Representing Fractions Using Different Bases

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July 2009

Many peoples' mathematics education in elementary school emphasized procedures such as "carrying" while adding and "regrouping" while subtracting with little emphasis on genuinely understanding our number system and the power of a place value system for representing numbers. Students blindly learn to follow rules they are given, assuming there is really no basis behind them, that this is just the way mathematics is done. With practice, these basic computations become easy, despite having no understanding as to why they work. I was one of these students. Fortunately, through experiences with Math Matters and Math in the Middle, I have learned a lot about the place value system we use – the base ten number system.

To thoroughly understand our base ten system, we need to understand how a place value system works. Often, this is best done by examining how our base ten numbers would be represented using other bases. Civilizations in the past have used place value systems using bases other than ten. For example, the Babylonians used base 60 and the Mayans used base 20. Today, computers use a binary (base two) or a hexadecimal (base sixteen) system. In this paper, we will explore how fractions are represented in different bases.

Our customary base ten number system uses the idea that when counting, we can visualize "how many" by grouping objects into groups of ten. Thus, we have single digit representations for the numbers zero to nine, but we do not have a solitary digit to represent ten. Instead, we write 10, meaning 1 ten and 0 ones. We continue to make groups of ten, groups of groups of ten, etc. as we count, using place values based on powers of ten: the ones, tens, hundreds . . . places. This works well since we have ten fingers to count with. Thus, we use place, and the value assigned to each place, to represent whole numbers as "polynomials in ten." For example, 234 is a compact representation for $2(10)^2 + 3(10)^1 + 4$, and 8400908 represents $8(10)^6 + 4(10)^5 + 9(10)^2 + 8$.

Our base ten place value system can be used, of course, to represent fractions. Because any positive fraction can be thought of as a "mixed number", i.e. a whole number plus a fraction between zero and one and any decimal number can be thought of as a whole number plus a decimal between zero and one, we will focus our attention on fractions between zero and one. Extending the idea of powers of ten to negative powers, we can represent fractions as follows:

$$\frac{1}{4} = 0.25 = 2*10^{-1} + 5*10^{-2}$$

$$\frac{25}{99} = 0.\overline{25} = 0.252525... = 2*10^{-1} + 5*10^{-2} + 2*10^{-3} + 5*10^{-4} + 2*10^{-5} + 5*10^{-6} + ...$$

$$0.2\overline{5} = 0.2555... = \frac{23}{90} = 2*10^{-1} + 5*10^{-2} + 5*10^{-3} + 5*10^{-4} + ...$$

These three fractions and their decimal representations provide examples of three types of decimal representations: **terminating**, *totally* **repeating**, and *partially* **repeating**. A decimal

(less than one) is **terminating** if it involves only a finite number of non-zero terms, **totally repeating** if there is a finite sequence of numbers which repeat and which begin immediately after the decimal point, and **partially repeating** if it has a positive but finite number of terms that precede the sequence of numbers that repeats.

A place value system that uses a different base follows the same grouping method as base ten, except that the grouping is done in powers of the base that is used. For example, in base five, we only have single digit representations for the numbers 0, 1, 2, 3 and 4. Thus, when we count one more than four, we run out of room in the units column as there is no single digit that can represent five in base five. Instead, we put a 0 in the units column and a 1 in the 5's column. Thus, $5_{ten} = 10_{five}$. We continue adding one each time, counting in base 5: 11, 12, 13, 14. Since there is no single digit to represent 5, we place a 0 in the units and add another 1 to the 5s column and count 20, i.e., two groups of five. So, counting in base five looks like: 0, 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, . . .

While it is not too difficult to count in different bases, it can at first be tricky to convert whole numbers from one base to another. One method to change any number from base ten to another base is:
1. List the powers of the base you are converting to.
2. Subtract the largest power you can from the base ten number as many times as you can.
3. Continue subtracting the next largest power from your result, keeping track of how many of each power you subtract.
4. Write how many of each power you subtract in the corresponding base's place value.

