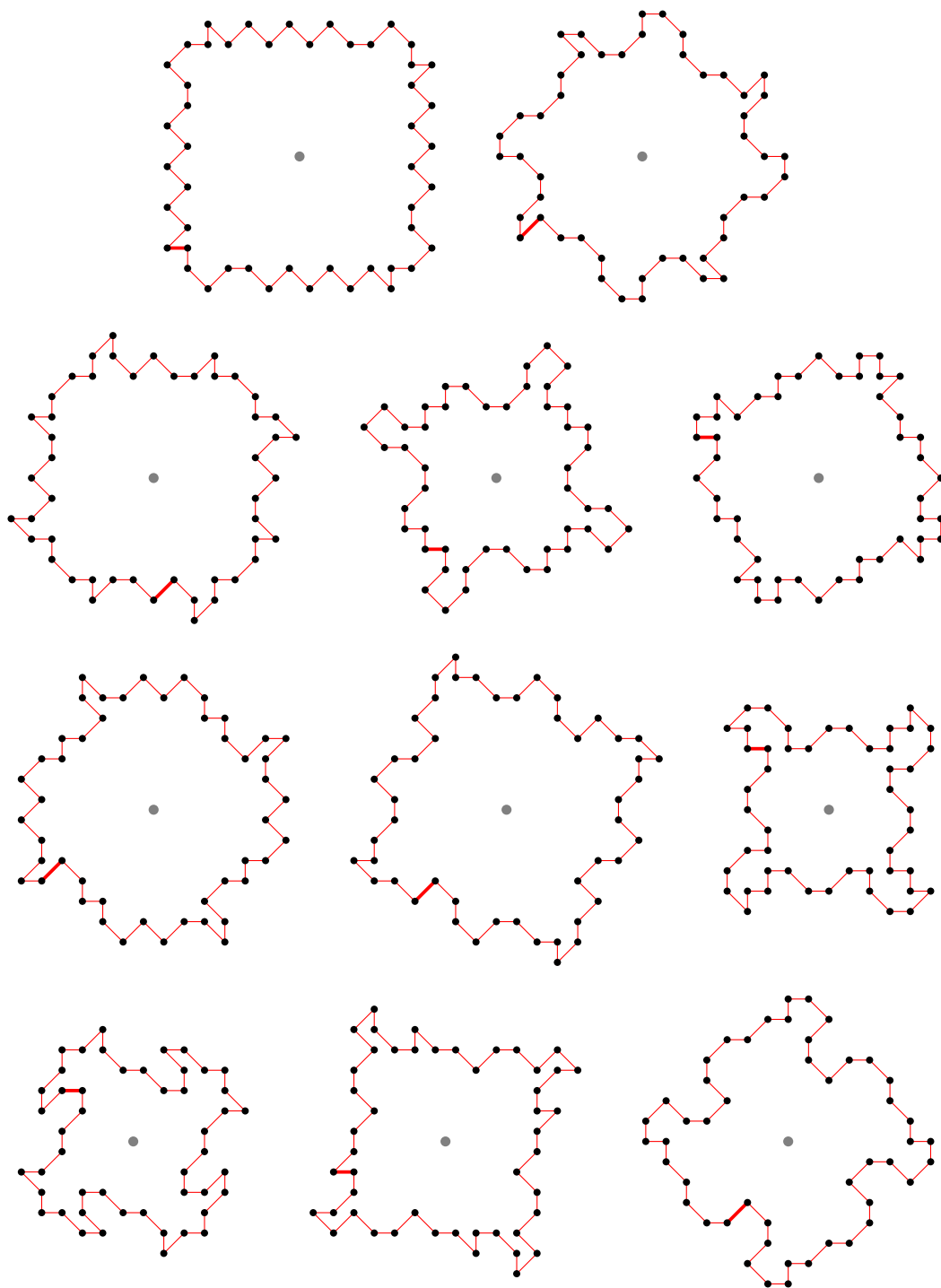
5.12.2 With largest area (432) there is 1 polygon

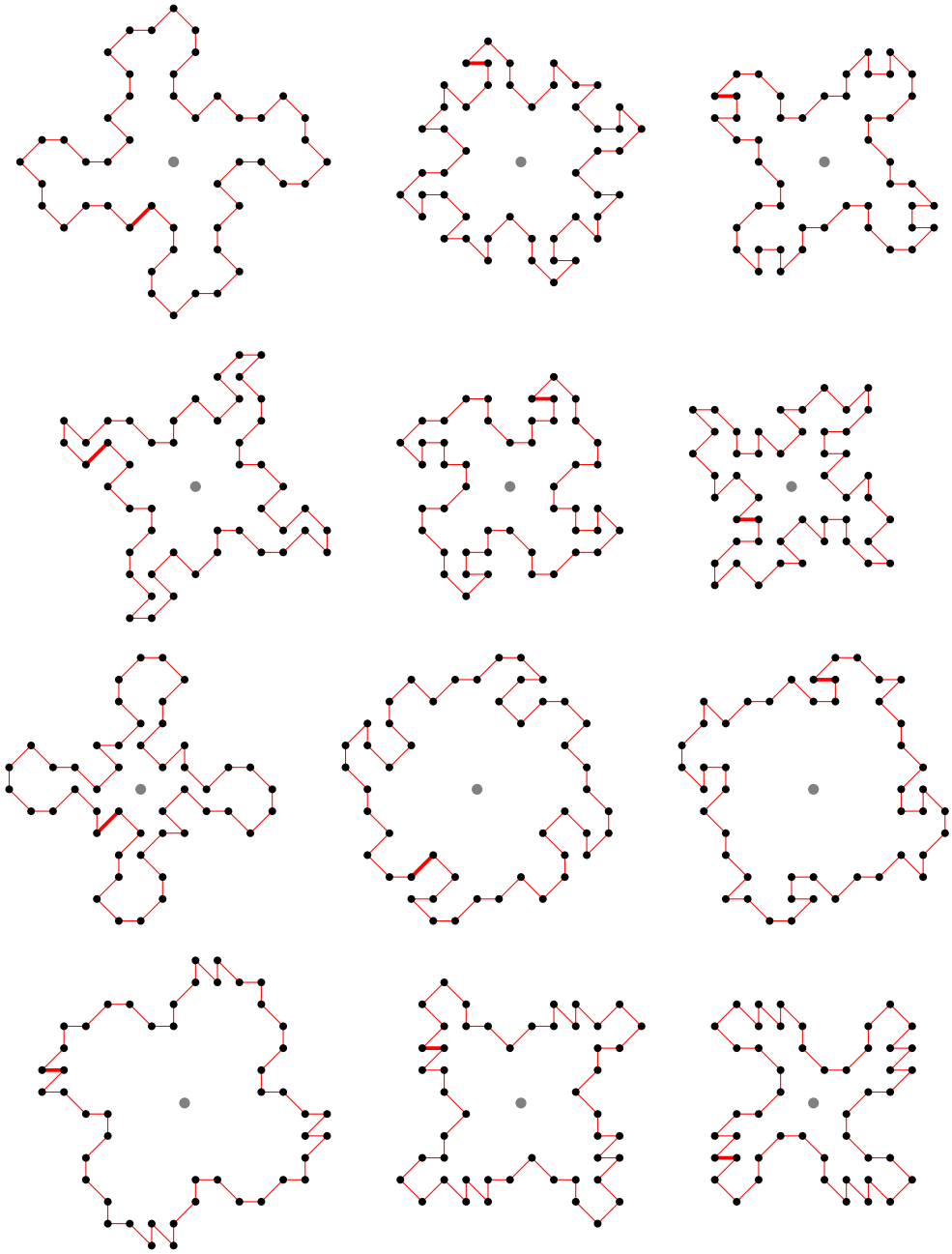


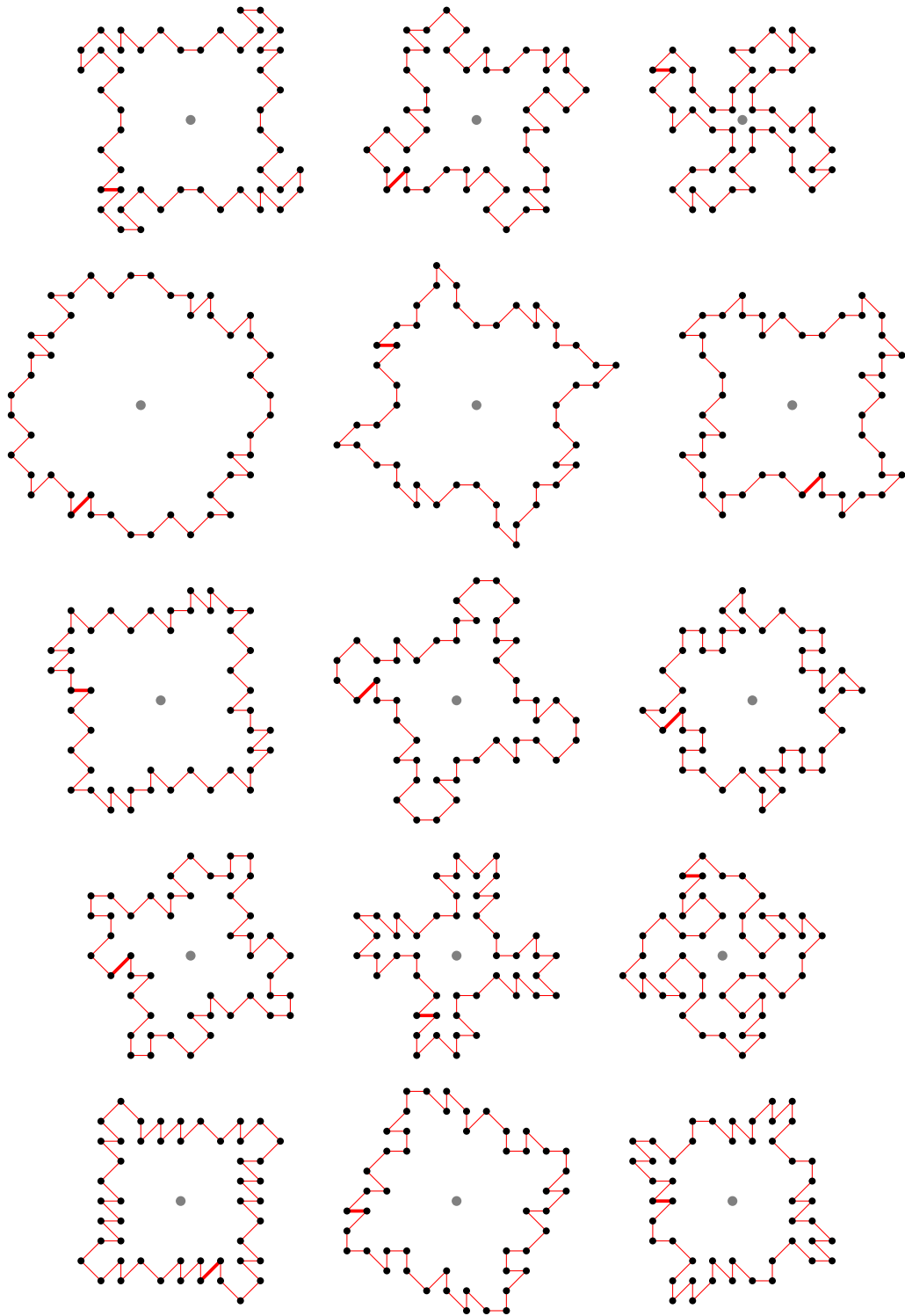
5.12.3 For other area sizes there are 1478051 polygons

