

# Welfare Effects of Prey-Refuges in Fisheries\*

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## Abstract

This paper uses a tractable predator-prey model with endogenous harvesting to assess the impact of a prey refuge on fishery performance. Using a two-stage game framework, where the prey refuge consistently protects part of the environment from predators, this study investigates: (i) how fishers adjust their behavior in response to the prey refuge and (ii) the conditions under which the prey refuge enhances social welfare. Results indicate that reducing species interaction intensity through the prey refuge decreases fishing pressure on both prey and predator populations. Additionally, while full prey protection maximizes prey fishing payoffs, it does not necessarily minimize predator fishing payoffs. Necessary and sufficient conditions for the existence of a positive cooperative surplus are derived. Numerical examples reveal that more depleted prey stocks require larger prey refuges, whereas more depleted predator stocks require smaller ones. Moreover, the overall efficiency of the fishery, as influenced by the prey refuge, depends heavily on fishers' willingness to wait for its benefits. Specifically, when fishers are sufficiently patient, the prey refuge enhances social welfare. Finally, prey refuge implementation proves beneficial even when fishers coordinate their fishing strategies. Overall, this paper proposes an ecosystem-based approach to enhance fishery efficiency without relying on direct economic incentives or restrictions.

**Keywords:** Prey-refuge, Predator-Prey, Two-stage game, Cooperative games

**JEL Codes:** C61, C72, Q22, Q28

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# 1 Introduction

Prey refuges are habitats where prey species can evade natural predators. These refuges are either inaccessible to larger predators or offer prey effective hiding spots. These refuges include natural habitats such as mangroves, coral reefs, and seagrass meadows, as well as human-made structures like electrified reefs, decommissioned offshore platforms, and anti-predator nets. These artificial structures replicate an important feature of natural habitats—predation avoidance—though they are generally not intended for direct fishery economic purposes. Instead, they are often implemented for habitat restoration, as seen in the recent Buccoo Reef restoration project in Tobago, or emerge from other economic activities, such as the “rigs to reefs” program, which repurposes decommissioned offshore platforms into artificial reefs. These structures create externalities in fisheries by altering population dynamics and, consequently, shifting fishing incentives. While ecologists and mathematicians have extensively studied prey refuges, their economic implications are relatively underexplored. This paper integrates a prey refuge into a standard two-species predator-prey model with endogenous fishing to examine the effects of prey refuges on fishing behavior, fishery efficiency, and propose a rationale for implementing artificial prey refuges.

This paper connects two key strands of the fishery economic literature. The first is the literature on predator-prey models with endogenous and strategic fishing. This is a relatively underdeveloped area due to the mathematical complexities arising from multi-species interactions and non-linear dynamics. Seminal works, such as [Hannesson \(1983\)](#), [Flaaten \(1991\)](#) or even [May et al. \(1979\)](#), introduced fishing into two-species models, typically focusing on optimal long-run solutions and examining the effects of species interactions and relative fish market prices on economic outcomes. However, these models are limited by their inability to account for non-linear decision-making and, consequently, overlook the dynamic consequences of fishing and strategic behavior. A smaller body of research, including early contributions by [Fischer and Mirman \(1992\)](#) and [Fischer and Mirman \(1992\)](#), addresses strategic fishing in a discrete-time setting. More recent work by [Koulovatianos \(2024\)](#) and [Bataille \(2024\)](#) shows that under a specific parametric constraint, an explicit solution for strategic and specialized fishing in predator-prey models can be obtained. This paper builds on these contributions by incorporating the concept of prey refuge into a tractable model.

The second strand of relevant literature concerns marine habitat considerations in fisheries management. Several studies, such as [Foley et al. \(2012\)](#) and [Moberg \(2016\)](#), endogenize existing biological parameters from single-species models, including intrinsic growth rates and carrying capacities. Others examine the role of habitat in shaping optimal management strategies. For instance, [Holland and Schnier \(2006\)](#) investigates the use of habitat quotas

to meet conservation objectives, while [Udumyan et al. \(2010\)](#), [Long et al. \(2020\)](#), and [Jean-Marie and Tidball \(2024\)](#) introduce habitat dynamics, incorporating fishing-induced habitat damage. These studies show, through both steady-state and fully dynamic analysis, the critical role habitat plays in optimal harvesting policy design. Building on these findings, the creation of marine reserves as non-market-based instruments has been explored, starting with [Sanchirico and Wilen \(2001\)](#), followed by [Smith and Wilen \(2003\)](#), and more recently [Moeller and Neubert \(2013\)](#). These works show that combining marine reserves with harvest taxes, particularly when accounting for habitat damage, can be an effective management strategy. My approach differs from this body of literature by redefining the role of habitat. Rather than considering habitat as merely the place where a species lives, I focus on its function in providing protection from natural predation. Specifically, I develop a two-species predator-prey model in which the predator is excluded from certain areas of the environment, limiting its access to a portion of the prey population. This is a new approach to the marine reserve literature, where access restrictions do not apply to fishers but to natural predators.

Finally, this paper draws on insights from theoretical ecology and mathematics. This literature introduces prey refuges in various forms, from modifications of predator functional responses ([Ma et al., 2009](#)) to spatially structured predator-prey models ([Ma et al., 2013](#)). Several non-economic studies have explored the stabilizing effects of prey refuges on the dynamics of predator-prey interactions and their role in preventing prey extinction ([McNair, 1986](#); [Ruxton, 1995](#); [Sih, 1987](#)), this study aims to investigate the potential of prey refuges to enhance the economic efficiency of a multi-species fishery. To maintain simplicity and tractability, I assume that the prey refuge consistently protects a fixed proportion of the total environment from predator access. Consequently, at any given time, the refuge protects a constant share of the prey population, while predators can only exploit the unprotected portion.

This paper bridges an important gap by combining literature from multiple fields, marking a significant step toward Ecosystem-Based Fisheries Management (EBFM) policies. Prey refuges significantly influence population dynamics by altering species interactions; specifically, they protect a portion of the prey stock, reducing predation risk and weakening inter-species relationships. This, in turn, affects the fishing strategies of those targeting either prey or predators. While it is intuitive to expect that prey refuges benefit prey fishers, the effects on predator harvesting and overall fishery efficiency are more complex. On one hand, prey refuges reduce the immediate availability of prey to predators, potentially harming both predator populations and predator harvests. However, by promoting prey regeneration, refuges could increase the availability of prey outside the refuge in the future, which may ultimately support predator fishing. This paper investigates two important questions: (i) How do fishers adjust

their fishing activities in response to the presence of the prey refuge? and, (ii) Under what conditions the prey refuge can increase the overall social welfare of the fishery?

The model considered is a two-species predator-prey fishery in which two specialized fishers target either the prey or the predator. To investigate changes in fishing behavior and fishery performance, I use a two-stage game framework. In the first stage, the fishers engage in a static cooperative game, deciding whether to sign an agreement to implement an artificial prey refuge (e.g., a electrified reef) that protects a specific area from predator access. The bargaining focuses on the size of the prey refuge, specifically the portion of the environment inaccessible to predators, with no option for cooperation on catch levels. Once an agreement is reached, the refuge is implemented, and the fishers engage in a standard differential game over catches. The establishment of the prey refuge requires a unanimous and binding agreement between both fishers; it is not a control variable for either party. Therefore, once implemented, the refuge remains permanent, and fishers can only adjust their strategies based on its size.

The model is solved through backward induction. First, I solve the non-cooperative game for a given prey refuge size, deriving equilibrium strategies and payoff flows that summarize all future welfare implications from first stage's decisions. Based on these results, I determine the necessary and sufficient conditions under which the cooperative payoffs—the maximum joint payoff obtained from implementing refuge—can be achieved, along with the conditions for surplus-sharing. A key feature of the model is that it assumes a self-enforcing mechanism for the implementation of the prey refuge; no direct restrictions are imposed on the fishers' behavior. The second stage seeks Markov-perfect Nash equilibria, implying time-consistent strategies along the equilibrium path.

Although the model provides explicit solutions, the conditions for the existence of cooperative surplus in the first stage could be difficult to interpret economically. Intuitively, some parameters should be of particular interest in this context. To clarify these relationships, I present stylized examples that, while not meant to represent specific real-world scenarios, highlight the characteristics a fishery must exhibit to achieve—or fail to achieve—a cooperative surplus based on prey refuge implementation. Among the many relevant factors, this study focuses on two: initial stock values and the discount rate. Initial conditions—such as heavily depleted stocks, which often drive cooperative efforts—can affect fishers' willingness to support prey protection through a refuge. Additionally, since prey refuges may require time to yield benefits via population recovery, fishers' valuation of future payoffs is expected to be particularly significant in determining prey refuge implementation.

The main results of this paper are as follows. First, I show the existence of a unique

Markov-Perfect Nash equilibrium in linear strategies, where fishing pressure on both prey and predator decreases as the prey-refuge size increases. The impact of the prey refuge on fishers' payoffs is notably asymmetric. Since prey fishers access the entire prey stock, their welfare is maximized when the prey population is fully protected. Interestingly, a prey refuge can also enhance predator fishers' welfare, both with and without compensations. Initial conditions and the common discount rate are important model parameters. More depleted prey (resp. predator) populations before cooperation require a higher (resp. lower) degree of protection. Additionally, as fishers become more patient, the prey-refuge size that maximizes joint welfare increases. Lastly, even if fishers coordinate their strategies (e.g., within a Regional Fishery Management Organization), a prey refuge policy can still serve as part of a broader cooperative management strategy.

The structure of the paper is as follows. Section 2 introduces the framework. In Section 3, I analyze strategic choices in a differential game with a prey refuge and present comparative statics. Section 4 examines a static cooperative game over prey refuge size and outlines efficiency conditions. Numerical examples illustrating the results are provided in Section 5, followed by a discussion of fishery policies in Section 6. Finally, Section 7 offers concluding remarks.

## 2 The Model

Building on recent contributions by Koulovatianos (2024) and Bataille (2024), I consider an ecological system,  $\mathcal{S}_n$ , with two species: the prey population  $x(t)$  and the predator population  $y(t)$ . Time,  $t$ , is continuous, and the inter-species dynamics are governed by the following equations:

$$\mathcal{S}_n \begin{cases} \dot{x}(t) = A_x x(t)^\theta - \delta_x x(t) - b_x y(t)^{1-\theta} x(t)^\theta, & x(0) > 0, \\ \dot{y}(t) = A_y y(t)^\theta - \delta_y y(t) + b_y x(t)^{1-\theta} y(t)^\theta, & y(0) > 0. \end{cases} \quad (1)$$

Here,  $\dot{x}(t)$  and  $\dot{y}(t)$  represent the time variation of the prey and predator populations, respectively. Each species  $s \in \{x, y\}$  has intrinsic growth represented by  $A_s s(t)^\theta$  and natural mortality represented by  $\delta_s s(t)$ . The competition and saturation effects lead to a population growth that, without interactions, has an inverted U-shape. The parameter  $0 < \theta < 1$  scales the reproductive rate of species and represents the degree of concavity in species growth. Both prey and predator populations are thus bounded by their respective carrying capacities, i.e., the maximum quantity of a species that the environment can sustain.

