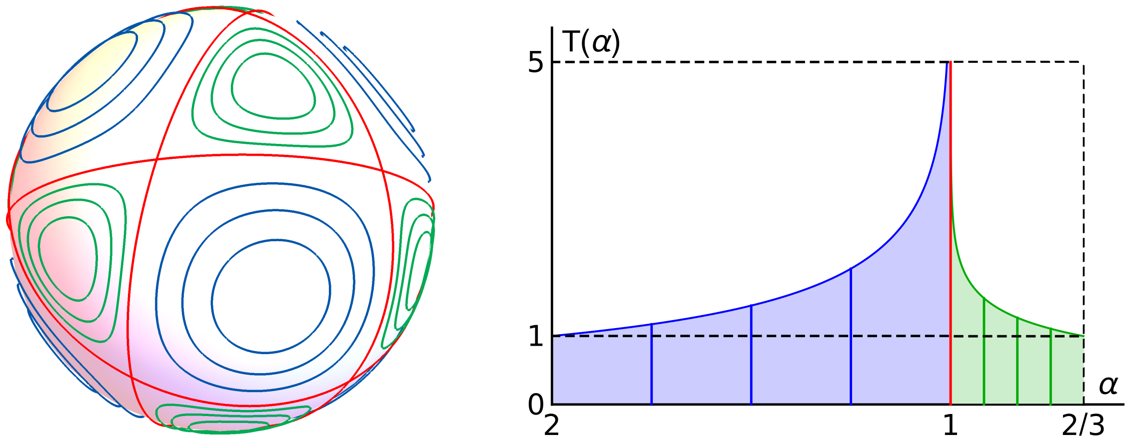

FIG. 1. Octahedral-symmetric Hamiltonian family of algebraic sphere curves (left), and the corresponding period function (right). The $\mathbf{J}$ vector precesses around curve $\mathcal{C}_\alpha$ with period $T(\alpha)$. Both graphs depict values: $\alpha = \frac{7}{4}, \frac{6}{4}, \frac{5}{4}$ (blue), $\alpha = 1$ (red), and $\alpha = \frac{11}{12}, \frac{10}{12}, \frac{9}{12}$ (green).

The coordinate variables $\mathbf{J} = (J_x, J_y, J_z)$ span an angular momentum space $\mathbb{R}^3$, where we take a particular interest in curves drawn on the surface of a sphere. Octahedral-symmetric energy surface $H(\mathbf{J}) = 2(J_x^4 + J_y^4 + J_z^4)$ determines a Hamiltonian family of algebraic sphere curves,

$$\mathcal{C}_\alpha = \{\mathbf{J} \in \mathbb{R}^3 : 1 = \mathbf{J} \cdot \mathbf{J} \ \ \& \ \ \alpha = H(\mathbf{J})\},$$

with dimensionless parameter $\alpha \propto E_0/(J_0)^4$, scaled relative to invariant energy and angular momentum, E_0 and J_0 respectively. Tangent geometry determines time evolution,

$$\dot{\mathbf{J}} = \partial_{\mathbf{J}} H \times \mathbf{J} \implies \frac{d}{dt}\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = 8\begin{bmatrix} J_y J_z (J_y^2 - J_z^2) \\ J_z J_x (J_z^2 - J_x^2) \\ J_x J_y (J_x^2 - J_y^2) \end{bmatrix}.$$

Choosing to integrate around the octahedral vertex axis J_z, we introduce a phase angle γ such that,

$$\tan(\gamma) = J_y/J_x \implies \dot{\gamma} = \frac{\dot{J}_y J_x - J_y \dot{J}_x}{(1 - J_z^2)} = \frac{4J_z(2J_z^2 - \alpha)}{(1 - J_z^2)}.$$

Phase angular velocity $\dot{\gamma}$ depends only on α and J_z. Considering the action-angle relation

$\partial_\alpha J_z = \dot{\gamma}^{-1}$, any derivative $dt^{(j)} = (\partial_\alpha^{j+1} J_z)\ \dot{\gamma}\ dt$ also depends only on α and J_z,

$$\begin{bmatrix} dt^{(0)} \\ dt^{(1)} \\ dt^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\left(10J_z^4 - 3(\alpha+2)J_z^2+\alpha\right)}{\left(4J_z^2(2J_z^2-\alpha)^2\right)} \\ \frac{3\left(84J_z^8 - 48(\alpha+2)J_z^6, +(7\alpha^2+40\alpha+28)J_z^4 - 4\alpha(\alpha+2)J_z^2+\alpha^2\right)}{\left(4J_z^2(2J_z^2-\alpha)^2\right)^2} \end{bmatrix} dt,$$

what a fortunate simplification!

Period derivatives $T^{(j)}(\alpha) = \oint dt^{(j)}$, evaluated around a loop of $\mathcal{C}_\alpha$, satisfy a Picard-Fuchs-type linear differential equation,

$$\sum_{j=0}^{2}\sum_{k=0}^{3} \mathcal{A}_{j,k}\ \alpha^k\ T^{(j)}(\alpha) = 0.$$

After a few computations, we obtain the integer annihilation matrix,

$$\mathcal{A} = \begin{bmatrix} -54 & 45 & 0 & 0 \\ 192 & -352 & 144 & 0 \\ -64 & 192 & -176 & 48 \end{bmatrix}.$$

Under the integral sign, summation over the elements of $\mathcal{A}$,

$$\sum_{j=0}^{2}\sum_{k=0}^{3} \mathcal{A}_{j,k}\ \alpha^k\ dt^{(n)} = \frac{d}{dt}\Omega(\mathbf{J}),$$

produces an exact differential on the right hand side. The function,

$$\Omega(\mathbf{J}) = \dot{J}_z\left(\frac{2J_z^2(5\alpha-6) - \alpha(3\alpha-2)}{\left(16J_z^3(2J_z^2-\alpha)^3\right)}\right)$$

is said to certify the annihilation coefficients $\mathcal{A}$. Given the proof data $\{\mathcal{C}_\alpha, \mathcal{A}, \Omega(\mathbf{J})\}$ a simple symbolic algorithm verifies the annihilation zero-sum for $T(\alpha)$.

Factorization of the coefficient to $T^{(2)}(\alpha)$,

$$\sum_{k=0}^{3} \mathcal{A}_{2,k}\alpha^k = 16(3\alpha-2)(\alpha-1)(\alpha-2).$$

determines the singular values $\alpha = 2/3, 1, 2$. Singular points occur along the symmetry axes of an octahedron. The separatrix $\mathcal{C}_1$, a product of four great circles, divides the energy range into two parts, $\alpha \in [1,2]$ and $\alpha \in [2/3, 1]$. The range $[1,2]$ contains square symmetric curves, while the range $[2/3, 1]$ contains triangular symmetric curves. Figure 1 plots a few curves $\mathcal{C}_\alpha$ alongside the period function $T(\alpha)$.

-Bradley Klee, August 23, 2018.