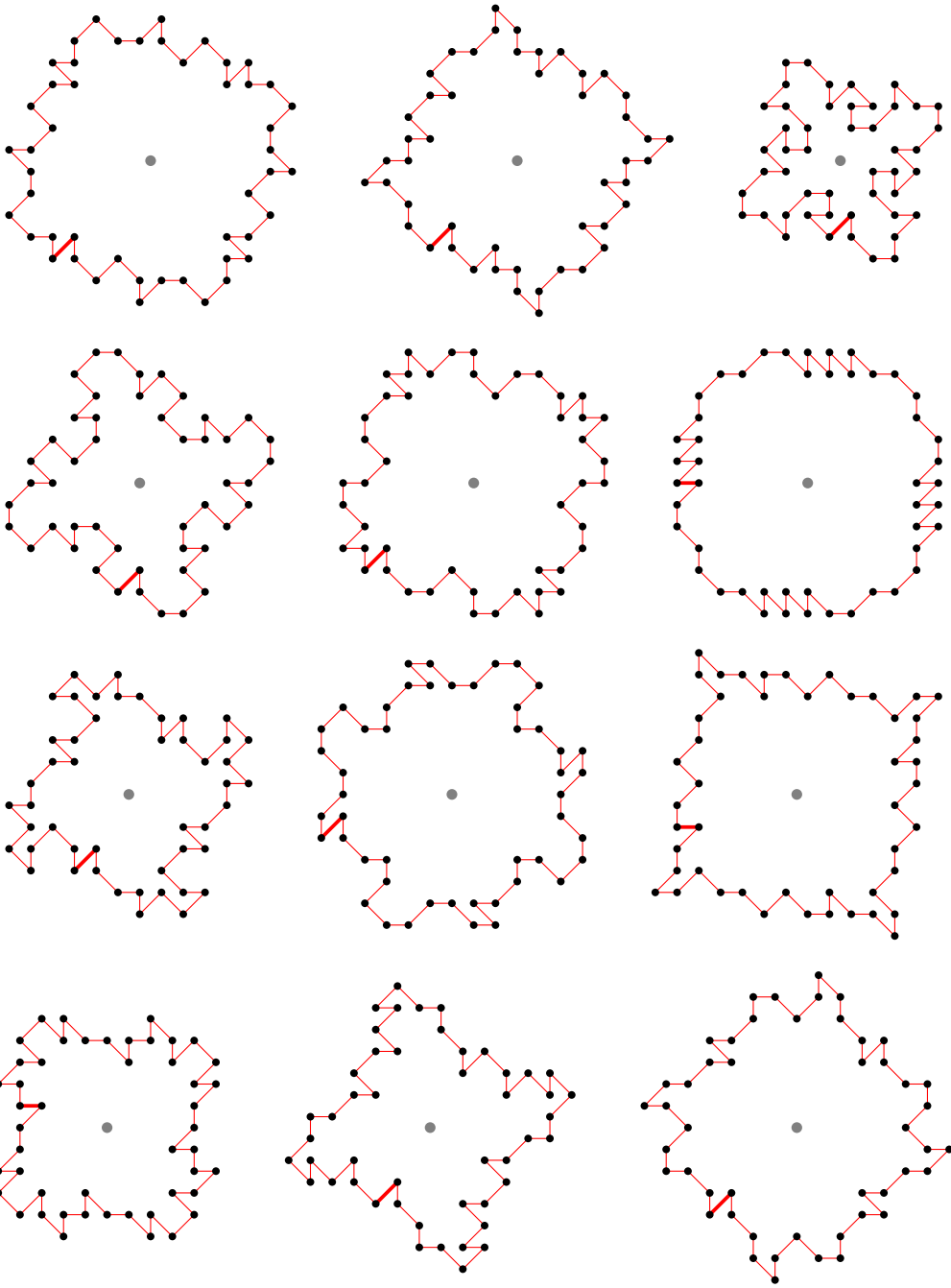
(100 samples)