Predator-prey interaction effects are additive and separable from intrinsic growth and

natural mortality. Specifically, the additional predator growth due to prey consumption is additively separable from natural growth, and the additional prey mortality due to predator consumption is additively separable from the prey’s natural death rate. The intake rate of the predator, i.e., the functional response, is an increasing and concave function of the prey stock, belonging to the class of Holling type II functional responses (see [Holling, 1959](#)). This matching technology scales the predator’s ability to match with prey according to the relative abundance of both populations<sup>1</sup>. The prey mortality induced by predator consumption is determined by the product of the predator’s consumption rate,  $b_x$ , and the number of matches,  $x(t)^\theta y(t)^{1-\theta}$ . Similarly, the predator’s growth due to prey consumption is the product of the predator’s conversion rate,  $b_y$ , and the number of matches,  $y(t)^\theta x(t)^{1-\theta}$ .

The ecological system  $\mathcal{S}_n$  in (1) admits four distinct steady-state equilibria: complete extinction, extinction of either one of the two species, or co-existence. To ensure non-negative population levels, the following restriction is imposed:

**Assumption 1**  $-\delta_y A_x + b_x A_y < 0$ .

**Prey-refuge.** Theoretically, prey refuges can be characterized as any strategy aimed at reducing predation risk ([Sih, 1987](#)). Drawing on extensive biological literature such as [Ko and Ryu \(2006\)](#), [Wang and Wang \(2012\)](#), and [Ma et al. \(2009\)](#), I introduce the refuge in its simplest form. I assume that the prey-refuge protect a constant share of the environment, and consequently a constant share of the prey population, denoted by  $r$ , from predator access at any given time. Additionally, it is assumed that the intrinsic reproduction rate of prey inside and outside the refuge is identical. Predator-prey interactions occur only outside the refuge. Consequently, at any time  $t$ , the predator’s functional response is determined solely by the number of predators,  $y(t)$ , and the prey population available outside the refuge,  $(1 - r)x(t)$ . This formulation facilitates straightforward comparisons and distinctions with the standard case, i.e, when  $r = 0$ . Given that the intensity of species interactions are not fixed in practice, the introduction of a refuge allows for the incorporation of different degrees of species interdependence into standard fishery models. In the remainder of the paper,  $r$  denotes the size of the refuge. This is the proportion of prey stock protected rather than the absolute quantity of prey.

**Economic activities.** I assume the presence of two specialized fishers (or two representative agents), each allowed to catch only one type of fish—either the prey species or the predator species. These fishers employ selective and environmentally respectful technology,

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<sup>1</sup>The Holling type II functional response captures the predator’s decelerating intake rate as prey density increases, modeling a satiation effect and allowing for more realistic trophic interactions.

avoiding by-catch and habitat damage. Specialized fishing, often driven by technological or legal constraints, is commonly practiced in small-scale fisheries.

The discounted present value of the benefit obtain from the harvesting activities,  $h_s(t)$ , has standard properties and is expressed as follows:

$$U_s((h_s(\cdot))_{t \geq 0}) = \int_0^\infty \frac{h_s(t)^{1-\nu}}{1-\nu} e^{-\rho t} dt, \quad \forall s \in x, y, \quad (2)$$

where  $\rho$  is the common discount factor, and  $\nu$  represents the elasticity of marginal utility.

Predator-prey models with endogenous harvesting are typically challenging to analyze due to additional state variables and nonlinear dynamics. However, recent contributions, such as Koulovatianos (2024) and Bataille (2024), shows that the problem becomes more tractable under a specific parametric restriction that links individual payoffs to species dynamics.

**Assumption 2**  $\nu = \theta$ .

This assumption states that the elasticity of marginal utility matches the degree of concavity in the species growth function. While this assumption is technical, it allows for closed-form solutions and provides clear economic intuitions. Relaxing this assumption would introduce additional complexities that are beyond the scope of this paper. Instead, I aim to use a sufficiently tractable model to theoretically explore the role of a prey refuge, an aspect that, to my knowledge, has not yet been fully examined.

**The two-stages game theoretic setting.** To investigate the welfare implications of prey-refuges within a fishery, I propose a two-stage game framework. In the first stage, a static and simultaneous membership game is played, where both prey and predator fishers decide whether to sign an agreement to protect a share of the prey population. In the second stage, players engage in a dynamic harvesting game given the fixed refuge size chosen during the first stage. There is no possibility of withdrawing from the agreement during the second stage since the share of protected prey is not a control variable for either player. The game is solved using backward induction.

(i) The second stage

Solving the infinite horizon dynamic fish harvesting game allows to derive, for each fishers, the discounted present value of their payoff as a function of the refuge size. The strategy space is restricted to stationary, linear, species-specific Markovian strategies of the following form:

$$\phi_s(s(t)) = \omega_s s(t), \quad \forall s = x, y, \quad (3)$$

where  $\omega_s$  stands for the catch rate of fisher  $s$ . Given the species' dynamics and the opponent's Markovian strategies, the prey fisher selects a harvesting path  $h_x(t)$  by solving the following maximization problem:

$$\mathcal{V}_x(r) = \max_{h_x(t) \geq 0} \int_0^\infty \frac{h_x(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to:

$$\begin{aligned} \dot{x}(t) &= A_x x(t)^\theta - \delta_x x(t) - b_x(1-r)^\theta y(t)^{1-\theta} x(t)^\theta - h_x(t), \\ \dot{y}(t) &= A_y y(t)^\theta - \delta_y y(t) + b_y(1-r)^{1-\theta} x(t)^{1-\theta} y(t)^\theta - \omega_y y(t), \\ x(0), y(0) &> 0. \end{aligned} \tag{4}$$

Likewise, the problem of the predator fisher is as follows :

$$\mathcal{V}_y(r) = \max_{h_y(t) \geq 0} \int_0^\infty \frac{h_y(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to:

$$\begin{aligned} \dot{x}(t) &= A_x x(t)^\theta - \delta_x x(t) - b_x(1-r)^\theta y(t)^{1-\theta} x(t)^\theta - \omega_x x(t), \\ \dot{y}(t) &= A_y y(t)^\theta - \delta_y y(t) + b_y(1-r)^{1-\theta} x(t)^{1-\theta} y(t)^\theta - h_y(t), \\ x(0), y(0) &> 0. \end{aligned} \tag{5}$$

(ii) The first stage

In the first stage, the decision to sign the agreement hinges on two factors: (i) the cooperative payoff attainable from selecting a specific prey-refuge size, and (ii) the sharing strategy that ensures individual rationality in signing the agreement. If players agree on a positive degree of protection,  $r^* > 0$ , this agreement is assumed to incur additional costs, represented by a cost function,  $\mathcal{C}(r)$ . This increasing and convex cost function is given by:

$$\mathcal{C}(r) = \frac{\xi r^2}{2}, \tag{6}$$

where  $\xi \in \mathbb{R}_+^2$ . The agreeability of the contract is formalized in the following definition:

**Definition 1** *A contract is said to be agreeable if and only if the following conditions are satisfied:*

(i) *There exists a non-empty subset of  $r \in [0, 1]$  such that:*

$$\phi(r) = \sum_{s \in x, y} [\mathcal{V}_s(r) - \mathcal{V}_s(0)] - \mathcal{C}(r) \geq 0. \tag{7}$$



(ii) Within the subset,  $r^*$  providing the highest surplus is chosen:

$$r^* = \arg \max_{r \in [0,1]} \phi(r). \quad (8)$$

After determining the optimal prey-refuge size, the distribution of the cooperative payoff between the fishers is addressed. For this, I use the cooperative Nash bargaining framework (see [Nash Jr, 1950](#)). This approach provides a unique payoff allocation that ensures individual and collective rationality in signing the agreement in the first stage. According to [Nash Jr \(1950\)](#), the sharing strategy is determined by solving the following static maximization problem:

$$\begin{aligned} & \max_{\mathcal{V}_x(r), \mathcal{V}_y(r)} [(\mathcal{V}_x(r) - \mathcal{V}_x(0))^\eta \cdot (\mathcal{V}_y(r) - \mathcal{V}_y(0))^{1-\eta}] \\ & \text{subject to:} \end{aligned} \quad (9)$$

$$\sum_{s \in \{x,y\}} (\mathcal{V}_s(r) - \mathcal{V}_s(0)) \leq \max_{r \in [0,1]} \phi(r)$$

where  $\eta$  represents the bargaining power of the prey fishing sector. In symmetric cases, the Nash bargaining solution typically leads to equal sharing of the surplus. However, due to player asymmetry—since they target different fish species—the solution is likely to be unequal.

The literature highlights concerns about dynamic time inconsistencies in Nash Bargaining solutions under commitment ([Haurie, 1976](#)). In this context, however, players commit to the size of the prey refuge, not catch flows, and the refuge size is not a control variable for either player. As a result, it is unclear what it would mean for a player to "leave the coalition". However, renegotiation of the refuge size at a later stage would likely yield a different outcome than initially agreed. This arises because the prey refuge impacts both current and future population dynamics. This creates incentives for players to renegotiate, implying potential time-inconsistency in the agreement. However, the focus of this paper is not on determining a time-consistent sharing rule, but rather on the existence of welfare gains from implementing a prey refuge. To keep the analysis focus, I will not address time-inconsistency directly. A simple approach would be to apply dynamic cooperative game theory, as in [Jørgensen and Zaccour \(2002\)](#) or [Petrosjan and Zaccour \(2003\)](#). This would involve decomposing total welfare over time, ensuring that each player's payoff at any moment exceeds their outside option, which evolves from prior negotiations. Alternatively, recursive Nash bargaining, as proposed by [Sorger \(2006\)](#), could be also be used. With the framework set, the next step involves explicitly characterizing the equilibrium of this two-stage game.

### 3 The differential game with prey-refuge

This stage is purely competitive. Both fishers, targeting distinct fish types, strategically make harvesting decisions given an unknown prey-refuge size  $r$ . The section proceeds as follows: First, I prove the existence of a unique linear Markov-Perfect Nash Equilibrium (MPNE) solving (4) and (5) simultaneously. This is followed by comparative statics on exploitation rates relative to prey-refuge size and its implications for individual welfare.

**Existence and Uniqueness of a linear MPNE.** The considered differential game with a prey-refuge remains standard. Using results from Dockner et al. (2000) and Van Long (2010), as well as recent work by Bataille (2024), dynamic programming techniques are rigorously applied. Given that agents specialize in specific fish types, the goal is to identify a pair of linear Markovian strategies that are best responses to one another. The following results are presented.