For example, convert 847 from base ten to base five:

Step 1: $5^0 = 1$
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$

Step 2:   847
$\underline{- 625}$ Subtract one 5^4
222

Step 3: 222
$\underline{- 125}$ Subtract one 5^3
97

97
$\underline{- 75}$ Subtract three 5^2
22

$$22$$
$$\underline{-\ 20}$$
$$2$$
Subtract four 5^1

$$2$$
$$\underline{-\ 2}$$
$$0$$
Subtract two 5^0

Step 4: $\underline{1\ \ 1\ \ 3\ \ 4\ \ 2} = 11342_{\text{five}}$
$5^4\ 5^3\ 5^2\ 5^1\ 5^0$

We can also convert numbers written in any other base to base ten by using the following process:

1. Starting with the rightmost digit, we call this position 0. We number the positions 1, 2, 3, etc. as we move from left from one place to the next.
2. We determine the value of each place by taking the base to the power of the number of the position.
3. Multiply all of the digits in each place by the value of that place.
4. Find the sum of the numbers that result from step 3.

For example, convert 4110_{eight} to base ten:

Step 1: $\underline{4\ \ 1\ \ 1\ \ 0}$
$8^3\ 8^2\ 8^1\ 8^0$

Step 2: $4*8^3 + 1*8^2 + 1*8^1 + 0*8^0$

Step 3: $2048 + 64 + 8 + 0 = 2120_{\text{ten}}$

We can also add, subtract, multiply and divide in different bases using standard algorithms and a single digit addition or multiplication table. We just need to remember what base we are working with and make sure we are considering place value. We can add 10111 + 10101 in base two, for example, remembering that $1 + 1 = 10_{\text{two}}$ because no place will have a digit for two or more. That is, the answer will never include a digit for two, as there is no single digit in base two that represents 2.

$$111$$
$$10111_{\text{two}}$$
$$\underline{+\ 10101_{\text{two}}}$$
$$101100_{\text{two}}$$

Subtracting in base six, we need to remember that when we "borrow" we are borrowing a six, not a ten. For example, when subtracting 455_{six} from 3211_{six}, we cannot subtract five from one in the units place without borrowing from the sixes' column. When borrowing from the one, we are getting six, not ten, since we are in base six. Thus, six plus the one in the units column gives us seven and seven take away five is two. We continue this line of thinking, getting:

$$\begin{array}{r} {\scriptstyle 11} \\ 210_1 \\ 3\!\!\!\not2\!\!\!\not1\!1_{six} \\ -\ 455_{six} \\ \hline 2312_{six} \end{array}$$

When multiplying in other bases, it is easy to first think in base ten and then convert to the base you are working with. For example $4_{five} * 3_{five}$ is 12 base ten and in base five this will be 22_{five}. Similarly, 5^2 is 25 in base ten and 25 in base five is 100_{five}. It is interesting to see that independent of the base we choose, the correct representation of the base squared is 100. Keeping this in mind helps us when working with large numbers in other bases.

Finally, dividing in other bases is probably the most difficult since you not only must divide, but also multiply and subtract, all in a different base! For example, take $5430_{six} \div 13_{six}$. First, note that 13 does not go into five. 13 goes into 54 three times since 13 * 3 in base six is 43. 54 take away 43 is 11 and bringing down the 3 is 113. 13 goes into 113 five times in base six. 13 times five in base six is 113. Finally, 0 divided by 13 is 0. $5430_{six} \div 13_{six} = 350_{six}$. Visually, this is shown as:

$$\begin{array}{r} 350 \\ 13\overline{)5430} \\ \underline{43} \\ 113 \\ \underline{113} \\ 0 \end{array}$$

As indicated earlier, this paper explores representing fractions using different bases. Before continuing, a review of how to work with fractions and decimals in base ten is helpful. If one understands how to work with these numbers in base ten, then working with fractions in other bases becomes more straightforward.