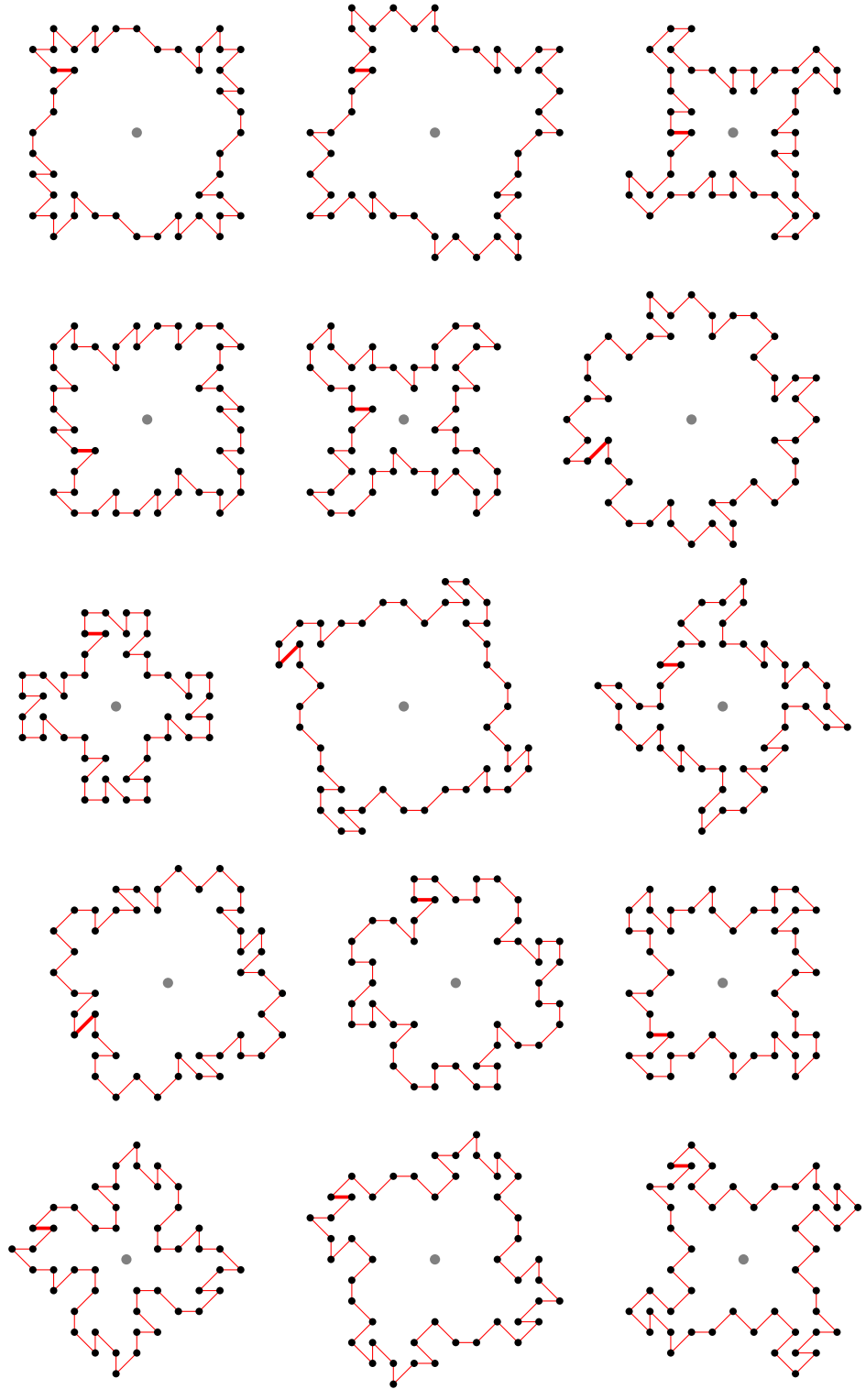


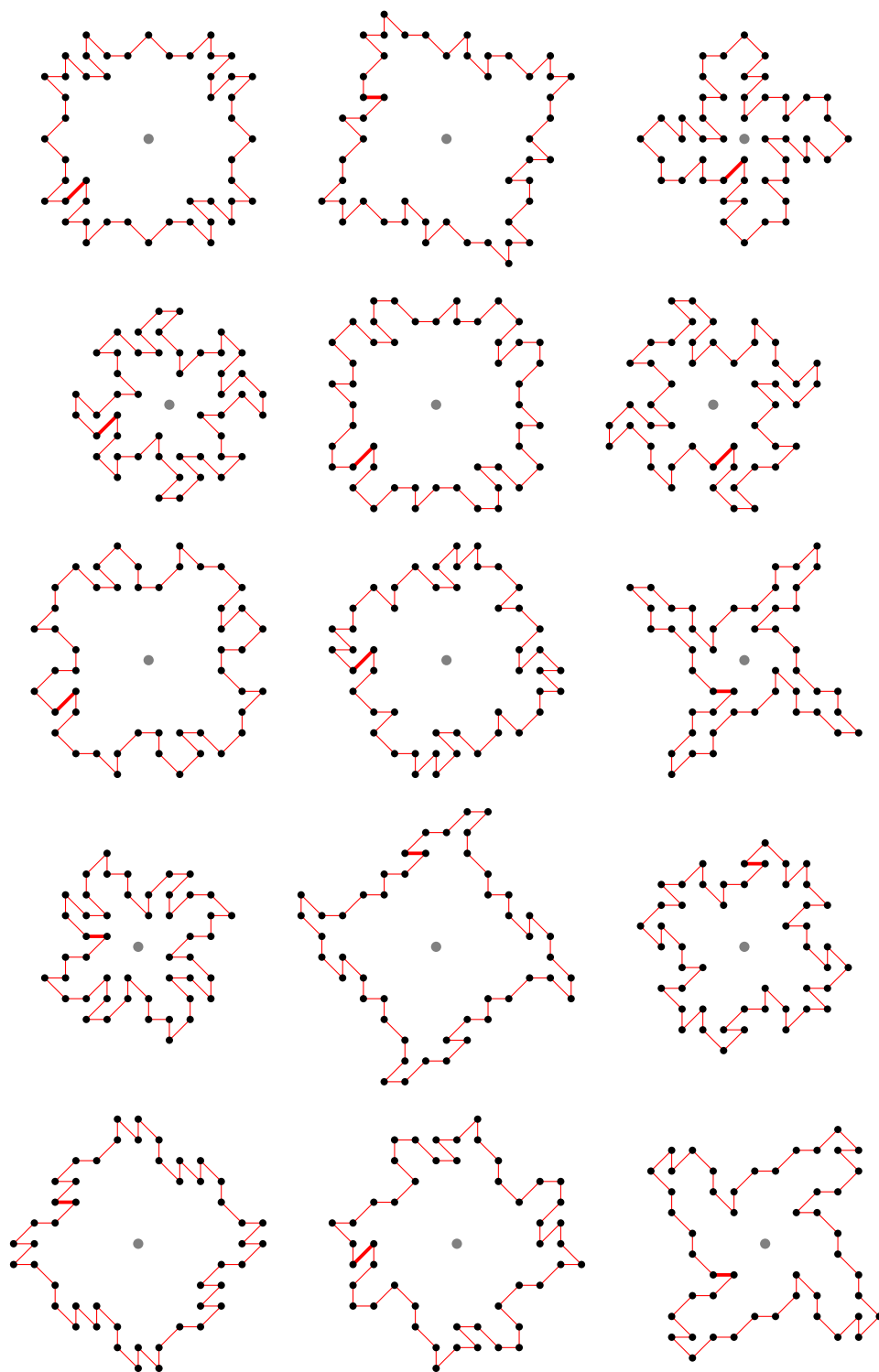


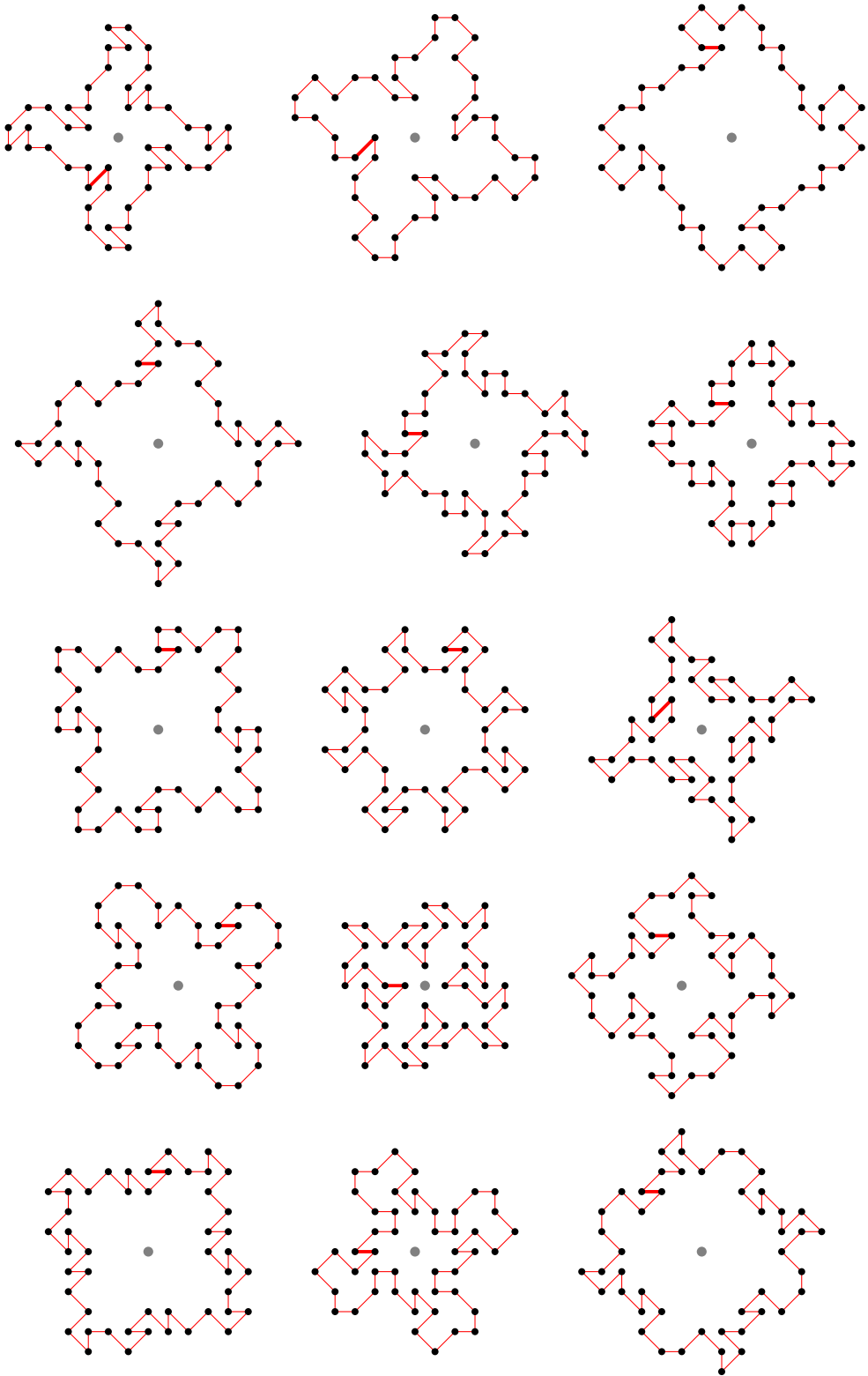


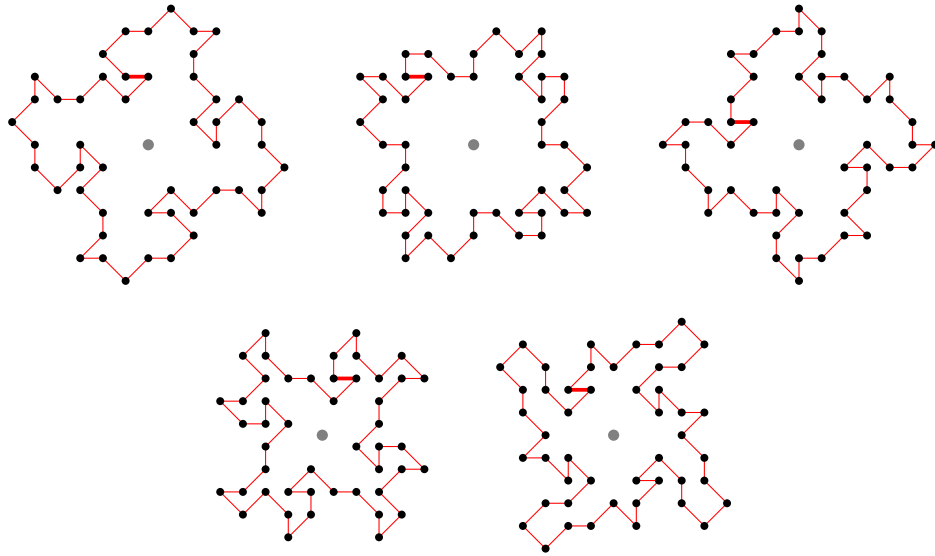








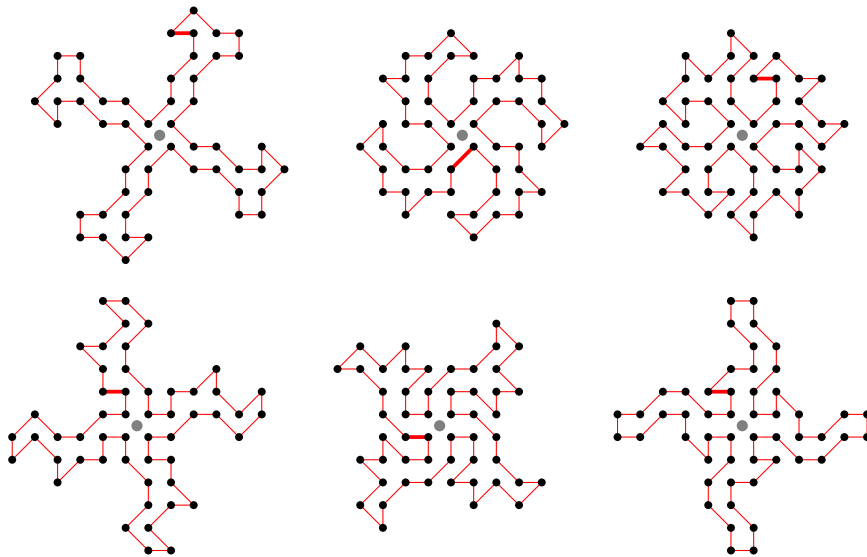




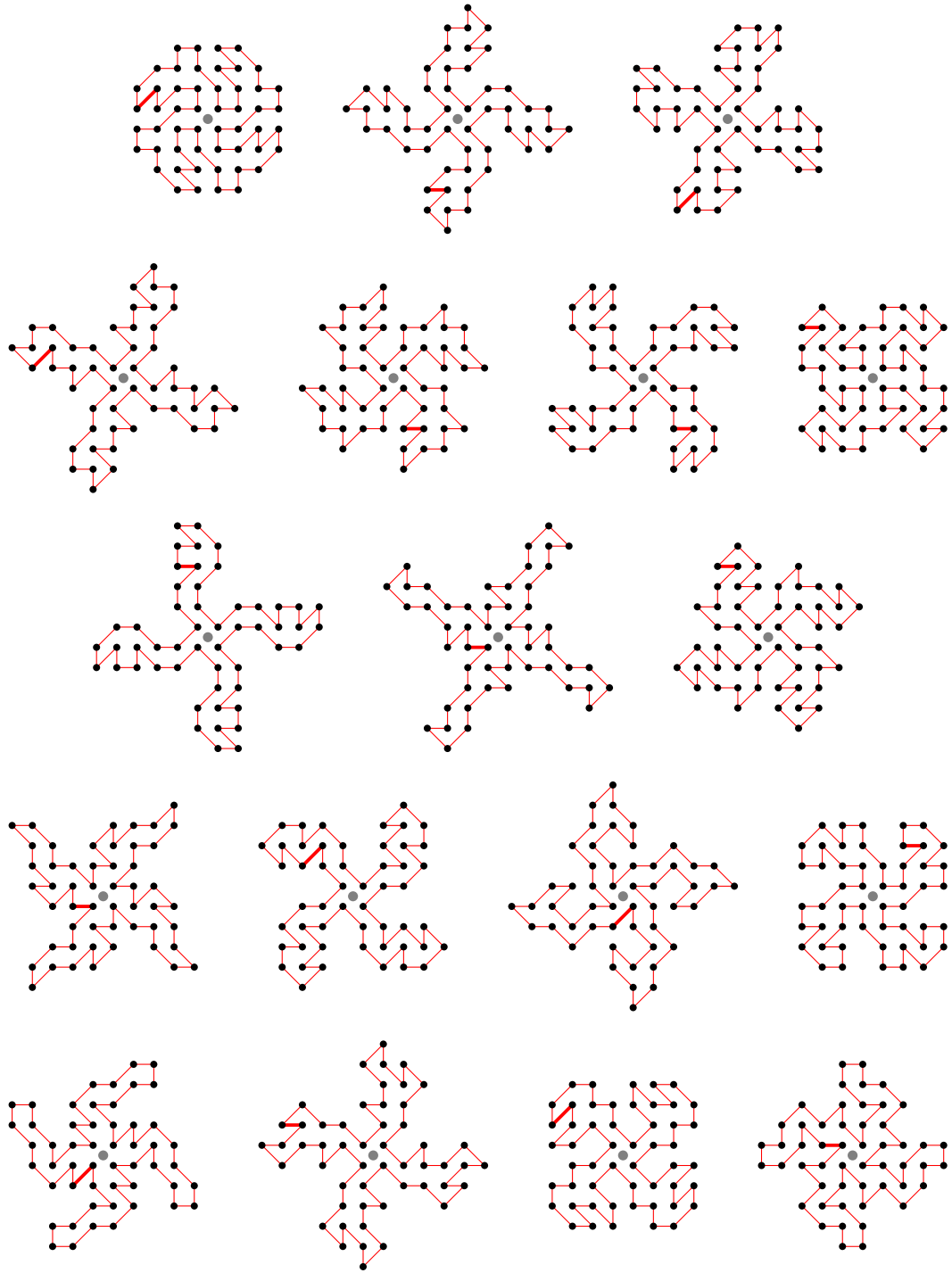
5.13 For  $n = 52$  there are 6414999 polygons