**Proposition 1** *Given parameters  $\delta_s, A_s, b_s, \theta, \rho$ , there exists a unique Markov-Perfect Nash Equilibrium (MPNE) in linear strategies where the harvest flows are  $h_s(t, r) = \omega_s(r)s(t, r)$  with fishing pressure  $\omega_s(r)$  :*

$$\omega_s(r) = \frac{1}{2} \left[ \frac{\rho}{1-\theta} + \delta_s \right] \Psi(r), \quad s = x, y, \quad (10)$$

where

$$\Psi(r) = \sqrt{\left(\frac{2\theta-1}{\theta}\right)^2 + \frac{4(1-\theta)}{\theta} \left(1 + \frac{b_x b_y (1-r)}{\left(\frac{\rho}{1-\theta} + \delta_x\right) \left(\frac{\rho}{1-\theta} + \delta_y\right)}\right)} - \left(\frac{2\theta-1}{\theta}\right), \quad (11)$$

and stock trajectories  $s(t, r)$  solving (1) under linear harvesting strategies. The present value of the equilibrium payoff flows is:

$$\mathcal{V}_s(r) = \frac{1}{\rho} \left[ \beta_s(r) \left( A_x + \frac{\rho}{1-\theta} x(0)^{1-\theta} \right) + \gamma_s(r) \left( A_y + \frac{\rho}{1-\theta} y(0)^{1-\theta} \right) \right] > 0, \quad (12)$$

where  $\omega_x(r) = \beta_x(r)^{-1/\theta}$  and  $\omega_y(r) = \gamma_y(r)^{-1/\theta}$ , determined by (10). Additionally,  $\gamma_x(r)$  and  $\beta_y(r)$  are:

$$\gamma_x(r) = \frac{\beta_x(r)}{b_y(1-r)^{1-\theta}} \left[ \frac{\rho}{1-\theta} + \delta_x - \frac{\theta}{1-\theta} \beta_x(r)^{-1/\theta} \right] \leq 0, \quad (13)$$

and

$$\beta_y(r) = -\frac{\gamma_y(r)}{b_x(1-r)^\theta} \left[ \frac{\rho}{1-\theta} + \delta_y - \frac{\theta}{1-\theta} \gamma_y(r)^{-1/\theta} \right] \geq 0. \quad (14)$$

**Proof.** The proof of this proposition is presented in Appendix A. ■

The value function in Equation (12) and the identified coefficients  $\gamma_s$  and  $\beta_s$  offer insights into the exploitation mechanism in the predator-prey model. These coefficients can be interpreted as follows:  $\gamma_s(r)$  and  $\beta_s(r)$  measure the effect of changes in predator and prey stocks on the lifetime utility of player  $s = x, y$ . As expected, an increase in predator stock lowers the prey fisher's payoffs ( $\gamma_x(r) \leq 0$ ) and raises the predator fisher's payoffs ( $\gamma_y(r) > 0$ ). Similarly, an increase in prey stock benefits both the prey fisher ( $\beta_x(r) > 0$ ) and the predator fisher ( $\beta_y(r) \geq 0$ ), consistent with standard predator-prey interactions. Furthermore,  $\omega_x(r) = \beta_x(r)^{-1/\theta}$  and  $\omega_y(r) = \gamma_y(r)^{-1/\theta}$  indicate that the marginal utility gains from stock increases ( $\beta_x$  or  $\gamma_y$ ) are inversely related to the fishing pressure on the respective species ( $\omega_x$  or  $\omega_y$ ). Thus, greater pressure on a stock reduces the marginal utility from further stock increases, reflecting the usual trade-offs in resource harvesting.

To fully characterize the equilibrium, several technical details must be addressed. First, the system dynamics under Markovian strategies must have a unique and continuous solution. The natural system dynamics (1), with  $h_s(t, r) = \omega_s(r)s(t, r)$ , form a two-dimensional Bernoulli-type differential system. These equations can be solved exactly by applying a substitution that transforms the nonlinear system into a linear one. Furthermore, under Assumption 1, the system's unique globally stable steady state is strictly positive.

Second, the transversality conditions must be satisfied, as they are crucial for problems extending over an infinite time horizon with unbounded payoffs. Since inter-temporal utility flows are determined by stock dynamics under linear strategies, the convergence of payoffs along the equilibrium path depends on the properties of these dynamics. It can be shown that, with the functional forms used, the transversality conditions hold regardless of the behavior of stock flows (see Bataille (2024)).

Finally, since the differential game is autonomous and has no terminal date, each subgame is of equal length, and the equilibrium strategies are independent of the initial conditions. Once the prey-refuge size is determined, the second-stage strategies are both subgame-perfect and time-consistent.

**The effect of the prey-refuge.** The model's economic outcomes are influenced by the strength of species interactions, and therefore by the prey-refuge size  $r$ . When there is no prey-refuge, i.e.,  $r = 0$ , interaction intensity is maximal and all prey are accessible to

predators, whereas when  $r = 1$ , the prey are fully protected, creating a complete sanctuary from natural predation.

It is clear that fishing pressures,  $\omega_s(r)$ , decrease as the prey-refuge size  $r$  increases (see Equation (10)). Since fishers target different species, their strategic behaviors are only influenced by biological interactions. Each fisher neglects the impact of their actions on the exploitation choice of non-targeted species. However, the prey-refuge reduces the intensity of species interactions and therefore mitigate the strategic externality. As a result, both fishers lower their fishing pressure. In the limiting case where the prey-refuge fully protects the prey, interactions vanish, and fishing pressures approach the levels that maximize yields in single-species models (Clark, 2010).

Comparing fishing intensities and their derivatives with respect to refuge parameters reduces to comparing their mortality rates. Prey and predator catch rates are linked as follows:

$$\omega_s(r) = \left( \frac{\rho}{1-\theta} + \delta_s \right) \left( \frac{\rho}{1-\theta} + \delta_{-s} \right)^{-1} \omega_{-s}(r) \quad \forall s = x, y \text{ and } -s \neq s. \quad (15)$$

Differentiating this equation with respect to  $r$  yields:

$$\partial_r \omega_s(r) = -\frac{b_x b_y}{2\omega_s(r)} \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_s \right) \left( \frac{\rho}{1-\theta} + \delta_{-s} \right)^{-1} < 0 \quad \forall s = x, y \text{ and } -s \neq s. \quad (16)$$

Thus,  $\delta_x \leq \delta_y \iff \omega_x(r) \leq \omega_y(r)$  and  $\partial_r \omega_x(r) \leq \partial_r \omega_y(r)$ . This implies that while fishers targeting species with higher mortality rates extract more, the reduction in fishing pressure due to prey-refuge effects is also more important.

To analyze how the reduction in fishing pressure varies with the prey-refuge size, I examine the concavity of these functions. Differentiating (16) yields:

$$\partial_r^2 \omega_s(r) = -\left[ \frac{b_x b_y}{2} \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_s \right) \left( \frac{\rho}{1-\theta} + \delta_{-s} \right)^{-1} \right]^2 \frac{1}{\omega_s^3(r)} < 0 \quad \forall s = x, y \text{ and } -s \neq s. \quad (17)$$

Fishing pressures are concave function of the prey-refuge size. This indicates that the reduction in fishing pressure becomes more pronounced as the prey-refuge size become large. Figure 1 summarize these results.

The prey-refuge parameter affects not only the fishing pressure but also the inter-temporal

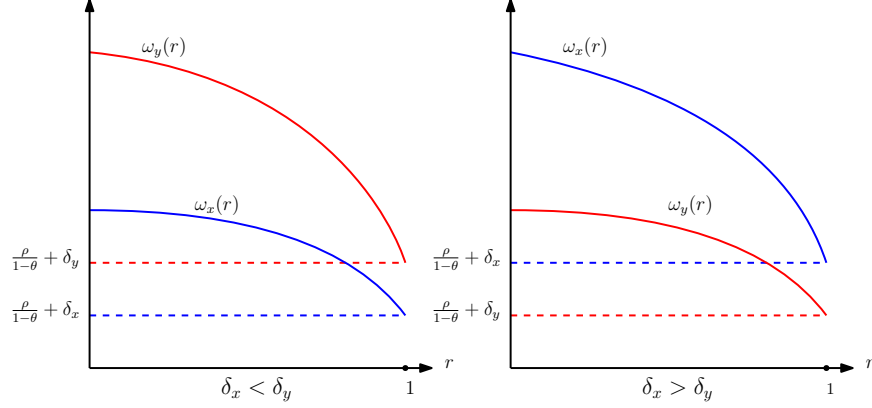


Figure 1: Fishing pressure as a function of the share of protected prey.

lifetime utility of the players through the interacting parameters  $\gamma_x(r)$  and  $\beta_y(r)$ . When prey are fully protected, i.e., when  $r = 1$ , we have  $\gamma_x(1) = 0$  and  $\beta_y(1) = 0$ , and the interactions no longer have any effect on the inter-temporal utility. However, for any  $r \neq 1$ , the parameters  $\gamma_x(r) < 0$  and  $\beta_y(r) > 0$  have opposite signs. Given that  $\beta_x(r)$  and  $\gamma_y(r)$  are increasing functions of the prey-refuge size  $r$ , the following corollary can be derived:

**Corollary 1** *The present value of the inter-temporal lifetime utility flows has the following properties:*

- (i) *Full prey protection, i.e.,  $r = 1$ , maximizes the prey fisher's welfare.*
- (ii) *No prey protection, i.e.,  $r = 0$ , does not necessarily maximize the predator fisher's welfare.*

**Proof.** The proof of (ii) will be provided later with an example (see Figure 11). ■

The first result is intuitive: as the refuge reduces predation intensity on the prey stock, competition between the prey fishers and the natural predators decreases, leading prey fishers to always benefit from greater prey protection. However, the second result stems from the interaction of two opposing effects: reduced access to prey can diminish the positive marginal impact that the prey population has on the predator fisher's payoff, which must be weighed against the benefits derived from a reduction in predator fishing intensity.

The key takeaways are twofold: First, prey refuges can benefit predator fishers, and second, even when they do not, the most significant negative effects occur at larger refuge sizes due to the concavity of  $\omega_s(r)$ . This raises an important question about the global efficiency of prey refuges in species-specific harvesting, which is the focus of the next section.

## 4 The cooperative solution over prey-refuge size

The prey refuge affects fishing pressure and equilibrium outcomes differently due to the asymmetry in fishers targeting distinct types of fish (see Corollary 1). To examine the global efficiency of the prey refuge, I introduce a cooperative stage where players can agree on the degree of protection over the prey stock, specifically choosing its size  $r^*$ . This constitutes a singular, binding agreement made at  $t = 0$ . The chosen prey refuge size is time-independent, representing a fixed proportion of the environment protected from predator access. It cannot be individually altered over time, as it is not a controllable variable. Fishers can only choose their best strategies in the second stage, given the refuge established in the initial stage.

This stage addresses the following question: are there efficient and equitable agreements on the size of the refuge? To explore this, I proceed in two steps: (i) Are there collective gains in protecting part of the prey stock from predators? If so, (ii) how should this surplus be shared among fishers? I use a cooperative axiomatic approach, specifically applying the Nash Bargaining solution (see Nash Jr, 1950). To focus on conditions leading to self-enforced agreements, the bargaining process is not modeled using a strategic form game as in Rubinstein (1982), but instead relies solely on a set of axioms with the Nash Bargaining solution applied.