We know that a fraction is terminating if, when in lowest terms, its denominator is a product of only 2's and 5's. For example, $\frac{3}{12}$ reduces to $\frac{1}{4}$, and the prime factorization of 4 is

2^2. Thus, there are only 2's in the denominator, and so $\dfrac{3}{12}$ will terminate. One can use long division to check this and see that $3 \div 12$ is 0.25 or $2*10^{-1} + 5*10^{-2}$. Conversely, one can write any terminating decimal as a fraction by simply using a denominator made of 2s and 5s. For example, the decimal 0.83564125 can be written as a fraction whose denominator is $2^8 * 5^8$ or 100,000,000, giving us the fraction $\dfrac{83,564,125}{100,000,000}$.

Another type of fraction is one that totally repeats. Looking at the fraction $\dfrac{1}{3}$, we notice that the denominator will not factor into a product of only 2's and 5's, meaning this fraction will not terminate when written in decimal form. We can use long division here to see that the result will be a decimal that repeats after only a few digits (0.33333. . .). The pigeonhole principle, combined with long division, can be used to explain why any fraction that does not terminate must repeat. We can also put an upper bound on the length of the sequence of repeating digits. (The essence of the pigeonhole principle can be explained by this example: "if you have 43 letters to distribute to 42 mailboxes, at least one mailbox will get two letters"). For example, the fraction $\dfrac{27}{43}$ has a denominator that is not a product of 2s and 5s. Thus, it will not have a terminating decimal representation. Using long division to find a decimal representation, we can make a list of the remainders. Note that zero can never be a remainder, because the decimal does not terminate. Thus, since there are only 42 possible remainders (1,2,3, . . . ,42), by the time you have collected 43 remainders, there must be at least one repeat in the list. When this happens, the corresponding sequence of terms in the quotient will start repeating as well.

On the other hand, if we want to convert a repeating decimal to a fraction, there is an algorithm for how to find the corresponding fraction. First, set the decimal equal to a variable and multiply both sides of the equation by the power of ten equal to the length of the repeating string. For example, we can write 0.237237237. . . as a fraction by setting it equal to x, then multiplying both sides by 1000 (since the length of the repeating string is 3) and solving by taking away x and the original decimal, leaving a whole number:

$$
\begin{aligned}
1000x &= 237.237237... \\
- \quad x &= \quad .237237... \\
\hline
999x &= 237
\end{aligned}
$$

$$x = \dfrac{237}{999}$$

Note that for this totally repeating decimal, the denominator equals $10^3 - 1 = 999$.

We can work with a partially repeating decimal in the same manner. For example, the decimal 0.4373737. . . begins with a 4 that is not part of the repeating term, and then has a repeating string of length two. Thus, we will multiply both sides by 100, then solve in the manner above:

$$100x = 43.7373737\ldots$$
$$\underline{-\ \ x = \ \ \ \ .4373737\ldots}$$
$$99x = 43.3$$

$$x = \frac{43.3}{99}$$

With one final step, we multiply the numerator and denominator by 10 to get rid of the decimal, leaving us with the fraction:

$$x = \frac{433}{990}$$

This time, note that $.4373737\ldots = .4 + .0373737\ldots = \dfrac{4}{10} + \dfrac{1}{10} \bullet \dfrac{37}{99} = \dfrac{4 \bullet 99}{990} + \dfrac{37}{990} = \dfrac{433}{990}.$

We also note that there are decimal numbers that do not repeat (i.e. irrational numbers such as 0.12345678900112233445566778899000111222333. . . .)

Using methods similar to those discussed above, it is possible to represent fractions (typically written using base ten) using our place value system but with a different base. The word *basimal* is used instead of *decimal* when discussing "decimals" in a different base because the "dec" communicates the idea that we are using base ten.