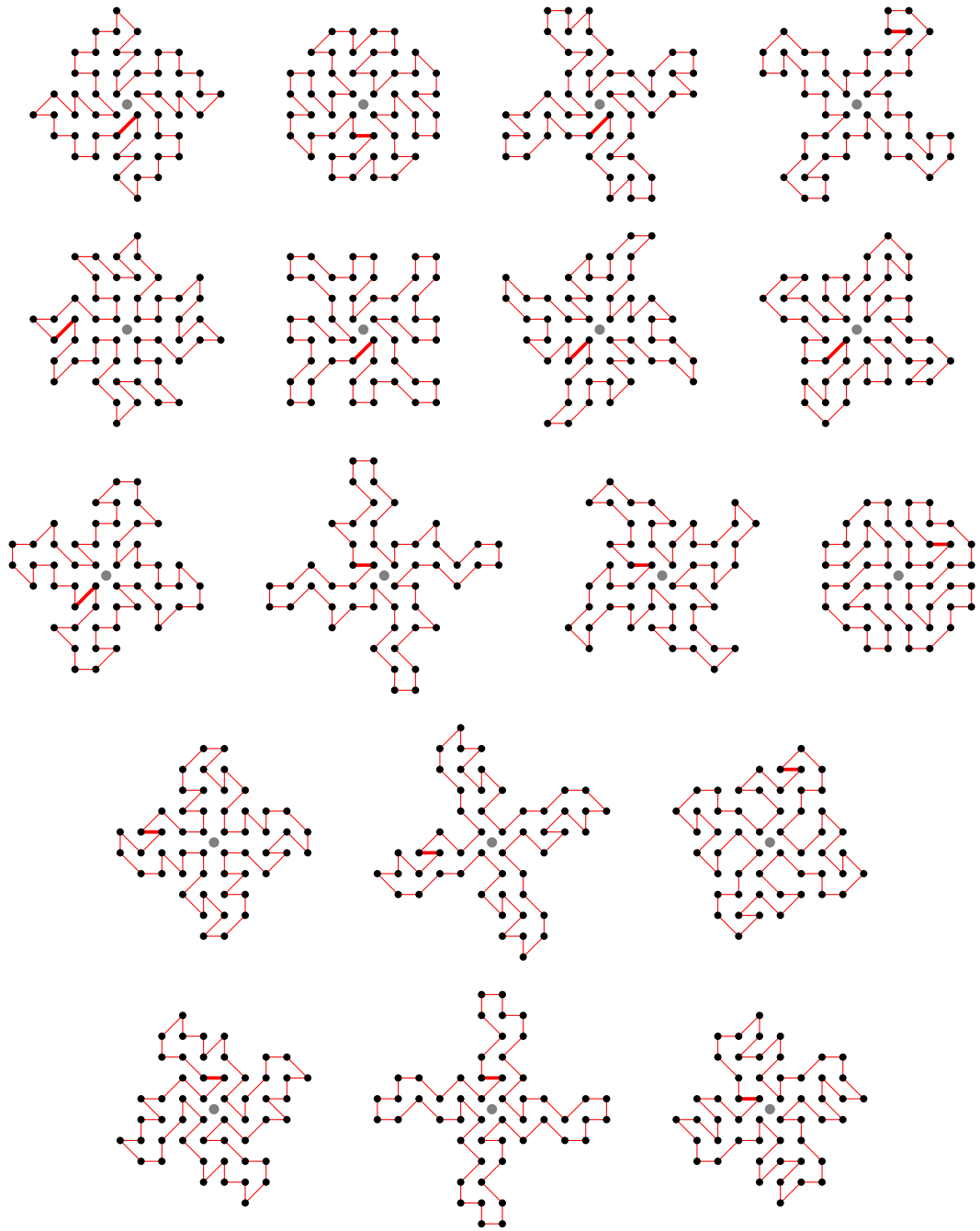
5.13.1 With smallest area (50) there are 16733 polygons

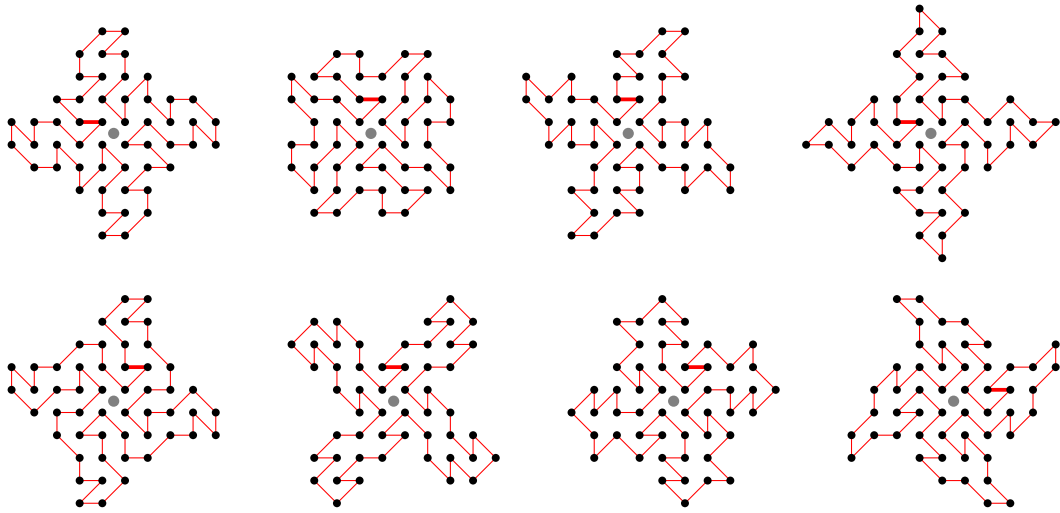
(50 samples)



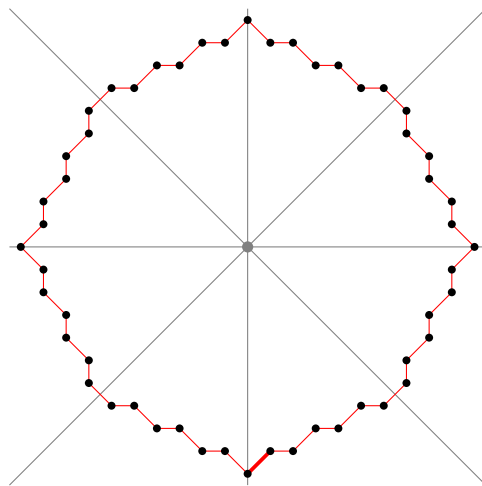






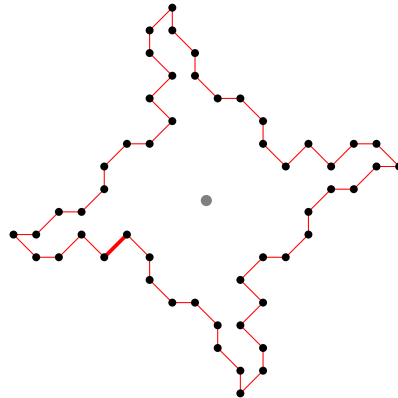
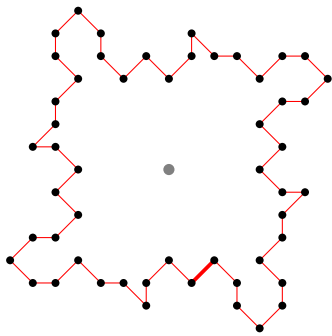
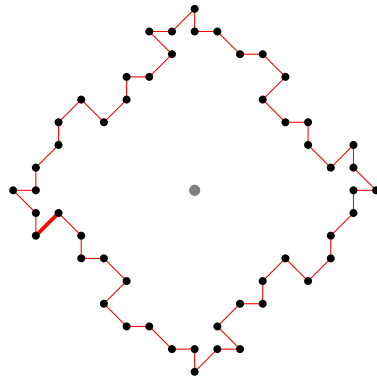
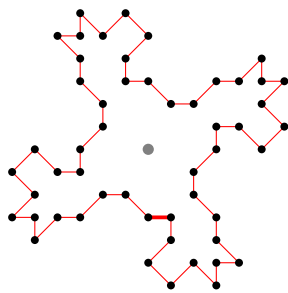
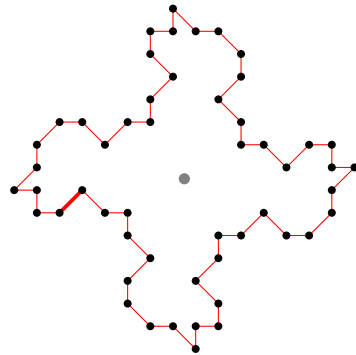
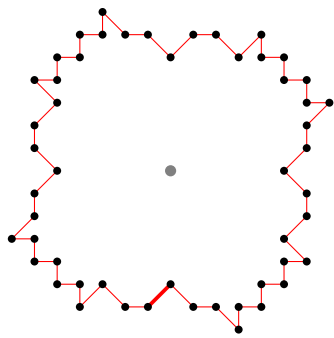