**Existence of surplus.** What is the cooperative surplus generated when the predator and prey fishers agree to protect a portion  $r^*$  of the prey stock? The methodology mainly consist in exploring the condition in (7). Using the equilibrium payoff in (12) and the cost function in (6), this condition is expressed as:

$$\begin{aligned} \phi(r) = & \frac{1}{\rho} \left[ \left( A_x + \frac{\rho}{1-\theta} x(0)^{1-\theta} \right) \left( \beta_x(r) + \beta_y(r) - \beta_x(0) - \beta_y(0) \right) \right. \\ & \left. + \left( A_y + \frac{\rho}{1-\theta} y(0)^{1-\theta} \right) \left( \gamma_x(r) + \gamma_y(r) - \gamma_x(0) - \gamma_y(0) \right) \right] \\ & - \frac{\xi r^2}{2} \geq 0 \end{aligned} \tag{18}$$

where  $\beta_s(r), \gamma_s(r)$  are defined in Proposition 1. A necessary and sufficient condition is derived from the behavior of the  $\phi(r)$  function at the status quo.

**Proposition 2** *A strictly positive prey-refuge size  $r^* > 0$  ensures agreeability if and only if:*

- (i).  $[\partial_r \phi(r)]_{r=0} > 0$ , or

(ii).  $[\partial_r \phi(r)]_{r=0} \leq 0$  and  $\exists \bar{r} \in (0, 1)$  such that  $\phi(\bar{r}) = \phi(0)$ ,

holds. If either condition holds, the optimal prey-refuge is given by:

$$r^* = \arg \max_{r \in [0, 1]} \phi(r). \quad (19)$$

These parameter-dependent conditions define cases of global efficiency for the selected prey-refuge size. The economic interpretation of conditions (i) and (ii) presents complexities, and specific stylized examples to illustrate these results are provided in Section 5. Prior to discussing these examples, I address the sharing mechanism.

**The sharing mechanism.** Let us assume that either condition (i) or (ii) holds and the optimal prey refuge is as in (19). The next step is to determine how the fishers will share the cooperative payoffs generated by agreeing on a specific prey-refuge size. To address this, I apply the Nash bargaining solution. Solving the program in (9) yields the following result:

**Proposition 3** *Suppose the parameters  $A_s, \delta_s, b_s, \rho, \theta, \xi$  are such that either condition (i) or (ii) is satisfied. Then, there exists a unique payoff allocation  $(\mathcal{V}_x^{\mathcal{NB}}, \mathcal{V}_y^{\mathcal{NB}})$ , which solves the Nash bargaining problem in (9), and is given by:*

$$\begin{pmatrix} \mathcal{V}_x^{\mathcal{NB}} \\ \mathcal{V}_y^{\mathcal{NB}} \end{pmatrix} = \begin{pmatrix} \mathcal{V}_x(0) \\ \mathcal{V}_y(0) \end{pmatrix} + \begin{pmatrix} \nu \\ 1 - \nu \end{pmatrix} \left( \max_{r \in [0, 1]} \phi(r) \right) \quad (20)$$

The Nash Bargaining solution allocates to each fisher the present value of their status-quo utility plus a share of the cooperative surplus based on their bargaining power. This agreement is both efficient and equitable, and, importantly, the results are robust to the choice of solution concept.

## 5 Stylized examples

Thus far, the behavioral changes resulting from prey refuge implementation, along with the conditions for economic gains, have been established. However, the conditions in Proposition 2 are challenging to interpret due to the complexity of the function  $\phi(r)$ . To support the theoretical model, this section presents a set of stylized numerical examples illustrating the potential (non-)existence of cooperative surplus. These examples are not intended to model specific real-world cases but rather to outline key characteristics a two-species fishery should satisfy to yield economic gains from prey-refuge implementation. Among several relevant

parameters, I focus on one biological parameter—the initial stock levels—and one economic parameter—the discount factor.

Initial stock levels are expected to play a crucial role, as heavily depleted stocks before negotiation could imply higher marginal gains or reduced marginal losses from implementing a prey refuge. Similarly, the discount factor is anticipated to be significant, as prey refuges may take time to yield benefits through population dynamics. Consequently, the players’ discounting of future returns will strongly influence their willingness to wait for these benefits.

The methodology proceeds as follows. I first present a benchmark case satisfying condition (i) in Proposition 2 to illustrate the mechanism behind prey refuge implementation. Next, I explore cases where either prey or predator populations are significantly depleted prior to negotiation. Then, I explore the role of the discount rate, specifically when players became more impatient. Lastly, I provide an example where agreement can occur even when utility is not transferable.

**Benchmark with(out) Costs.** Consistent with Assumption 1 and Proposition 2, the specific parameter values used are detailed in Table 1:

$\xi$	$\rho$	$\theta$	$\delta_x$	$\delta_y$	$A_x$	$A_y$	$b_x$	$b_y$	$x(0)$	$y(0)$
0	0.05	0.5	0.1	0.05	0.1	0.06	0.1	0.15	0.8	0.3

Table 1: Parameter values

While these parameters are not intended to represent a specific predator-prey fishery, they provide an illustrative two-species predator-prey model. In this example, the predator encounters prey at a relatively low rate ( $b_x$ ) but efficiently converts biomass ( $b_y$ ), with a low intrinsic growth rate, making the prey its primary food source. The prey population is initially more abundant than the predator population ( $x(0) > y(0)$ ), aligning with typical trophic biomass distributions. The discount factor ( $\rho$ ) is set to reflect that fishers value future payoffs.

As an initial scenario, consider a costless prey refuge implementation,  $\xi = 0$ . Figure 9 displays the cooperative surplus,  $\phi(r)$ , as a function of prey-refuge size, measured as the sum of individual welfare gains minus their outside options. Figure 10 shows individual welfare changes relative to refuge size.

The cooperative surplus exhibits an inverted U-shaped pattern, increasing with prey-refuge size until reaching a threshold, after which it declines. This non-linear relationship reflects the trade-off between the economic benefits of protection and the opportunity costs of restricting predator access. Figure 10 shows that prey fishers consistently gain from the refuge, with their welfare rising sharply, while predator fishers experience declining welfare.



## Cooperative Surplus and Individual Welfare when $\xi = 0$

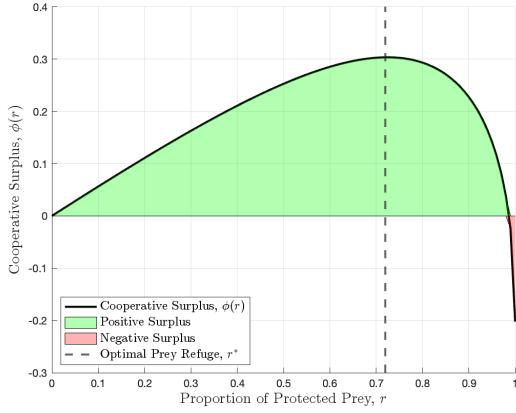


Figure 2: Cooperative Surplus

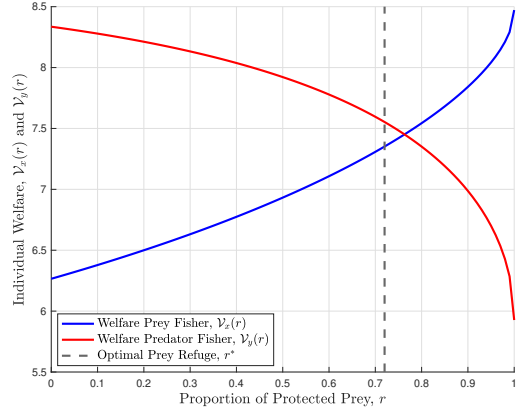


Figure 3: Individual Welfare

Similar to changes in the catch rate shown in Figure 1, the largest marginal welfare shifts occur at higher refuge sizes.

With transfers between players, they reach an agreement on an optimal refuge size of approximately  $r^* \approx 72\%$ , where the marginal benefit to prey fishers equals the marginal loss to predator fishers. At this equilibrium, prey fishers compensate predator fishers for welfare losses resulting from the refuge.

To extend this example, consider a scenario where prey-refuge implementation incurs a cost,  $\xi = 0.2$ , while all other parameters remain unchanged. As expected, this cost reduces the cooperative surplus, as illustrated in Figure 4. The optimal prey-refuge size correspondingly decreases to  $r^* \approx 61\%$ .

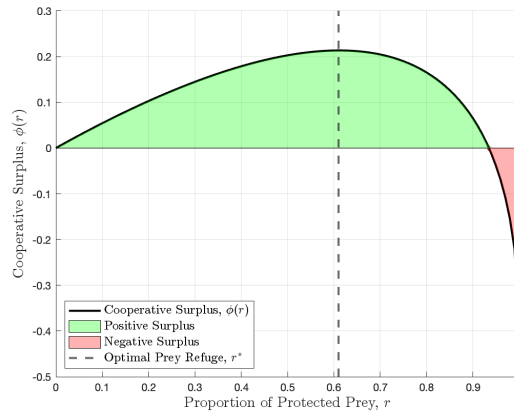


Figure 4: Cooperative surplus with  $\xi = 0.2$ .

**The Role of Initial Conditions  $x(0), y(0)$ .** This section examines the influence of initial prey and predator stock values on the optimal refuge size. Cooperative management often aims to stabilize depleted stock levels, so it is valuable to assess how an initial scarcity of prey or predator might affect the optimal prey refuge. When prey stocks are initially low, I anticipate changes in the marginal effect of prey refuge for both fishers, likely resulting in a different optimal refuge size. Using the parameter values in Table 1, the figures below illustrate the cooperative surplus and individual welfare when the prey stock decreases substantially from  $x(0) = 0.8$  to  $x(0) = 0.1$ .

Cooperative Surplus and Individual Welfare for Lower  $x(0)$

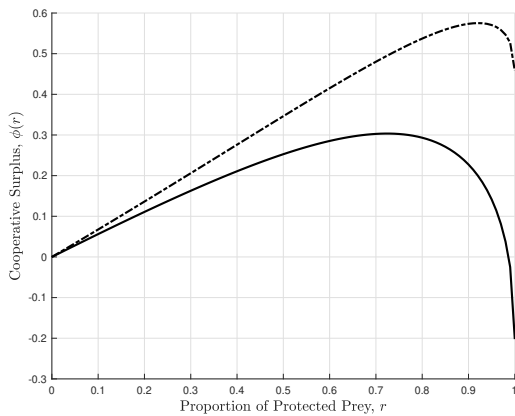


Figure 5: Cooperative Surplus

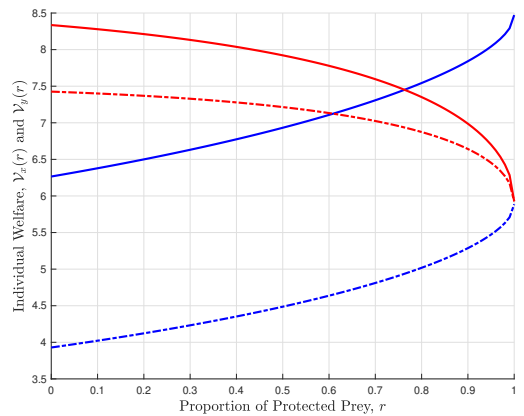


Figure 6: Individual Welfare

The associated legend is provided in Table 2:

Table 2: Effect of Lower  $x(0)$  on Cooperative Surplus and Value Functions

$x(0)$	$\phi(r)$	Value Function Prey $\mathcal{V}_x(r)$	Value Function Predator $\mathcal{V}_y(r)$
0.8	—————	—————	—————
0.1	- - - - -	- - - - -	- - - - -

As anticipated, the optimal prey-refuge size increases as the prey stock becomes severely depleted, rising from 0.72 to 0.93. While the present value of welfare declines for both players, prey fishers maintain the same marginal benefits from the refuge, whereas predator fishers experience lower marginal losses across all refuge sizes. With lower initial prey stocks, protecting a larger portion of the prey reduces negative impacts on predator fishers. The resulting agreement is still obtain by balancing marginal benefits and (lower) marginal losses.