Our first observation is that a fraction that has a terminating decimal representation may have a repeating basimal representation in a different base. This emphasizes the idea that the answer to "which fractions terminate" depends on the base being used. In the following work, when a number is written as a fraction, we are using base ten, unless specifically stated otherwise.

1) $\dfrac{1}{2} = 0.111\ldots_{\text{three}} = 3^{-1} + 3^{-2} + 3^{-3} + \ldots$

We convert from the basimal number back to a fraction using the same algorithm that is used above. Let $a = 0.111\ldots_{\text{three}}$. Here, the entry in the 3^{-1} spot is a repeating string of length one. Thus, multiply both sides by 10_{three} and solve. Remember that 10_{three} is 3 in base ten.

$$10_{\text{three}}a = 1.111\ldots_{\text{three}}$$

$$\begin{array}{r} - \quad a = \quad .111\ldots_{\text{three}} \\ \hline 2a = 1 \end{array}$$

$$a = \frac{1}{2}$$

We emphasize that in base three, the fraction $\frac{1}{2}$ is a repeating basimal.

We can also convert from the fraction $\frac{1}{2}$ to a base three basimal using long division, keeping in mind that we are dividing in base three.

$$\begin{array}{r} .111\ldots \\ 2\overline{)1.000\ldots} \\ \underline{2} \\ 10 \quad \leftarrow \text{The remainder repeats.} \\ \underline{2} \\ 1 \end{array}$$

Thus, $\frac{1}{2}$ is equal to $0.111\ldots_{\text{three}}$.

2) What is the base three basimal representation for 5/12?

We can convert from a fraction to a basimal using long division, remembering we are working in base three. First, convert 5/12 to a base three fraction, getting $12_{\text{three}}/110_{\text{three}}$.

$$\begin{array}{r} 0.1\overline{02} \\ 110\overline{)12.000} \\ \underline{110} \\ 100 \\ \underline{0} \\ 1000 \\ \underline{220} \\ 10 \text{ (first remainder that repeats)} \end{array}$$

We can check this answer by converting the basimal back to a fraction (remembering we are working in base three).

$$a = 0.1020202\ldots$$

$$\begin{array}{r} 100_{\text{three}}a = 10.2020202\ldots \\ - \quad a = \quad .1020202\ldots \\ \hline \end{array}$$

$$22_{three}a = 10.1_{three}$$

$$a = \frac{10.1_{three}}{22_{three}} = \frac{10.1_{three}}{22_{three}} \bullet \frac{10_{three}}{10_{three}} = \frac{101_{three}}{220_{three}} = \frac{3^2 + 1}{2 \bullet 3^2 + 2 \bullet 3} = \frac{10}{24} = \frac{5}{12}$$

Note that our answer was a partially repeating basimal in base three.

3) What is the base eight basimal representation for 13/16?

First, convert 13/16 to base eight, getting $15_{eight}/20_{eight}$. Next, divide 15_{eight} by 20_{eight} in base eight.

$$
\begin{array}{r}
0.64 \\
20\overline{)15.00} \\
\underline{140} \\
100 \\
\underline{100} \\
0
\end{array}
$$

Thus, 13/16 equals 0.64_{eight}. Check this by converting 0.64_{eight} to a fraction.

$$.64 = 6 * \frac{1}{8} + 4 * \frac{1}{8^2}$$
$$= \frac{6}{8} + \frac{4}{64}$$
$$= \frac{52}{64}$$
$$= \frac{13}{16}$$

4) Which fraction p/q, in base ten, is equal to 0.132_{four}?

Working in base four, we can use a similar technique involving place value to write a fraction. In base four, the value of the first place after the basimal is not 10^{-1}, but 4^{-1} or 1/4 and the second place after the basimal is 4^{-2} or 1/16 and so on. Thus, 0.132 in base four is:

$$(1 * \frac{1}{4}) + (3 * \frac{1}{16}) + (2 * \frac{1}{64})$$
$$= \frac{1}{4} + \frac{3}{16} + \frac{2}{64}$$
$$= \frac{16}{64} + \frac{12}{64} + \frac{2}{64}$$

$$= \frac{30}{64}$$

$$= \frac{15}{32}$$

Thus, $0.132_{\text{four}} = \dfrac{15}{32}$

5) Which fraction p/q, in base ten, is equal to $0.3\overline{12}_{\text{five}}$?