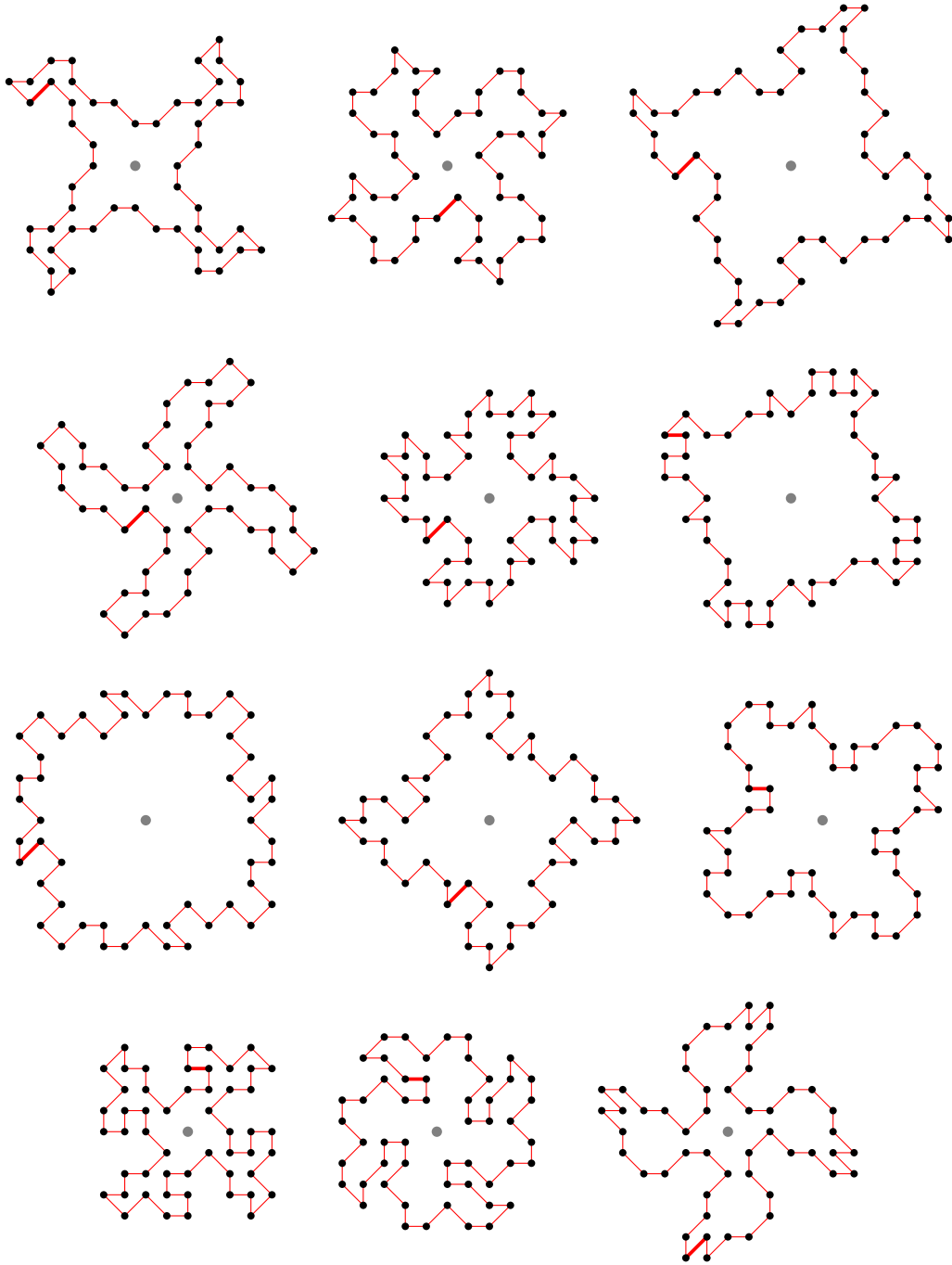
5.13.2 With largest area (508) there is 1 polygon

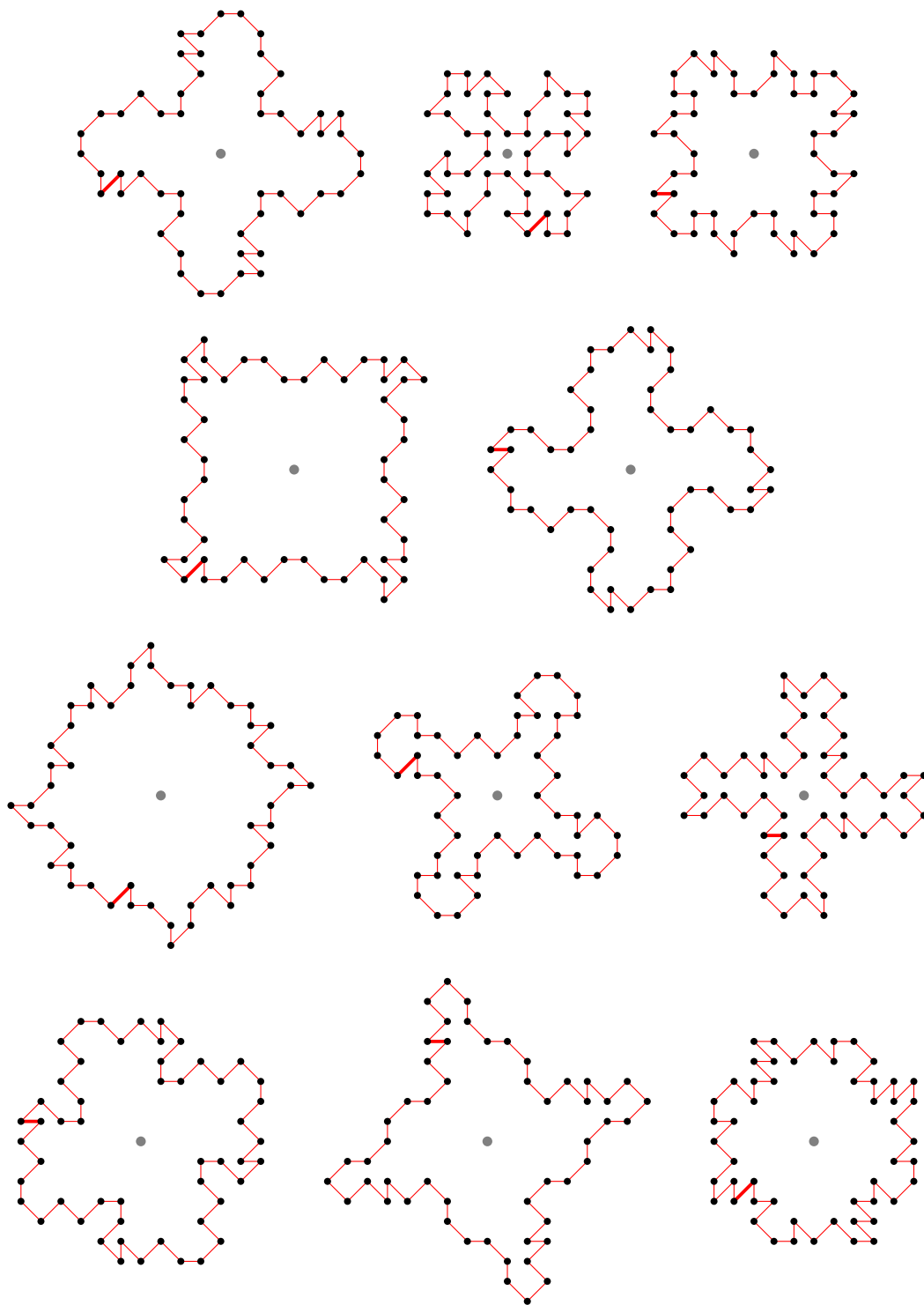


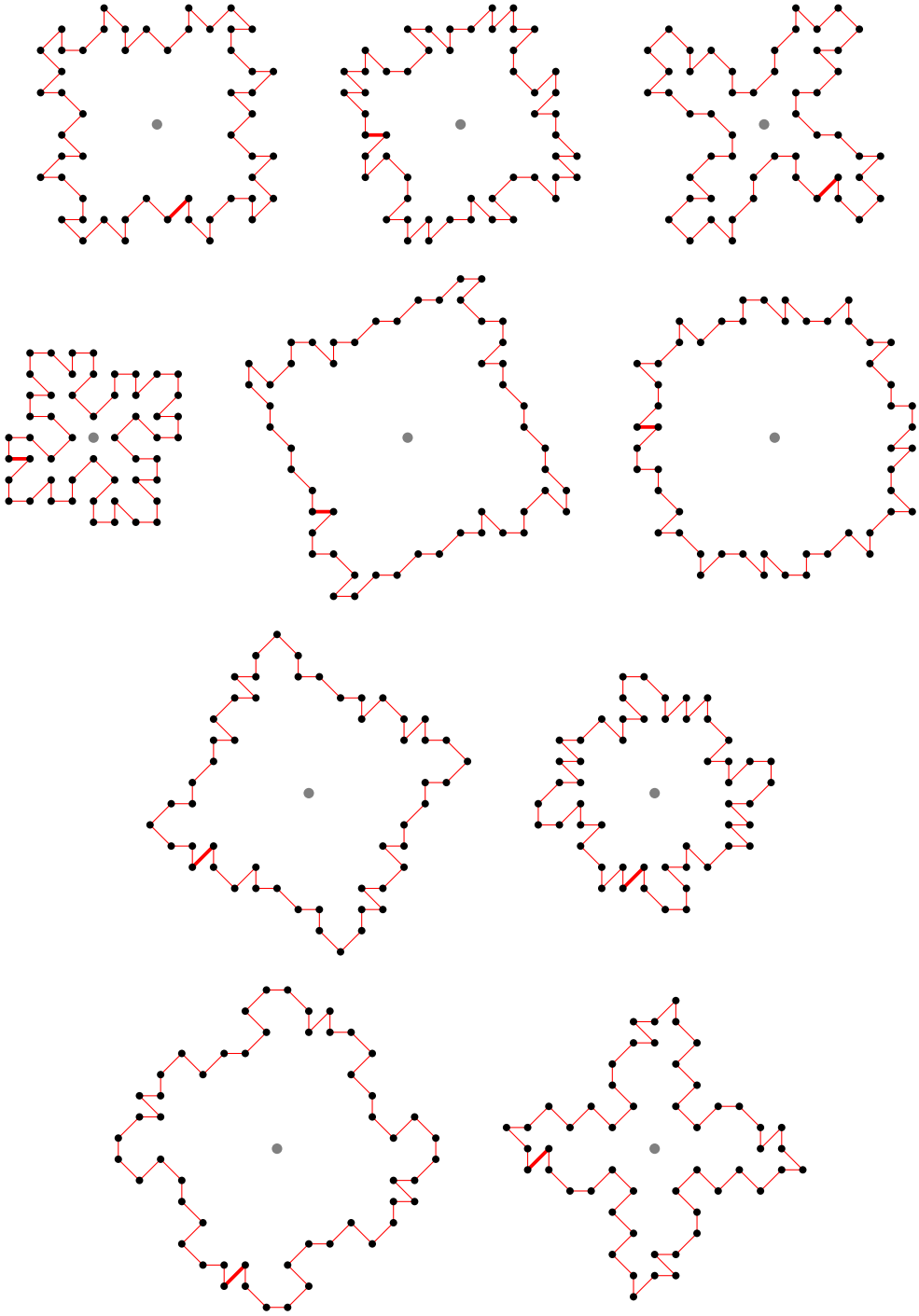
5.13.3 For other area sizes there are 6398265 polygons