Similarly, when exploring the trade-off with a lower initial predator population, we expect a decrease in predator fisher welfare and an increase in prey fisher welfare due to reduced

predator pressure on the prey stock. This shift should also affect the marginal impacts on welfare. The following figures show cooperative surplus and individual welfare as predator stock declines from  $y(0) = 0.3$  to  $y(0) = 0.05$ .

### Cooperative Surplus and Individual Welfare for Lower $y(0)$

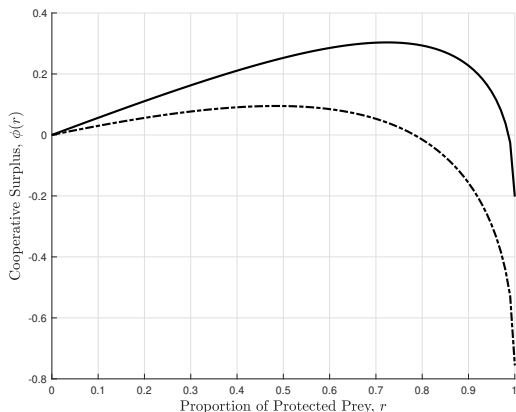


Figure 7: Cooperative Surplus

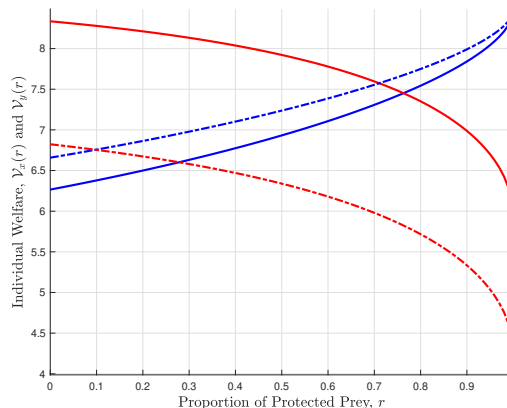


Figure 8: Individual Welfare

The associated legend is provided in Table 3:

Table 3: Effect of Lower  $y(0)$  on Cooperative Surplus and Value Functions

$y(0)$	$\phi(r)$	Value Function Prey $\mathcal{V}_x(r)$	Value Function Predator $\mathcal{V}_y(r)$
0.3	—————	—————	—————
0.05	- - - - -	- - - - -	- - - - -

While a lower initial predator population reduces predator fisher welfare, the marginal losses due to the refuge remain similar to the initial scenario. However, the prey fisher experiences both higher welfare and greater marginal gains from expanding the prey refuge. Consequently, balancing these marginal benefits and losses results in a smaller optimal refuge size when the initial predator population is lower.

**The Role of the Discount Factor  $\rho$ .** This section examines the role of intertemporal trade-offs in harvesting decisions, represented by  $\rho$ . By protecting a fixed portion of the prey stock from predators, the prey refuge alters the dynamics of both populations, having both immediate and long-term effects. In particular, the refuge reduces the immediate food available to predators but promotes prey recovery over time, implying a trade-off between short-term losses and long-term benefits. This dynamic nature of the refuge implementation leads to a lasting impact on both players' welfare, making the (common) discount factor  $\rho$  crucial to the existence of cooperative surplus.

Drawing from the fisheries economics literature, notably [Larkin et al. \(2001\)](#), higher discount rates reduce the net present value of future harvests, leading to lower equilibrium welfare as fishers prioritize immediate gains over future stock recovery. Here, I investigate whether a higher discount rate not only reduces equilibrium payoffs but also changes the marginal impact of the prey refuge on each player, thus affecting the optimal refuge size. Following [Larkin et al. \(2001\)](#), discount rates between 4% and 15% are appropriate for most applied fisheries. Keeping all parameters as in Table 1 except  $\rho$ , I present cooperative surplus and individual welfare results for three discount rates:  $\rho = 0.05$  (benchmark),  $\rho = 0.08$ , and  $\rho = 0.12$ .

Cooperative Surplus and Individual Welfare for  $\rho = 0.05, 0.08, 0.12$

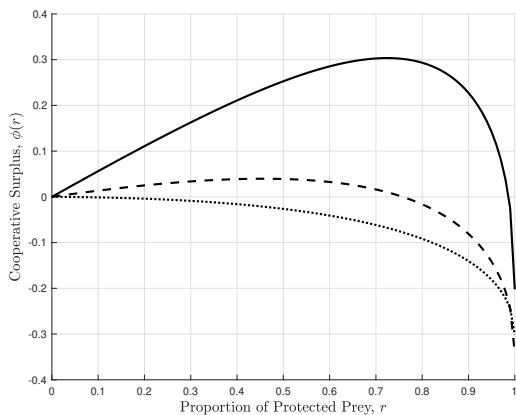


Figure 9: Cooperative Surplus

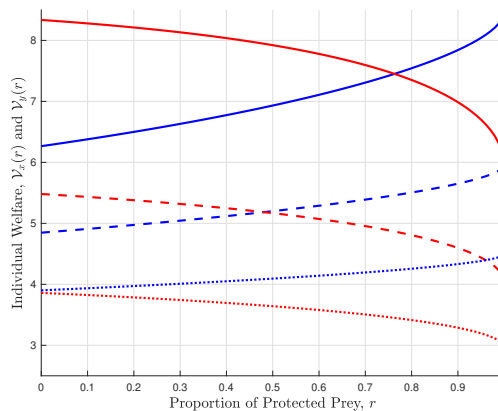


Figure 10: Individual Welfare

The associated legend is provided in Table 4:

Table 4: Legend for Different Values of  $\rho$

$\rho$ Value	$\phi(r)$	Value Function Prey $\mathcal{V}_x(r)$	Value Function Predator $\mathcal{V}_y(r)$
0.05	—————	—————	—————
0.08	- - - - -	- - - - -	- - - - -
0.12	.....	.....	.....

As expected, higher discount rates lead to lower equilibrium welfare, consistent with the fisheries literature. Notably, at higher  $\rho$ , the slope of  $\mathcal{V}_s(r)$  for all  $s = x, y$  declines, indicating that marginal gains or losses are smaller when fishers are more impatient. At  $\rho = 0.12$ , the marginal gains for prey fishers become insufficient to offset predator fishers' losses, resulting in a non-existent cooperative surplus. Thus, given the dynamic effects of the refuge, the discount factor is an important parameter, as excessive discounting can eliminate cooperative surplus altogether.

**Agreement in the Non-Transferable Utility (NTU) Case.** There are several reasons why compensations may not occur, such as political tensions or the absence of proper economic valuation for compensation. In this example, I emphasize that agreements can still be reached even when utility is not transferable between players. In practice, countries may be unwilling to pay, even if it would yield Pareto improvements. Here, agreement is possible when both prey and predator fishers directly benefit without transfers. As shown in previous examples, the likelihood of agreement depends heavily on the discounting of future returns. To illustrate, I adjust the common discount rate to a very low level,  $\rho = 0.001$ , representing high valuation of future payoffs.

In this NTU setting, the coalition’s worth is no longer defined by maximizing the sum of payoffs (as in definition 1). Instead, the bargaining set includes all refuge sizes that yield Pareto improvements over the status quo. The Pareto frontier becomes non-linear, and the contract curve—the segment of the Pareto frontier above the disagreement point—represents the acceptable refuge sizes. The figures below illustrate individual welfare and the Pareto frontier in this specific example.

Agreement without Transfers:  $\xi = 0$  and  $\rho = 0.001$

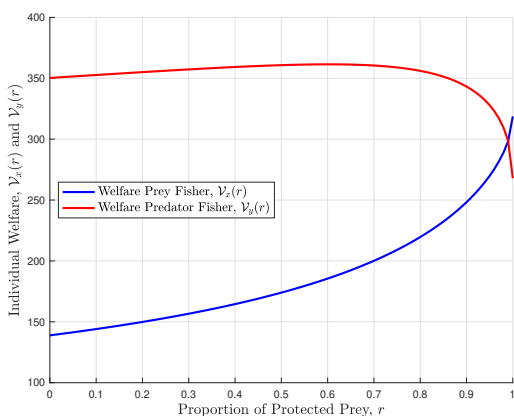


Figure 11: Individual Welfare

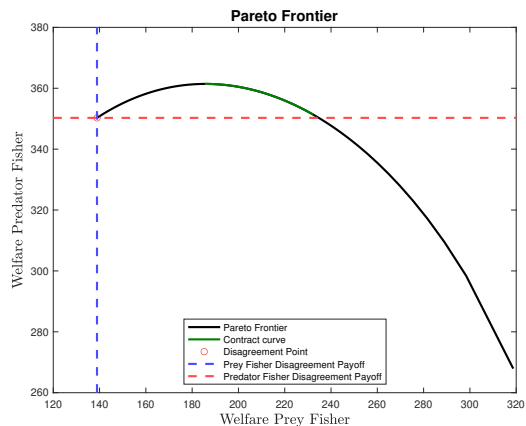


Figure 12: Pareto Frontier NTU

Only refuge sizes on the contract curve are viable for agreements. To determine the cooperative refuge size, the Nash Bargaining Solution, as defined in 9, is applied but modified to account for the non-linear Pareto frontier. The optimal size is chosen based on the relative bargaining power of the fishers.

In the NTU setting, the choice of solution concept influences the outcome: for example, the Kalai-Smorodinsky solution emphasizes aspirations, while the egalitarian solution ensures equal gains from the disagreement point. However, a detailed analysis of these solution concepts is beyond the scope of this paper.

## 6 First-Best vs. Second-Best Outcomes

Cooperative fishery management literature often focuses on how to allocate catch quotas among members to achieve first-best outcomes (see Grønbaek et al. (2020) for a recent overview). This paper takes a similar approach but proposes an alternative method to enhance the efficiency of multi-species fisheries through the implementation of an (artificial) prey refuge. Unlike catch quotas, this approach imposes no direct restrictions on fishers' behavior, instead altering incentives dynamically through population changes. Thus, it serves as a second-best policy, particularly when cooperation on catch quotas is infeasible.

In the model presented, if players agree to implement a prey refuge in the first stage, they subsequently enter a non-cooperative differential game, where each player strategically determines their harvesting flow based on the selected refuge size and their opponent's strategy. An interesting question arises as to whether the refuge still generates cooperative surplus if players can also cooperate over catch flows. This is relevant as many interacting fish stocks are managed by Regional Fishery Management Organizations (RFMOs), where quotas regulate harvests. Exploring the combined effects of prey refuge and coordinated harvesting could provide insights into the broader applicability of artificial refuges.