Since we are now working with a partially repeating basimal, we can no longer use the basimal expansion method applied in number 4. However, we can use the method of getting rid of the repeating string like in numbers 1 and 2. First, let $a = 0.312121212.\dots$. Next, multiply by 100_{five} since the repeating string is two digits long. Then subtract $a = 0.3121212.\dots$ remembering we are working in base five:

$$100_{\text{five}}\, a = 31.2121212\dots$$
$$-\qquad a = \quad .3121212\dots$$
$$44_{\text{five}}a = 30.4_{\text{five}}$$

$$= \frac{30.4_{\text{five}}}{44_{\text{five}}}$$

Next, convert this fraction to base ten:

$$30.4_{\text{five}} = 15\frac{4}{5} \text{ and } 44_{\text{five}} = 24.$$

Therefore, we get $\dfrac{79/5}{24} = \dfrac{79}{120}$ ten.

Thus, $0.3\overline{12}_{\text{five}} = \dfrac{79}{120}$ ten.

6) Find b so that in base b, $\dfrac{5}{24} = 0.113_b$.

Using the idea of expanded place value like in number 4, we can write 0.113_b as:

$$\frac{1}{b} + \frac{1}{b^2} + \frac{3}{b^3}$$

Next, make a common denominator of b^3:

$$\frac{1}{b}\left(\frac{b^2}{b^2}\right) + \frac{1}{b^2}\left(\frac{b}{b}\right) + \frac{3}{b^3} = \frac{b^2 + b + 3}{b^3}$$

Set this equal to $\dfrac{5}{24}$, then cross multiply, set the equation equal to zero, and use the calculator to solve for b:

$$\frac{5}{24} = \frac{b^2 + b + 3}{b^3}$$

$$5b^3 = 24b^2 + 24b + 72$$

$$5b^3 - 24b^2 - 24b - 72 = 0$$

$$b = 6$$

Thus, in base six, $\dfrac{5}{24} = 0.113_{\text{six}}$. One can check this by seeing if 0.113_{six} is $\dfrac{5}{24}$:

$$\frac{1}{6} + \frac{1}{6^2} + \frac{3}{6^3}$$

$$= \frac{36}{216} + \frac{6}{216} + \frac{3}{216}$$

$$= \frac{45}{216}$$

$$= \frac{5}{24}$$

7) Find b so that in base b, $\dfrac{37}{60} = 0.2\overline{13}_b$.

This problem is very similar to number 6, just now working with a repeating decimal. I can begin by working with the repeating decimal in the manner done in past problems. Let $a = 0.2131313\ldots$ then multiply both sides by 100_b, since the repeating string is two digits long. Next, subtract $a = 0.2131313\ldots$ from the resulting equation. Here, a problem arises when trying to subtract because we do not know what base we are subtracting in. $100a - 1a$ can be re-written as $(b^2 - 1)a$ because 100_b in any base is b^2. Therefore, when we subtract, we get:

$$100a = 21.31313\ldots$$
$$-\quad a = \quad .21313\ldots$$
$$(b^2 - 1)a = 21.1_b$$

By writing 21.1 in expanded form (using b again for the base we are in), we now have

$$(b^2 - 1)a = 2b + 1 + \frac{1}{b}$$

Next, solve for *a* and simplify:

$$a = \frac{2b + 1 + 1/b}{b^2 - 1}$$

$$a = \frac{2b^2 + b + 1}{b^3 - b}$$

Finally, set *a* equal to $\frac{37}{60}$, cross multiply then use the calculator to solve for *b*.

$$\frac{37}{60} = \frac{2b^2 + b + 1}{b^3 - b}$$

$$37b^3 - 37b = 120b^2 + 60b + 60$$

$$37b^3 - 120b^2 - 97b - 60 = 0$$

$$b = 4$$

So, $\frac{37}{60} = 0.2\overline{13}_{\text{four}}$, a partially repeating basimal.