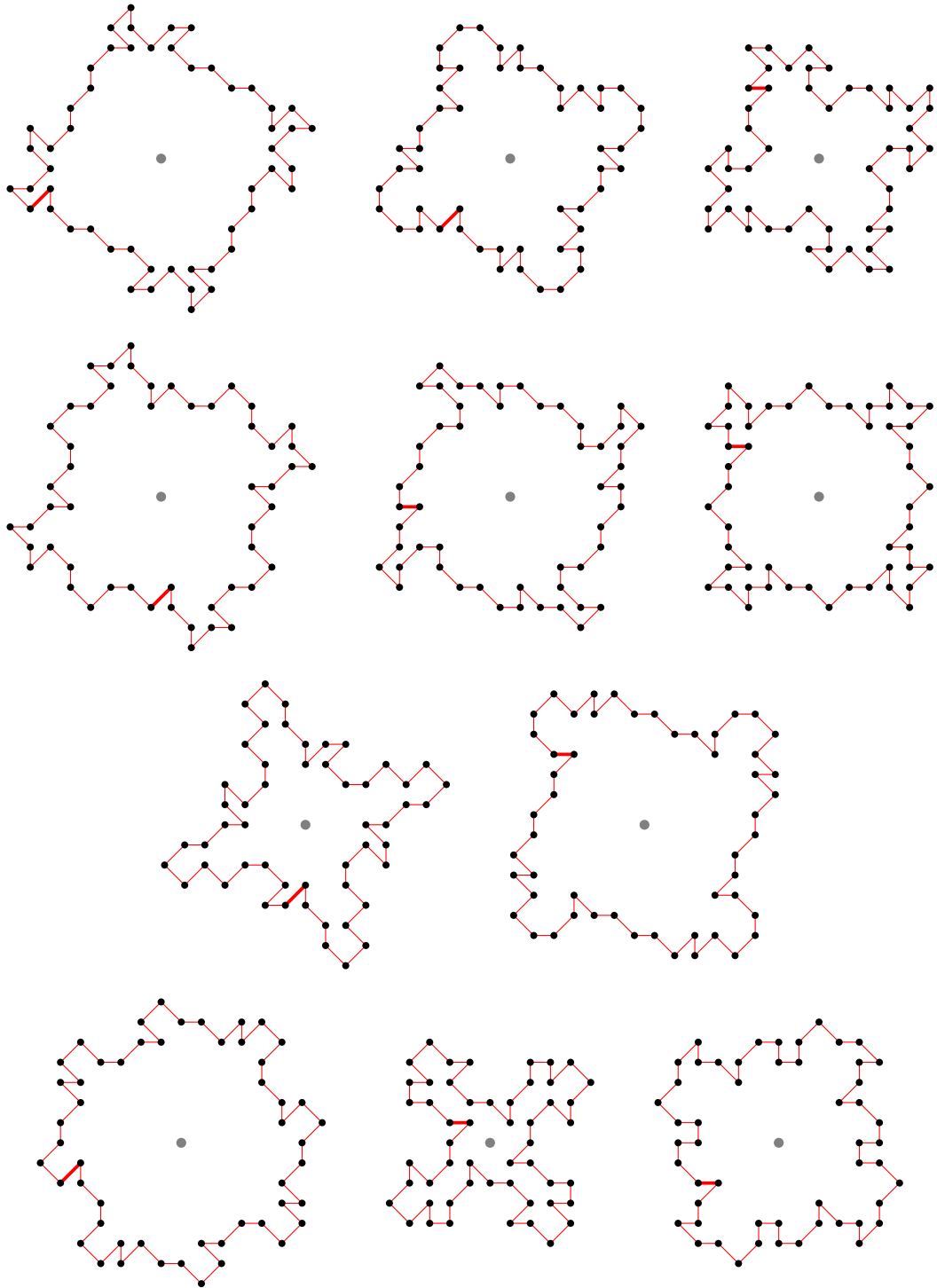
(100 samples)