To investigate this, consider a scenario where prey and predator fishers coordinate their harvests to maximize joint discounted payoffs. Let  $\bar{V}(r)$  denote the coalition's worth. The joint optimization problem is defined as follows:

$$\begin{aligned} \bar{V}(r) = \max_{h_x(\cdot), h_y(\cdot) \geq 0} & \int_0^\infty \left[ \frac{h_x(t)^{1-\theta}}{1-\theta} + \frac{h_y(t)^{1-\theta}}{1-\theta} \right] e^{-\rho t} dt \\ \text{s.t.} & \dot{x}(t) = A_x x(t)^\theta - \delta_x x(t) - b_x (1-r)^\theta y(t)^{1-\theta} x(t)^\theta - h_x(t) \\ & \dot{y}(t) = A_y y(t)^\theta - \delta_y y(t) + b_y (1-r)^{1-\theta} x(t)^{1-\theta} y(t)^\theta - h_y(t) \\ & x(0), y(0) > 0 \end{aligned} \quad (21)$$

The optimal control problem is solved using dynamic programming, with technical details in Appendix B. This yields the following results:

**Proposition 4** *For given parameters  $\delta_s, A_s, b_s, \theta, \rho$ , the optimal joint value function  $\bar{V}(r)$  is:*

$$\bar{V}(r) = \frac{1}{\rho} \left[ \beta(r) \left( A_x + \frac{\rho}{1-\theta} x(0)^{1-\theta} \right) + \gamma(r) \left( A_y + \frac{\rho}{1-\theta} y(0)^{1-\theta} \right) \right] > 0, \quad (22)$$

where  $\beta(r), \gamma(r)$  are the unique solutions to the following system:

$$\beta \left[ \frac{\rho}{1-\theta} + \delta_x \right] - \frac{\theta}{1-\theta} \beta^{-\frac{1-\theta}{\theta}} - \gamma b_y (1-r)^{1-\theta} = 0, \quad (23)$$

$$\gamma \left[ \frac{\rho}{1-\theta} + \delta_y \right] - \frac{\theta}{1-\theta} \gamma^{-\frac{1-\theta}{\theta}} + \beta b_x (1-r)^\theta = 0. \quad (24)$$

The optimal harvesting flows are:

$$\bar{h}_x(t, r) = \beta(r)^{-1/\theta} x(t, r) = \bar{\omega}_x(r) x(t, r) \quad \text{and} \quad \bar{h}_y(t, r) = \gamma(r)^{-1/\theta} y(t, r) = \bar{\omega}_y(r) y(t, r), \quad (25)$$

where  $x(t, r), y(t, r)$  solve the system dynamics in (1) under optimal harvesting.

To illustrate the mechanisms, I compare the joint optimal value function from Equation (22) with the sum of individual value functions in Equation (12). Given the importance of the discount rate in determining welfare gains, I provide two examples: one with  $\rho = 0.05$  (benchmark) and another with a lower discount rate,  $\rho = 0.02$ . The results are shown in the figures below.

First-Best versus Second-Best Outcomes for  $\rho = 0.05$  and  $\rho = 0.02$

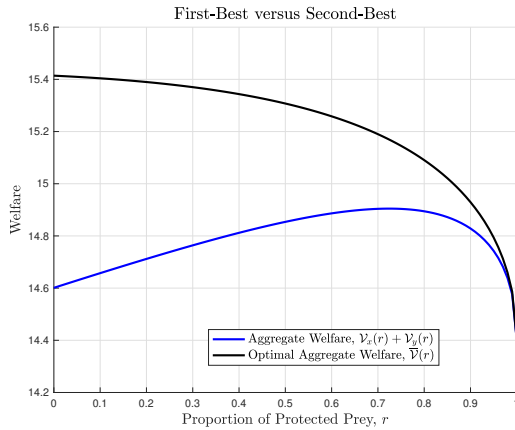


Figure 13:  $\rho = 0.05$

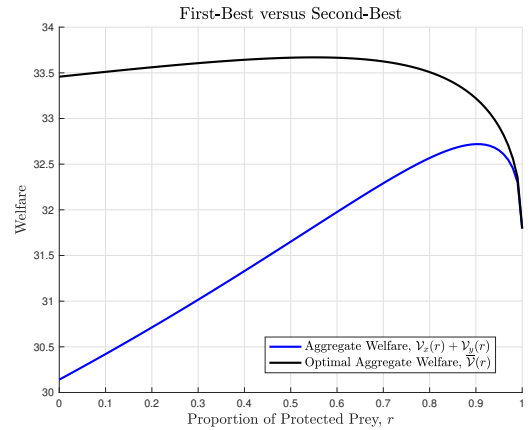


Figure 14:  $\rho = 0.02$

As shown previously, when  $\rho = 0.05$ , fishers are incentivized to implement a prey refuge, as it increases future cooperative surplus. However, when cooperation allow for a joint choice of harvest flows, the prey refuge becomes useless and does not enhance social welfare, serving only as a second-best policy. In contrast, at a lower discount rate ( $\rho = 0.02$ ), cooperative behavior in harvesting and prey refuge implementation can jointly improve welfare, with the refuge forming part of a broader cooperative management strategy.

## 7 Concluding remarks

This paper introduces a prey refuge into a standard predator-prey model, where a fixed portion of the environment is protected from natural predation. The refuge alters biological interactions, influencing population dynamics and fishing behaviors by reducing fishing pressure on both prey and predator stocks. Although complete prey protection maximizes prey fishing payoffs, it does not necessarily minimize predator fishing payoffs. Numerical examples highlight initial conditions and the discount factor as crucial parameters. Heavily depleted prey (resp. predator) populations require a larger (resp. smaller) optimal prey refuge. If players are too impatient, marginal gains may fail to offset marginal losses, eliminating potential welfare gains. Finally, the prey refuge can still enhance efficiency even when fishers coordinate catches under regional fishery management organizations. This paper thus contributes to the multi-species fishery management literature by providing insights into the use of artificial refuges to improve the economic efficiency of fisheries.

This paper has several important implications for fishery management. Perhaps most notably, it demonstrates that the economic efficiency of a multi-species fishery can be enhanced without directly restricting fishers' behavior. Ensuring compliance and monitoring in fishery policies is challenging, contributing to the rise in illegal, unreported, and unregulated (IUU) fishing. In this framework, incentives arise solely from changes in population dynamics, which could help address both compliance and monitoring challenges if the refuge is established by the regulator.

Secondly, this paper contributes to the ecosystem-based fishery management (EBFM) approach. Although habitat and fish stocks are inherently linked, few fishery management policies incorporate habitat considerations, especially for interacting species. This study helps bridge this gap and encourages future research in this direction.

Finally, the model offers a basis for evaluating economic losses or gains related to climate shocks on marine habitats. For instance, it can quantify the consequences of losing a key habitat function—protection from predation—due to climate-related disasters, highlighting how the deviation from optimal protection impacts fishery outcomes.

This study opens several avenues for future research. First, the model assumes fishing occurs within the refuge, yet some refuges are designated as Marine Protected Areas where fishing is also prohibited. An interesting extension would examine the trade-offs discussed here under conditions where all forms of predation—natural and human—are restricted. Although this would require significant adjustments to the current model, it represents a promising direction for further study.

Second, the prey refuge is assumed to protect a constant share of the environment.



Although questionable, this choice of modeling allow to keep tractability. Future research could explore alternative formulations, such as protecting a fixed number of prey or incorporating spatial components with patchy predator-prey models. These modifications would refine the model by introducing migration externalities, and could provide more complex but richer insights for fishery policies.

Finally, the implications of this research extend beyond fisheries. Offshore energy structures, such as wind farms, can function as artificial reefs, creating potential positive externalities for both energy production and fishing. Exploring these interactions in a richer model could be a valuable avenue for future research, with important policy implications.

## References

- Bataille, G. (2024). An explicit solution to harvesting behaviors in a predator–prey system. *Natural Resource Modeling*, page e12408.
- Clark, C. W. (2010). *Mathematical bioeconomics: the mathematics of conservation*, volume 91. John Wiley & Sons.
- Dockner, E., Jørgensen, S., Long, N., and Sorger, G. (2000). *Differential Games in Economics and Management Science* Cambridge University Press, Cambridge.
- Fischer, R. D. and Mirman, L. J. (1992). Strategic dynamic interaction: fish wars. *Journal of Economic Dynamics and Control*, 16(2):267–287.
- Flaaten, O. (1991). Bioeconomics of sustainable harvest of competing species. *Journal of Environmental Economics and Management*, 20(2):163–180.
- Foley, N., Armstrong, C., Kahui, V., Mikkelsen, E., and Reithe, S. (2012). A review of Bioeconomic Modeling of Habitat-Fisheries Interactions. *International Journal of Ecology*.
- Grønbaek, L., Lindroos, M., Munro, G., and Pintassilgo, P. (2020). *Game theory and fisheries management: Theory and Applications*. Springer.
- Hannesson, R. (1983). Optimal harvesting of ecologically interdependent fish species. *Journal of Environmental Economics and Management*, 10(4):329–345.
- Haurie, A. (1976). A note on nonzero-sum differential games with bargaining solution. *Journal of Optimization Theory and Applications*, 18(1):31–39.
- Holland, D. and Schnier, K. E. (2006). Individual habitat quotas for fisheries. *Journal of Environmental Economics and Management*, 51:72–92.
- Holling, C. S. (1959). Some characteristics of simple types of predation and parasitism<sup>1</sup>. *The canadian entomologist*, 91(7):385–398.
- Jean-Marie, A. and Tidball, M. (2024). Dynamic fishing with endogenous habitat damage. *Dynamic Games and Applications*, 14(1):33–56.
- Jørgensen, S. and Zaccour, G. (2002). Time consistency in cooperative differential games. In *Decision & control in management science: essays in honor of Alain Haurie*, pages 349–366. Springer.