8) Under what conditions will a fraction *p/q* between 0 and 1 will be terminating, totally repeating, or partially repeating base *b* basimal?

Earlier we stated that in base ten, we know a fraction will terminate if, in lowest terms, the denominator has only factors of 2 and 5. It is similar with fractions of other bases, in that a fraction will terminate if, in lowest terms, the denominator is a product of prime factors of the base. For example, in base three, $0.21221_{\text{three}} = \frac{21221}{100000}$ with $100,000_{\text{three}} = 3^5$. In base six, the denominator would factor to 2s and 3s, in base eight the denominator would factor to powers of 2 and so on.

As we defined earlier, a totally repeating basimal begins its repeating string right after the "basimal point". For example, $0.\overline{215}$ is a totally repeating decimal in base ten. Using methods described earlier, we can find $0.\overline{215}$ in base ten :

$$1000a = 215.215215\ldots$$
$$-\quad\ \ a = \quad\ .215215\ldots$$
$$999a = 215$$

$$a = \frac{215}{999}$$

The denominator can be re-written as 10^3-1, where the coefficient is the base and the exponent is the number of repeating digits in the decimal. Let's consider another base and find $0.\overline{215}_{\text{eight}}$. Using our algorithm to convert this to a fraction, we get $\frac{215}{8^3 - 1}$. In general, any totally repeating decimal can be written as

$$\frac{w}{b^m - 1}$$

where w is the whole number string that is being repeated, b is the base, and m is the number of repeating digits in the string.

A partially repeating decimal is one that does not begin repeating right after the decimal point. $4.65\overline{217}$, for example, is a partially repeating decimal in base ten. Partially repeating decimals have a part that terminates and a part that repeats. We have just shown, in general, how to represent a totally repeating decimal. Breaking down $4.65\overline{217}$ in base ten, we get:

$$4.65 \quad + \quad .00\overline{217}$$
$$\text{Terminates} \quad + \quad \text{repeats}$$

We can move the decimal in the repeating part over two places and then rewrite the repeating string using the information found above, giving us:

$$4.65 \quad + \quad \frac{1}{10^2} \quad * \quad \frac{217}{10^3 - 1}$$
$$\text{Terminating part} \quad + \quad \text{part to move the decimal} \quad * \quad \text{repeating part}$$

Using the same idea, we can represent partially repeating decimals in other bases in the same manner:

$$3.24\overline{321}_{\text{five}}$$
$$= 3.24_{\text{five}} + .00\overline{321}_{\text{five}}$$
$$= 3.24_{\text{five}} \quad + \quad \frac{1}{5^2} \quad * \quad \frac{321}{5^3 - 1}$$

In general, a fraction will be partially repeating if it is in the form of:

$$\text{Terminating Part} + \frac{1}{b^n} * \frac{w}{b^m - 1}$$

Where w is a whole number, b is the base, n is the number of places needed to move the decimal to get the repeating part next to the basimal point, and m is the number of repeating digits in the string.

Our work up to this point has demonstrated that if b is any whole number, we can use standard algorithms that we use with base ten to convert between fractions written using base ten and base b basimal numbers. For example, we showed that 13/16 in base ten is the same as 0.64_{eight}, where long division in base eight was the key. In part 2, the pigeonhole principal was the key to showing why every fraction must either terminate or repeat when written in a basimal representation, for any base b. Working in the opposite direction, we can convert any terminating basimal, such as 0.4356 in base eight, to a fraction in base ten as follows:

$$0.4356_{\text{eight}} = 4* 8^{-1} + 3*8^{-2} + 5*8^{-3} + 6*8^{-4}$$
$$= 4*\frac{1}{8} + 3*\frac{1}{64} + 5*\frac{1}{512} + 6*\frac{1}{4096}$$
$$= \frac{2048}{4096} + \frac{192}{4096} + \frac{40}{4096} + \frac{1}{4096}$$
$$= \frac{2281}{4096}$$