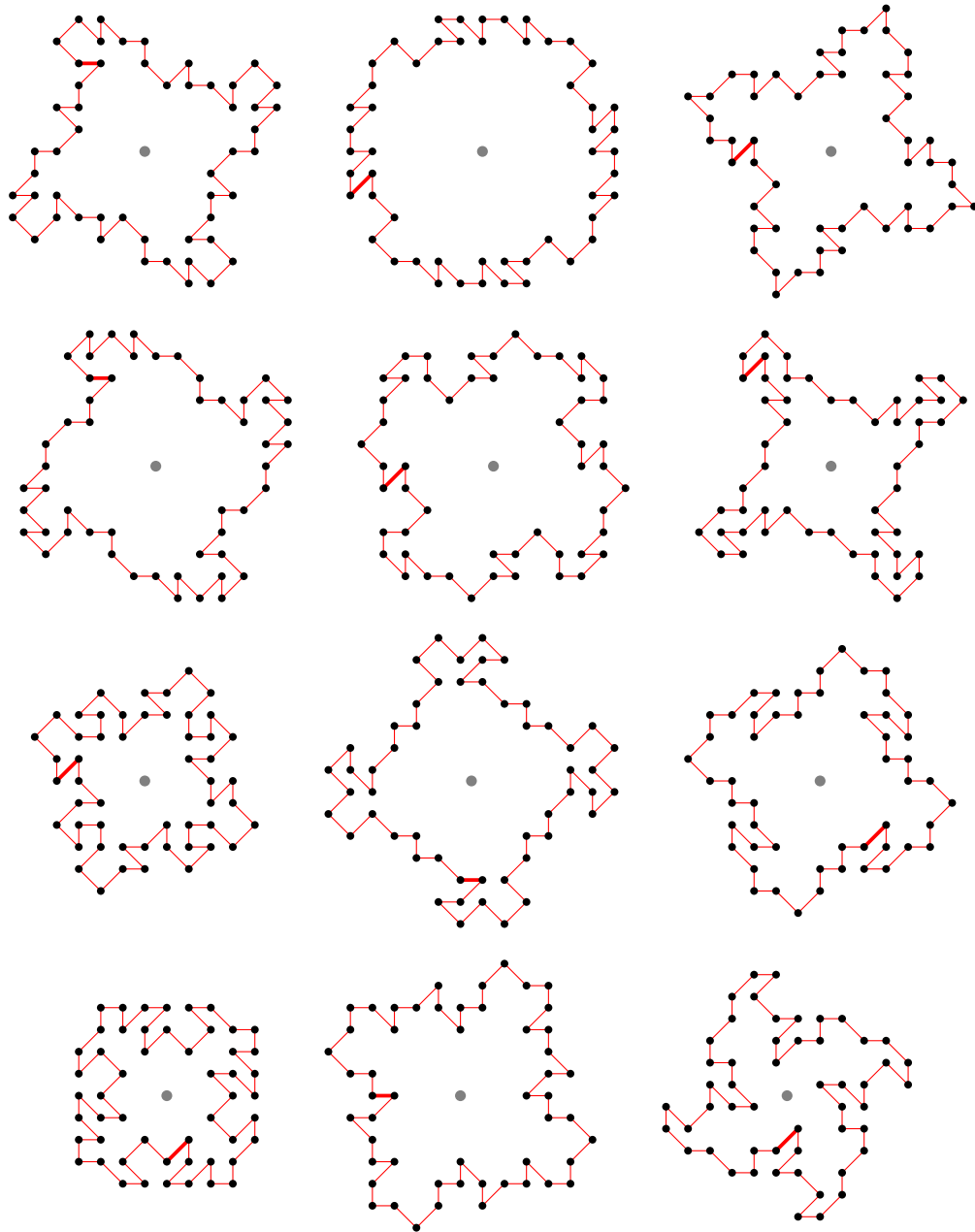


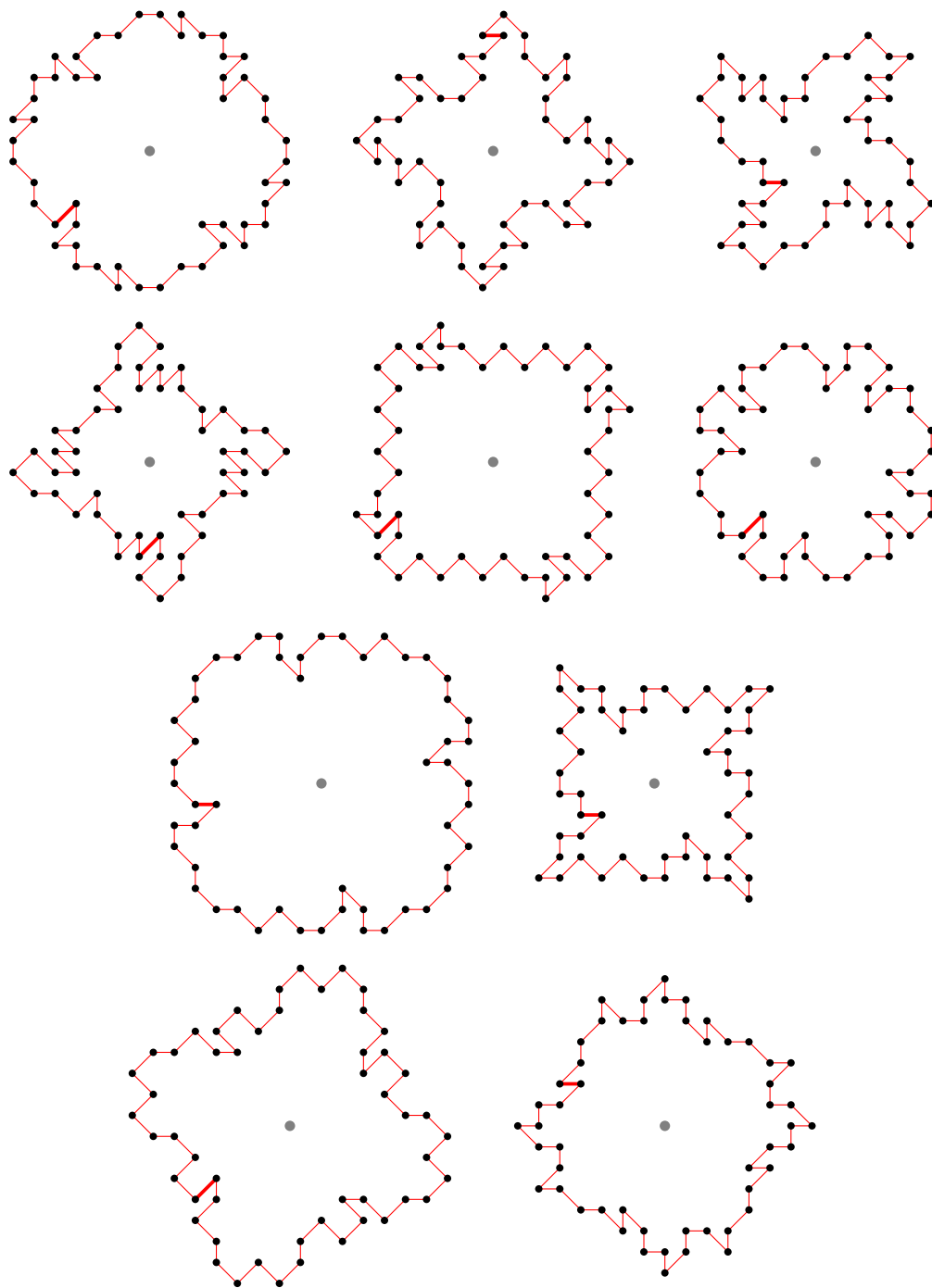


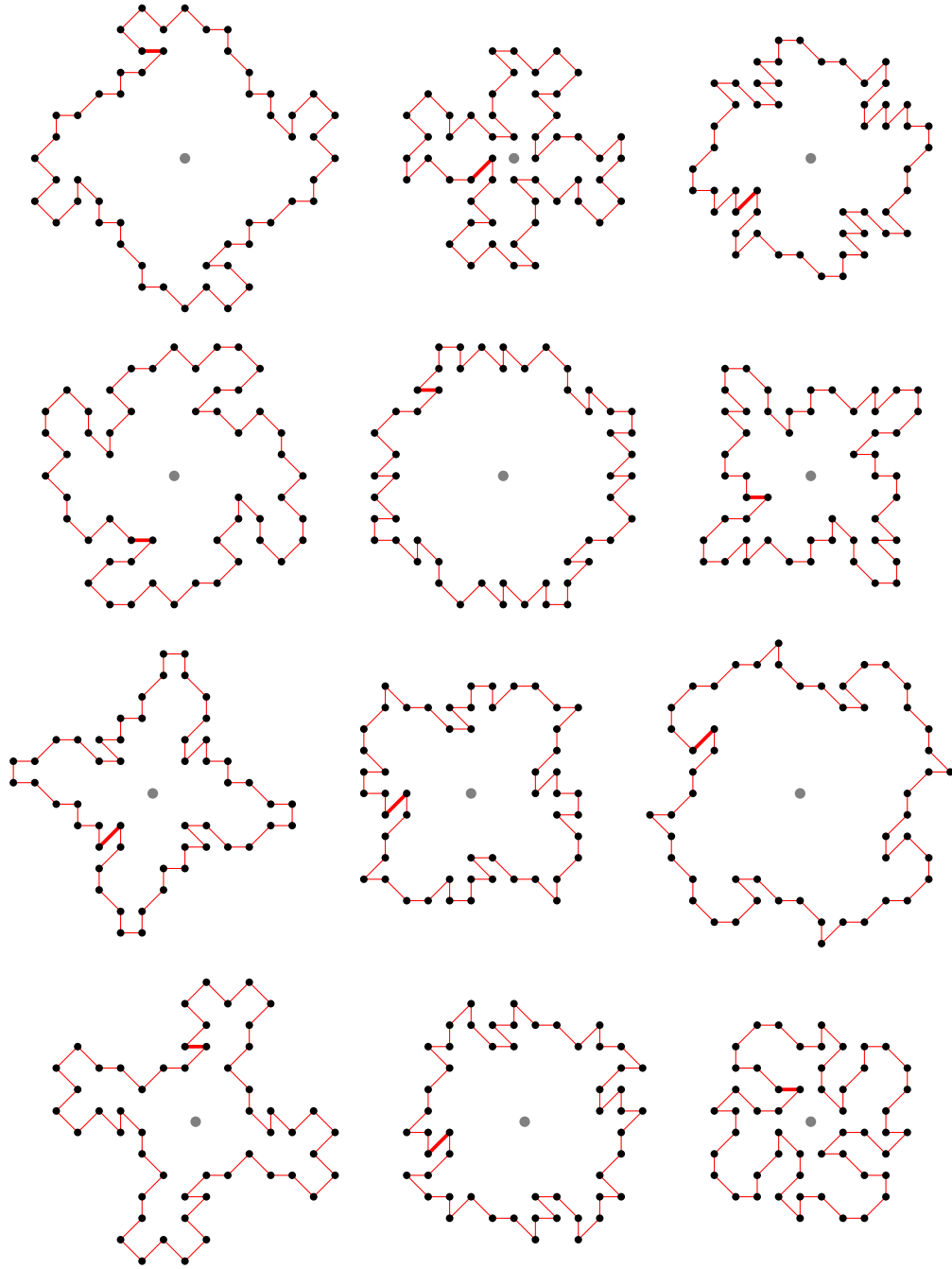


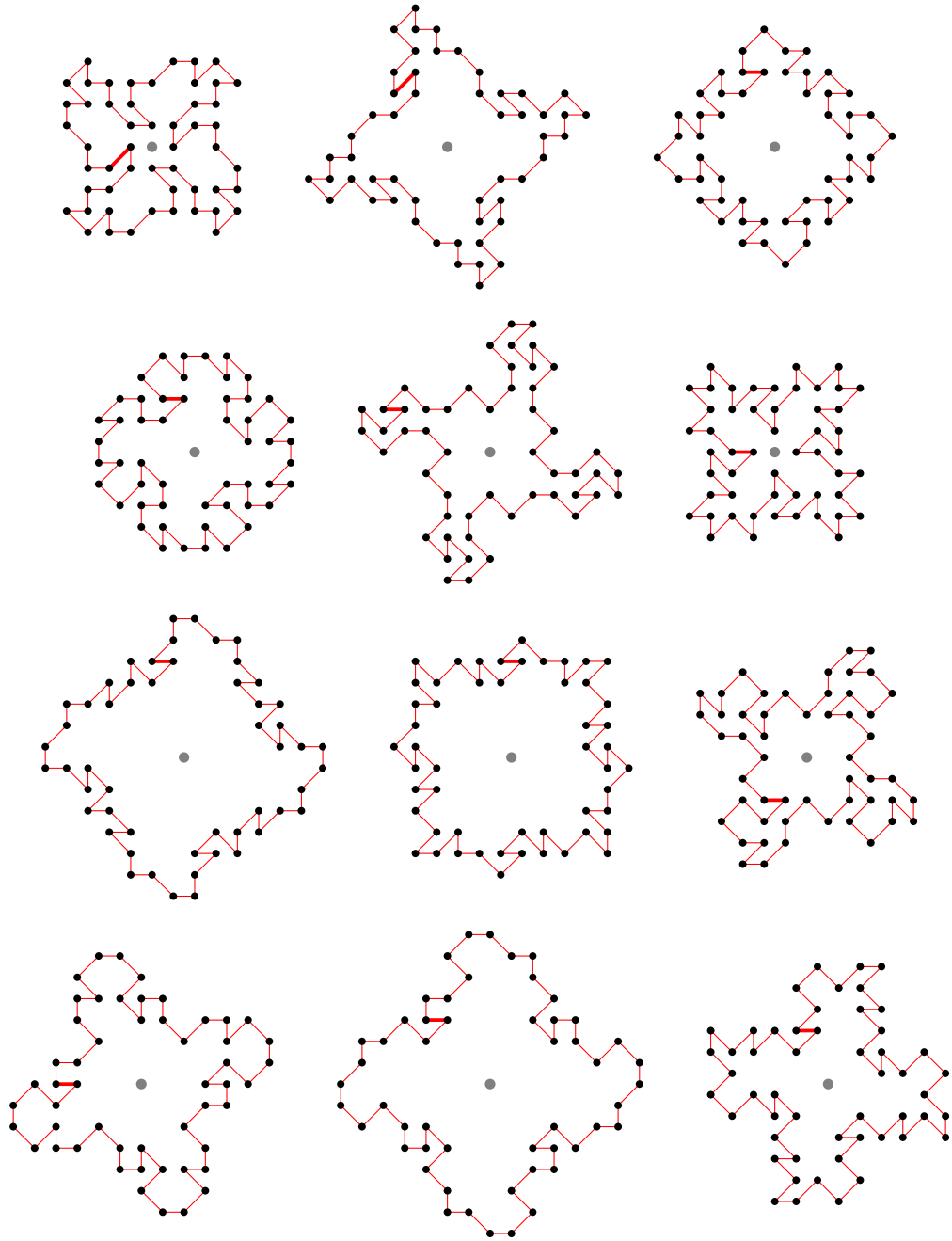


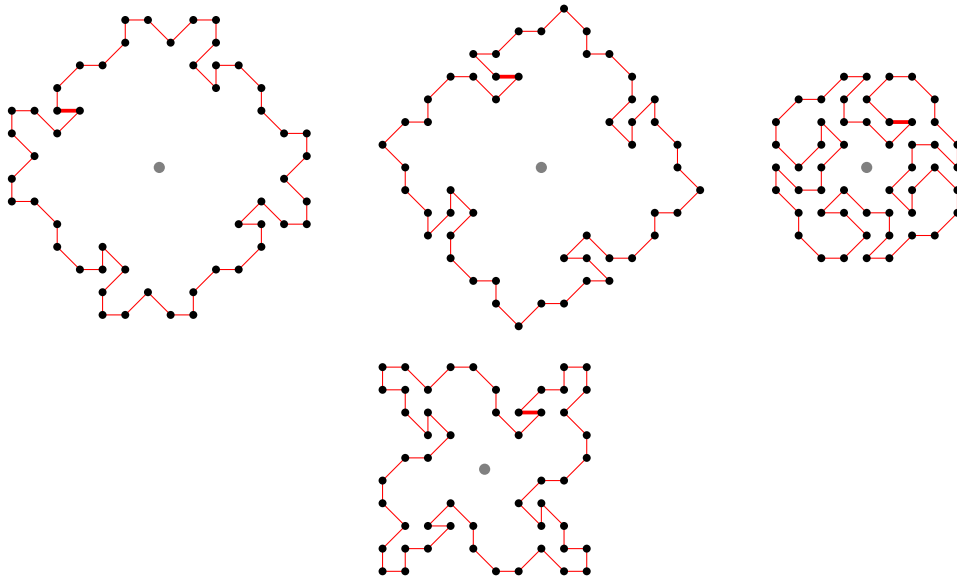








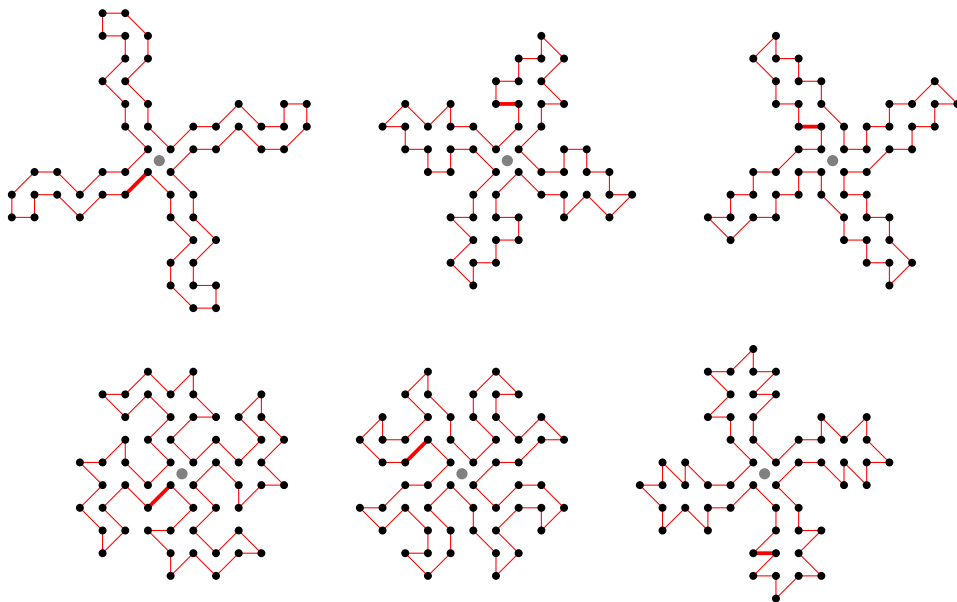


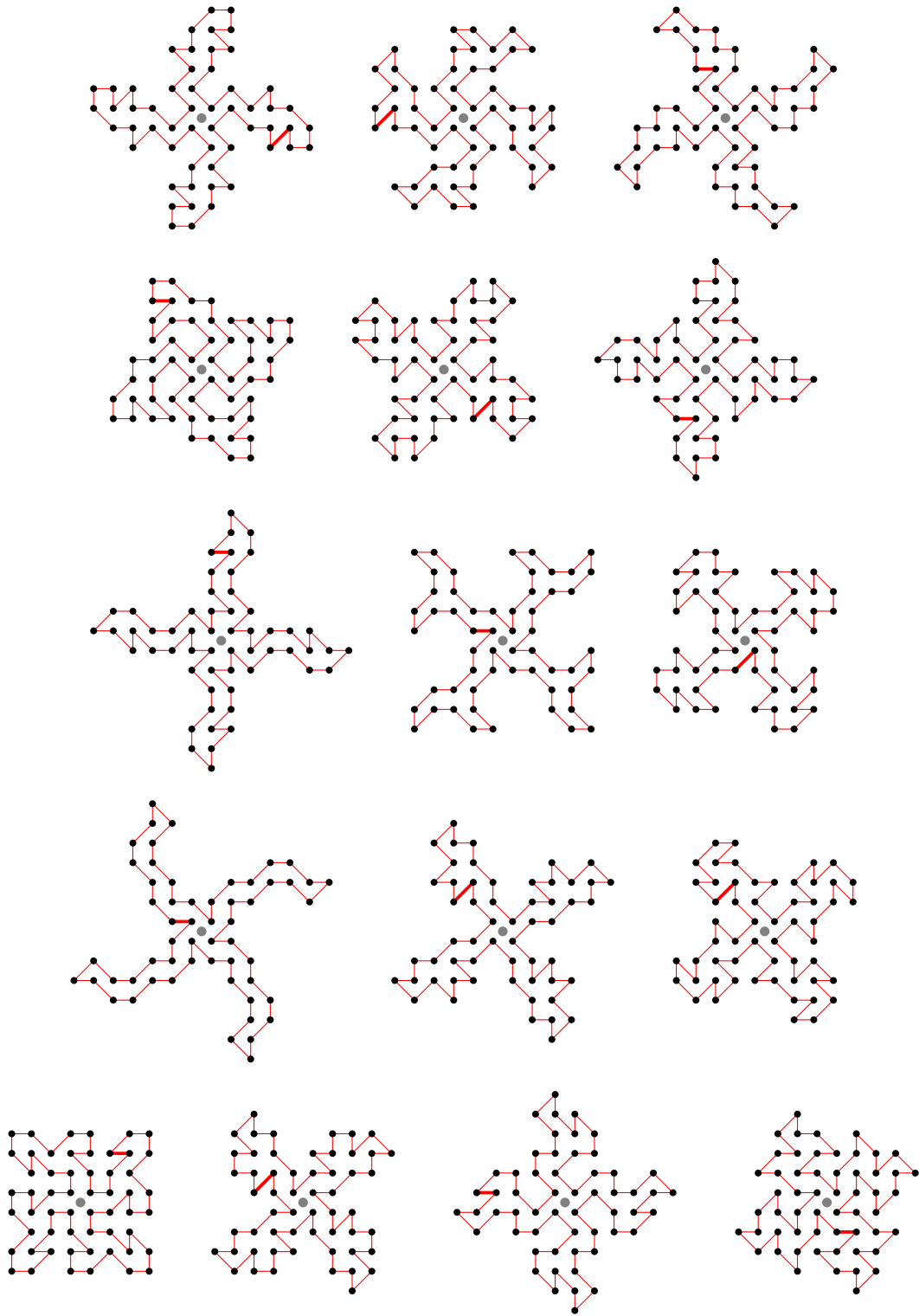


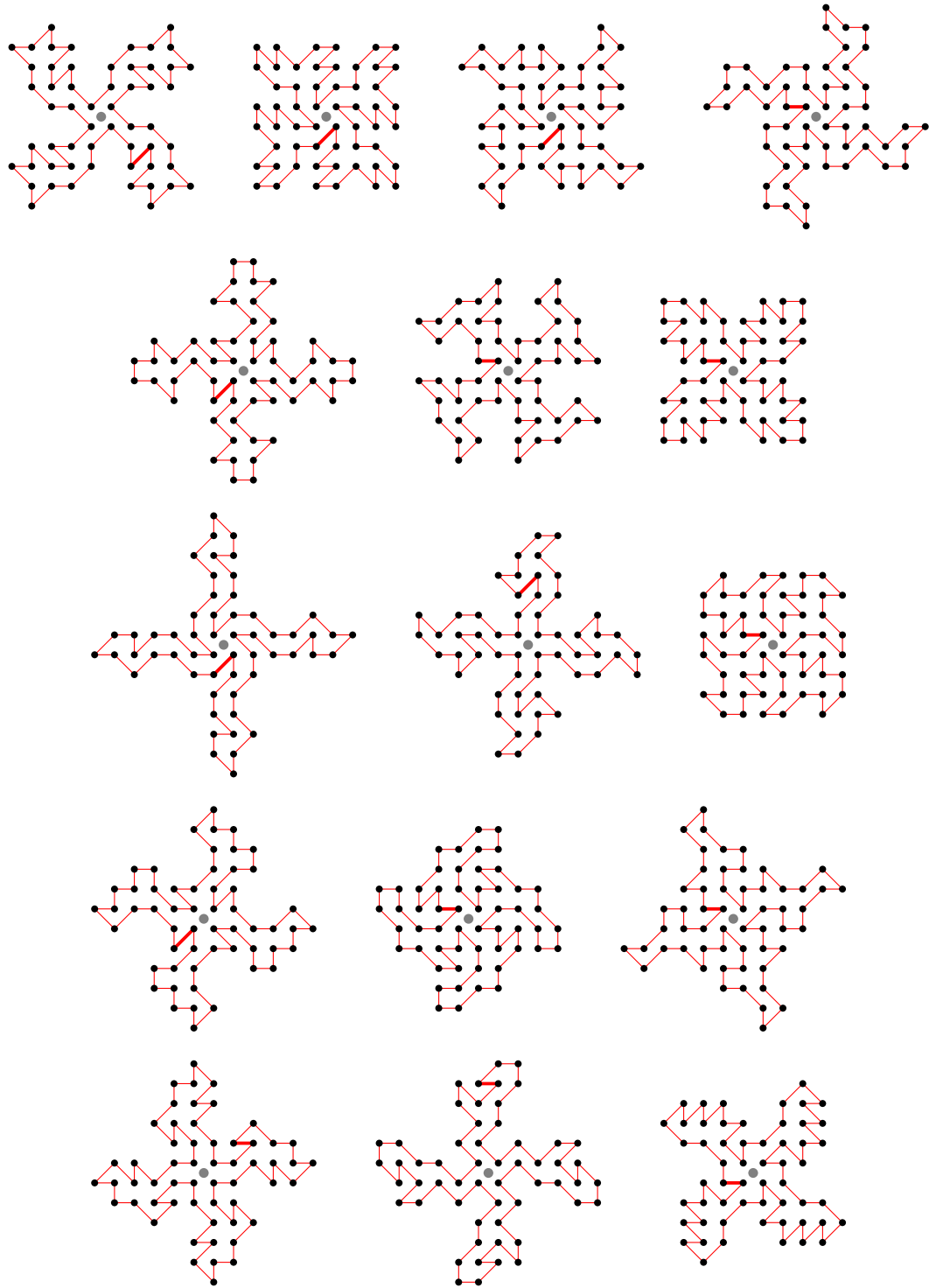