- Ko, W. and Ryu, K. (2006). Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a prey refuge. *J. Differ. Equ*, 231:534–550.
- Koulovatianos, C. (2024). Rational noncooperative strategic exploitation of species in a predator–prey ecosystem with random disturbances. *Dynamic Games and Applications*, 14(1):57–77.
- Larkin, S., Harte, M., Quigley, K., and Sylvia, G. (2001). Future generations, discount rates and the optimal harvest of fisheries resources. *IIFET 2000 Proceedings. International Institute of Fisheries Economics & Trade*.
- Long, N. V., Tidball, M., and Zaccour, G. (2020). Optimal harvesting and taxation when accounting for the marine environmental quality of the fishery. *Natural Resource Modeling*, 33(3):e12244.
- Ma, Z., Li, W., Zhao, Y., Wang, W., Zhang, H., and Li, Z. (2009). Effects of prey refuges on a predator–prey model with a class of functional responses: The role of refuges. *Mathematical Biosciences*, 218(73–79).
- Ma, Z., Wang, S., Li, W., and Li, Z. (2013). The effect of prey refuge in a patchy predator–prey system. *Mathematical Biosciences*, 243(1):126–130.
- May, R. M., Beddington, J. R., Clark, C. W., Holt, S. J., and Laws, R. M. (1979). Management of multispecies fisheries. *Science*, 205(4403):267–277.
- McNair, J. (1986). The effects of refuges on predator–prey interactions: a reconsideration. *Theor. Popul. Biol.*, 29(38).
- Moberg, E. (2016). Optimal bioeconomic management of changing marine resources. *Massachusetts Institute of Technology*.
- Moeller, H. V. and Neubert, M. G. (2013). Habitat damage, marine reserves, and the value of spatial management. *Ecological applications*, 23(5):959–971.
- Nash Jr, J. F. (1950). The bargaining problem. *Econometrica: Journal of the econometric society*, pages 155–162.
- Petrosjan, L. and Zaccour, G. (2003). Time-consistent shapley value allocation of pollution cost reduction. *Journal of economic dynamics and control*, 27(3):381–398.
- Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica: Journal of the Econometric Society*, pages 97–109.

- Ruxton, G. (1995). Short term refuge use and stability of predator–prey models. *Theor. Popul. Biol.*, 47(1).
- Sanchirico, J. N. and Wilen, J. E. (2001). A bioeconomic model of marine reserve creation. *Journal of Environmental Economics and Management*, 42(3):257–276.
- Sih, A. (1987). Prey refuges and predator-prey stability. *Theoretical Population Biology*, 31(1):1–12.
- Smith, M. D. and Wilen, J. E. (2003). Economic impacts of marine reserves: the importance of spatial behavior. *Journal of Environmental Economics and Management*, 46(2):183–206.
- Sorger, G. (2006). Recursive nash bargaining over a productive asset. *Journal of Economic Dynamics and Control*, 30(12):2637–2659.
- Udumyan, N., Ami, D., and Cartigny, P. (2010). Integrating habitat concerns into Gordon-Schaefer model. working paper or preprint.
- Van Long, N. (2010). *A survey of dynamic games in economics*, volume 1. World Scientific.
- Wang, Y. and Wang, J. (2012). Influence of prey refuge on predator–prey dynamics. *Theor. Popul. Biol.*, 67:191–201.

# Appendices

## A Proof of proposition 1

To prove the existence of a linear MPNE, I apply dynamic programming techniques. Let  $\mathcal{V}_s(x, y)$  be the value function of player  $s = x, y$  in state  $(x, y)$ . The associate Hamilton-Jacobi-Bellman (HJB) equations of problems (4) and (5) are as follows:

$$\begin{aligned} \rho \mathcal{V}_x(x, y) = \max_{h_x \geq 0} & \left\{ \frac{h_x^{1-\theta}}{1-\theta} + \partial_x \mathcal{V}_x(x, y) [A_x x^\theta - \delta_x x - b_x(1-r)^\theta y^{1-\theta} x^\theta - h_x] \right. \\ & \left. + \partial_y \mathcal{V}_x(x, y) [A_y y^\theta - \delta_y y + b_y(1-r)^{1-\theta} x^{1-\theta} y^\theta - \omega_y y] \right\} \end{aligned} \quad (26)$$

and,

$$\begin{aligned} \rho \mathcal{V}_y(x, y) = \max_{h_y \geq 0} & \left\{ \frac{h_y^{1-\theta}}{1-\theta} + \partial_x \mathcal{V}_y(x, y) [A_x x^\theta - \delta_x x - b_x(1-r)^\theta y^{1-\theta} x^\theta - \omega_x x] \right. \\ & \left. + \partial_y \mathcal{V}_y(x, y) [A_y y^\theta - \delta_y y + b_y(1-r)^{1-\theta} x^{1-\theta} y^\theta - h_y] \right\}. \end{aligned} \quad (27)$$

The first-order necessary conditions for the right-hand side of (26) and (27) are:

$$h_s^{-\theta} = \partial_s \mathcal{V}_s, \quad \forall s = x, y. \quad (28)$$

Given that the analysis is restricted to linear Markovian strategies as in (3), the player's value function is assumed to be:

$$\mathcal{V}_s(x, y) = \alpha_s + \frac{\beta_s x^{1-\theta}}{1-\theta} + \frac{\gamma_s y^{1-\theta}}{1-\theta}, \quad \forall s = x, y, \quad (29)$$

where the parameters  $\alpha_s$ ,  $\beta_s$ , and  $\gamma_s$  have to be identified using (26), (27), (28) and (29). Plugging the value functions (29) as well as the first order conditions (28) into (26) and (27) leads to :

$$\begin{aligned} \rho \left[ \alpha_x + \frac{\beta_x x^{1-\theta}}{1-\theta} + \frac{\gamma_x y^{1-\theta}}{1-\theta} \right] &= \frac{\beta_x^{-(1-\theta)/\theta} x^{1-\theta}}{1-\theta} + \beta_x x^{-\theta} [A_x x^\theta - \delta_x x - b_x(1-r)^\theta y^{1-\theta} x^\theta - \beta_x^{-1/\theta} x] \\ &+ \gamma_x y^{-\theta} [A_y y^\theta - \delta_y y + b_y(1-r)^{1-\theta} x^{1-\theta} y^\theta - \gamma_y^{-1/\theta} y]. \end{aligned} \quad (30)$$

and,

$$\rho \left[ \alpha_y + \frac{\beta_y x^{1-\theta}}{1-\theta} + \frac{\gamma_y y^{1-\theta}}{1-\theta} \right] = \frac{\gamma_y^{-(1-\theta)/\theta} y^{1-\theta}}{1-\theta} + \beta_y x^{-\theta} [A_x x^\theta - \delta_x x - b_x (1-r)^\theta y^{1-\theta} x^\theta - \beta_x^{-1/\theta} x] + \gamma_y y^{-\theta} [A_y y^\theta - \delta_y y + b_y (1-r)^{1-\theta} x^{1-\theta} y^\theta - \gamma_y^{-1/\theta} y]. \quad (31)$$

The values of  $\alpha_x$  and  $\alpha_y$  are directly obtained from the following equations:

$$\rho \alpha_x = \beta_x A_x + \gamma_x A_y, \quad (32)$$

$$\rho \alpha_y = \beta_y A_x + \gamma_y A_y. \quad (33)$$

Next, the marginal effects of the prey stock on the value functions of the prey and predator fishers are given by:

$$\frac{\rho \beta_x}{1-\theta} = \frac{\beta_x^{-(1-\theta)/\theta}}{1-\theta} - \beta_x (\delta_x + \beta_x^{-1/\theta}) + \gamma_x b_y (1-r)^{1-\theta}, \quad (34)$$

$$\frac{\rho \beta_y}{1-\theta} = -\beta_y (\delta_x + \beta_x^{-1/\theta}) + \gamma_y b_y (1-r)^{1-\theta}. \quad (35)$$

Similarly, the marginal effects of the predator stock on the value functions of the prey and predator fishers are described by:

$$\frac{\rho \gamma_x}{1-\theta} = -\beta_x b_x (1-r)^\theta - \gamma_x (\delta_y + \gamma_y^{-1/\theta}), \quad (36)$$

$$\frac{\rho \gamma_y}{1-\theta} = \frac{\gamma_y^{-(1-\theta)/\theta}}{1-\theta} - \beta_y b_x (1-r)^\theta - \gamma_y (\delta_y + \gamma_y^{-1/\theta}). \quad (37)$$

The coefficients  $\alpha_x$  and  $\alpha_y$  are identified through  $\beta_x$ ,  $\beta_y$ ,  $\gamma_x$ , and  $\gamma_y$ . This leads to a system of 4 equations with 4 unknowns. One approach to solving the system (34)-(37) simultaneously is to substitute (34) and (37) into (35) and (36). However, this substitution is valid only if  $(1-r) \neq 0$  or equivalently  $r \neq 1$ . Therefore, the system is first solved for the case  $r \neq 1$ , followed by solving it for  $r = 1$ . Finally, the continuity of the solution between these two cases is analyzed and established.

(i)  $r \neq 1$

Equations (34) and (37) can be reformulated as:

$$\gamma_x = \frac{\beta_x}{b_y(1-r)^{1-\theta}} \left[ \frac{\rho}{1-\theta} + \delta_x - \frac{\theta}{1-\theta} \beta_x^{-1/\theta} \right], \quad (38)$$

$$\beta_y = -\frac{\gamma_y}{b_x(1-r)^\theta} \left[ \frac{\rho}{1-\theta} + \delta_y - \frac{\theta}{1-\theta} \gamma_y^{-1/\theta} \right]. \quad (39)$$

Substituting (38) and (39) into (35) and (36) yields:

$$\left[ \frac{\rho}{1-\theta} + \delta_x - \frac{\theta}{1-\theta} \beta_x^{-1/\theta} \right] \left[ \frac{\rho}{1-\theta} + \delta_y + \gamma_y^{-1/\theta} \right] + b_x b_y (1-r) = 0, \quad (40)$$

$$\left[ \frac{\rho}{1-\theta} + \delta_y - \frac{\theta}{1-\theta} \gamma_y^{-1/\theta} \right] \left[ \frac{\rho}{1-\theta} + \delta_x + \beta_x^{-1/\theta} \right] + b_x b_y (1-r) = 0. \quad (41)$$

Using the first-order conditions (28),  $\omega_x = \beta_x^{-1/\theta}$  and  $\omega_y = \gamma_y^{-1/\theta}$ , and rearranging terms, we obtain:

$$\begin{aligned} -\frac{1-\theta}{\theta} \left[ \left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right) + b_x b_y (1-r) \right] + \left( \frac{\rho}{1-\theta} + \delta_y \right) \omega_x \\ - \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_x \right) \omega_y + \omega_x \omega_y = 0, \end{aligned} \quad (42)$$

and,

$$\begin{aligned} -\frac{1-\theta}{\theta} \left[ \left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right) + b_x b_y (1-r) \right] - \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_y \right) \omega_x \\ + \left( \frac{\rho}{1-\theta} + \delta_x \right) \omega_y + \omega_x \omega_y = 0. \end{aligned} \quad (43)$$

Subtracting Eq. (43) from Eq. (42) gives the following linear equation:

$$\left( \frac{\rho}{1-\theta} + \delta_y \right) \frac{\omega_x}{\theta} - \left( \frac{\rho}{1-\theta} + \delta_x \right) \frac{\omega_y}{\theta} = 0 \iff \omega_x = \left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right)^{-1} \omega_y. \quad (44)$$

Substituting this back into Eq. (43) results in the following polynomial equation in  $\omega_y$ :

$$\begin{aligned}
-\frac{1-\theta}{\theta} \left[ \left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right) + b_x b_y (1-r) \right] + \left( \frac{\rho}{1-\theta} + \delta_x \right) \left[ \frac{2\theta-1}{\theta} \right] \omega_y \\
+ \left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right)^{-1} \omega_y^2 = 0.
\end{aligned} \tag{45}$$