When converting a totally or partially repeating decimal from a base b into base ten, the method of setting up two equations and subtracting to get rid of the repeating string (as in number five), works well for all bases. In this method, we must first figure out how long the string of repeating digits is, then multiply by the base raised to a power equal to the length of the repeating string. If the basimal is 0.2525. . ., the repeating string is 25, which has length 2, and if the basimal is 0.34567856785678. . . the repeating string is 5678, which has length 4. By multiplying by the base raised to the power which equals the length of the string, we are able to "move the decimal over." Thus, when we subtract the initial equation from the second equation, we get rid of the repeating numbers. The most difficult part of this method, when working in other bases, is subtracting and converting fractions to other bases.

Another example of this is converting $0.232323. ..._{\text{four}}$ to a fraction:

Let $a = 0.232323. . .$

To eliminate the repeating part, multiply both sides by $4^2 = 100_{\text{four}}$, since the repeating string (23) is two digits long:
$$100a = 23.2323. . .$$

Next, subtract $a = 0.232323. . .$ from the new equation:

$$100a = 23.2323\ldots$$
$$-\quad a = 0.2323\ldots$$

Since we are subtracting $100a$ and a in base four, we need to stop and think that this is not 99, but 33_{four}. $23.2323\ldots_{\text{four}} - 0.2323\ldots_{\text{four}}$ is 23_{four}. We got rid of the repeating numbers! Now, we can divide each side by 33_{four} to solve for a.

$$100a = 23.2323\ldots$$
$$-\quad a = 0.2323\ldots$$
$$\frac{33_{\text{four}}a = 23_{\text{four}}}{33_{\text{four}}}$$
$$a = \frac{23_{\text{four}}}{33_{\text{four}}}$$

Finally, 23_{four} is 11 in base ten and 33_{four} is 15 in base ten, so our fraction is $\dfrac{11}{15}$.

As a final example, we examine an observation about an alternate representation of decimal numbers and thus a comparable representation of basimal numbers in any base. Earlier, we noted that in base ten:

$$0.25 = 2*10^{-1} + 5*10^{-2},$$

$$\frac{25}{99} = 0.\overline{25} = 0.252525\ldots = 2*10^{-1} + 5*10^{-2} + 2*10^{-3} + 5*10^{-4} + 2*10^{-5} + 5*10^{-6} + \ldots,$$

and $0.2\overline{5} = 0.2555\ldots = \dfrac{23}{90} = 2*10^{-1} + 5*10^{-2} + 5*10^{-3} + 5*10^{-4} + \ldots$

Similarly, in base b, $0.231_b = 2 + 3 \bullet b^{-1} + 1 \bullet b^{-2}$ and
$0.\overline{231}_b = 2 \bullet b^{-1} + 3 \bullet b^{-2} + 1 \bullet b^{-3} + 2 \bullet b^{-4} + 3 \bullet b^{-5} + 1 \bullet b^{-6} + \ldots$. More generally, any fraction p/q between 0 and 1 can be represented using negative powers of b. That is, it is possible to find whole numbers $c_1, c_2, c_3\ldots$, where each c_i is less than b, such that $\dfrac{p}{q} = c_1 b^{-1} + c_2 b^{-2} + c_3 b^{-3}\ldots$, where the coefficients c_i are all 0 past some point if the basimal terminates.

In summary, readers may observe that representing fractions using other bases requires (1) a good handle on fractions and decimals in base ten, (2) confidence that basic place value

algorithms will work in number systems with other bases, and (3) a good understanding of working with whole numbers in different bases.   With these ideas in place, it really is not that difficult to represent fractions using bases other than ten.

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