5.14 For  $n = 56$  there are 27895652 polygons

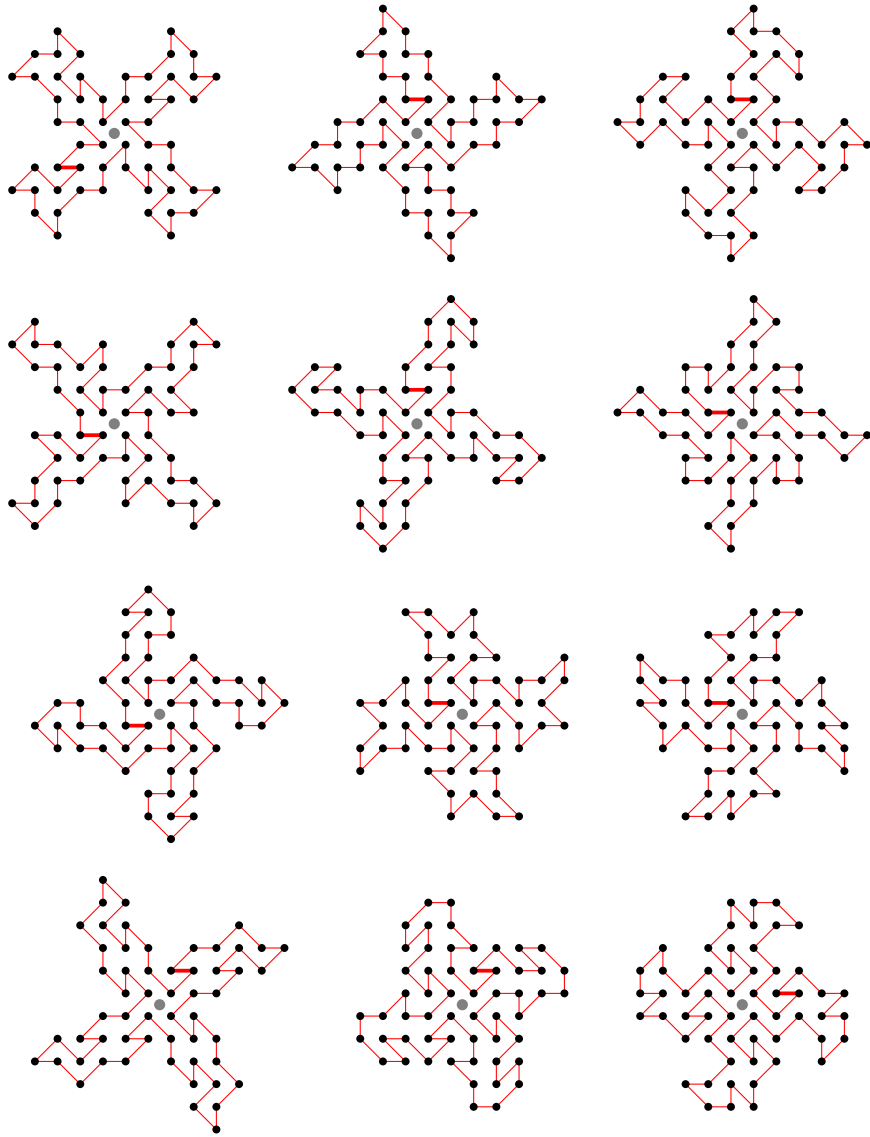
5.14.1 With smallest area (54) there are 48096 polygons

(50 samples)



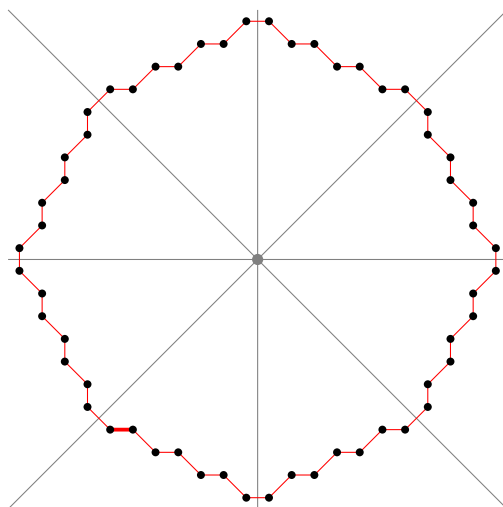








5.14.2 With largest area (590) there is 1 polygon



5.14.3 For other area sizes there are 27847555 polygons

(100 samples)