Similarly, reversing Eq. (44) yields a polynomial in  $\omega_x$ :

$$\begin{aligned}
-\frac{1-\theta}{\theta} \left[ \left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right) + b_x b_y (1-r) \right] + \left( \frac{\rho}{1-\theta} + \delta_y \right) \left[ \frac{2\theta-1}{\theta} \right] \omega_x \\
+ \left( \frac{\rho}{1-\theta} + \delta_y \right) \left( \frac{\rho}{1-\theta} + \delta_x \right)^{-1} \omega_x^2 = 0.
\end{aligned} \tag{46}$$

In both (45) and (46), the constant term is identical and negative, while the coefficients of the polynomials are positive. This ensures the existence of a unique positive solution  $\{\omega_x, \omega_y\}$ , given by:

$$\omega_x = \frac{1}{2} \left[ \frac{\rho}{1-\theta} + \delta_x \right] \left[ - \left( \frac{2\theta-1}{\theta} \right) + \sqrt{\left( \frac{2\theta-1}{\theta} \right)^2 + \frac{4(1-\theta)}{\theta} \left( 1 + \frac{b_x b_y (1-r)}{\left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right)} \right)} \right], \tag{47}$$

and

$$\omega_y = \frac{1}{2} \left[ \frac{\rho}{1-\theta} + \delta_y \right] \left[ - \left( \frac{2\theta-1}{\theta} \right) + \sqrt{\left( \frac{2\theta-1}{\theta} \right)^2 + \frac{4(1-\theta)}{\theta} \left( 1 + \frac{b_x b_y (1-r)}{\left( \frac{\rho}{1-\theta} + \delta_x \right) \left( \frac{\rho}{1-\theta} + \delta_y \right)} \right)} \right]. \tag{48}$$

This concludes the proof for the identification of parameters when  $r \neq 1$ .

(ii)  $r = 1$

In this case, Equations (34) and (37) are directly identified as:



$$\omega_x = \beta_x^{-1/\theta} = \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_x \right), \quad (49)$$

$$\omega_y = \gamma_y^{-1/\theta} = \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_y \right). \quad (50)$$

Substituting into Equations (35) and (36) gives:

$$\beta_y \left[ \frac{\rho}{1-\theta} + \delta_x + \omega_x \right] = 0, \quad (51)$$

$$\gamma_x \left[ \frac{\rho}{1-\theta} + \delta_y + \omega_y \right] = 0. \quad (52)$$

Since  $\beta_x > 0$  and  $\gamma_y > 0$ , this implies:

$$\beta_y = \gamma_x = 0. \quad (53)$$

This concludes the proof for the case where  $r = 1$ .

(iii) Continuity

This section examines potential system discontinuities when  $r \rightarrow 1$ . From the solutions in (47) and (48), it follows that:

$$\lim_{r \rightarrow 1} \omega_x = \lim_{r \rightarrow 1} \beta_x^{-1/\theta} = \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_x \right), \quad (54)$$

$$\lim_{r \rightarrow 1} \omega_y = \lim_{r \rightarrow 1} \gamma_y^{-1/\theta} = \frac{1-\theta}{\theta} \left( \frac{\rho}{1-\theta} + \delta_y \right). \quad (55)$$

Substituting these into (38) and (39) yields the indeterminate form:

$$\lim_{r \rightarrow 1} \gamma_x = \lim_{r \rightarrow 1} \beta_y = \frac{0}{0}. \quad (56)$$

To solve this, I apply L'Hôpital's rule. First, express  $\gamma_x$  and  $\beta_y$  as functions of  $\omega_x$  and  $\omega_y$  using the first-order conditions  $\omega_x^{-\theta} = \beta_x$  and  $\omega_y^{-\theta} = \gamma_y$ :

$$\gamma_x = \frac{1}{b_y(1-r)^{1-\theta}\omega_x^\theta} \left[ \frac{\rho}{1-\theta} + \delta_x - \frac{\theta}{1-\theta}\omega_x \right], \quad (57)$$

$$\beta_y = -\frac{1}{b_x(1-r)^\theta\omega_y^\theta} \left[ \frac{\rho}{1-\theta} + \delta_y - \frac{\theta}{1-\theta}\omega_y \right]. \quad (58)$$

Since  $\omega_x$  and  $\omega_y$  depend on  $r$  via their expressions in (47) and (48), the derivatives of the numerators and denominators are, for  $s = x, y$ :

$$\partial_r \left[ \frac{\rho}{1-\theta} + \delta_s - \frac{\theta}{1-\theta} \omega_s \right] = -\frac{\theta}{1-\theta} \partial_r \omega_s(r), \quad (59)$$

and for the denominators:

$$\partial_r [b_y(1-r)^{1-\theta} \omega_x^\theta] = -\omega_x^\theta b_y(1-\theta)(1-r)^{-\theta} + \partial_r \omega_x b_y(1-r)^{1-\theta} \theta \omega_x^{\theta-1}, \quad (60)$$

$$\partial_r [b_x(1-r)^\theta \omega_y^\theta] = -\omega_y^\theta b_x \theta (1-r)^{\theta-1} + \partial_r \omega_y b_x (1-r)^\theta \theta \omega_y^{\theta-1}. \quad (61)$$

Taking the derivatives of (47) and (48) with respect to  $r$  gives:

$$\partial_r \omega_x = \frac{(1-\theta)}{\theta} \frac{-b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_y\right)} \left[ \left(\frac{2\theta-1}{\theta}\right)^2 + \frac{4(1-\theta)}{\theta} \left(1 + \frac{b_x b_y(1-r)}{\left(\frac{\rho}{1-\theta} + \delta_x\right) \left(\frac{\rho}{1-\theta} + \delta_y\right)}\right) \right]^{-1/2}, \quad (62)$$

$$\partial_r \omega_y = \frac{(1-\theta)}{\theta} \frac{-b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_x\right)} \left[ \left(\frac{2\theta-1}{\theta}\right)^2 + \frac{4(1-\theta)}{\theta} \left(1 + \frac{b_x b_y(1-r)}{\left(\frac{\rho}{1-\theta} + \delta_x\right) \left(\frac{\rho}{1-\theta} + \delta_y\right)}\right) \right]^{-1/2}. \quad (63)$$

Evaluated at  $r = 1$ , these expressions become:

$$\left[ \partial_r \omega_x \right]_{r=1} = \frac{-(1-\theta) b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_y\right)}, \quad \left[ \partial_r \omega_y \right]_{r=1} = \frac{-(1-\theta) b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_x\right)}. \quad (64)$$

Thus, the limits are:

$$\lim_{r \rightarrow 1} \gamma_x = - \left[ \frac{\frac{\theta b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_y\right)}}{\left(\frac{1-\theta}{\theta} \left(\frac{\rho}{1-\theta} + \delta_x\right)\right)^\theta b_y (1-\theta) (1-r)^{-\theta}} \right]_{r=1} = - \left[ \frac{\frac{\theta b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_y\right)}}{\infty} \right] = 0^-, \quad (65)$$

$$\lim_{r \rightarrow 1} \beta_y = \left[ \frac{\frac{\theta b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_x\right)}}{\left(\frac{1-\theta}{\theta} \left(\frac{\rho}{1-\theta} + \delta_y\right)\right)^\theta b_x \theta (1-r)^{\theta-1}} \right]_{r=1} = \left[ \frac{\frac{\theta b_x b_y}{\left(\frac{\rho}{1-\theta} + \delta_x\right)}}{\infty} \right] = 0^+. \quad (66)$$

This concludes the proof.

## B Proof of proposition 4

The associated Hamilton-Jacobi-Bellman equation of problem (21) is:

$$\rho \bar{\mathcal{V}}(x, y) = \max_{h_x, h_y \geq 0} \left\{ \left[ \frac{h_x^{1-\theta}}{1-\theta} + \frac{h_y^{1-\theta}}{1-\theta} \right] + \partial_x \bar{\mathcal{V}}(x, y) (A_x x^\theta - \delta_x x - b_x (1-r)^\theta y^{1-\theta} x^\theta - h_x) \right. \\ \left. + \partial_y \bar{\mathcal{V}}(x, y) (A_y y^\theta - \delta_y y + b_y (1-r)^{1-\theta} x^{1-\theta} y^\theta - h_y) \right\}. \quad (67)$$

Maximizing the right hand side gives:

$$\bar{h}_x = \beta^{-1/\theta} x = \bar{\omega}_x x \quad \text{and} \quad \bar{h}_y = \gamma^{-1/\theta} y = \bar{\omega}_y y. \quad (68)$$

Assuming the value function takes the following form:

$$\bar{\mathcal{V}}(x, y) = \alpha + \frac{\beta}{1-\theta} x^{1-\theta} + \frac{\gamma}{1-\theta} y^{1-\theta}. \quad (69)$$

Plugging (69) and (68) into (67) leads to :

$$\rho \left( \alpha + \frac{\beta}{1-\theta} x^{1-\theta} + \frac{\gamma}{1-\theta} y^{1-\theta} \right) = \frac{(\beta^{-1/\theta} x)^{1-\theta}}{1-\theta} + \frac{(\gamma^{-1/\theta} y)^{1-\theta}}{1-\theta} \\ + \beta x^{-\theta} (A_x x^\theta - \delta_x x - b_x (1-r)^\theta y^{1-\theta} x^\theta - \beta^{-1/\theta} x) \\ + \gamma y^{-\theta} (A_y y^\theta - \delta_y y + b_y (1-r)^{1-\theta} x^{1-\theta} y^\theta - \gamma^{-1/\theta} y).$$

Identification of the left and right hand side directly gives :

$$\alpha = \frac{1}{\rho} (A_x \beta + A_y \gamma), \quad (70)$$

where  $\{\beta, \gamma\}$  is the solution to the following system of equations:

$$\Psi_1(\beta, \gamma) = \beta \left[ \frac{\rho}{1-\theta} + \delta_x \right] - \frac{\theta}{1-\theta} \beta^{-\frac{1-\theta}{\theta}} - \gamma b_y (1-r)^{1-\theta} = 0, \quad (71)$$

$$\Psi_2(\beta, \gamma) = \gamma \left[ \frac{\rho}{1-\theta} + \delta_y \right] - \frac{\theta}{1-\theta} \gamma^{-\frac{1-\theta}{\theta}} + \beta b_x (1-r)^\theta = 0. \quad (72)$$

It remain to show that  $\{\beta, \gamma\}$  admits a unique and strictly positive solution. Lets observe that  $\Psi_1(\beta, \gamma) = 0 \iff \gamma = f(\beta)$  and that  $\Psi_2(\beta, \gamma) = 0 \iff \beta = g(\gamma)$  provided

that  $r \neq 1$ . Moreover, it is straightforward to see that  $f'(\beta) > 0$ ,  $f''(\beta) < 0$ ,  $g'(\gamma) < 0$ ,  $g''(\gamma) > 0$  and that  $\lim_{\beta \rightarrow 0^+} \gamma = -\infty$ ,  $\lim_{\gamma \rightarrow 0^+} \beta = +\infty$ . Therefore, since  $f(\beta) = 0 \iff \beta = \frac{1-\theta}{\theta} (\frac{\rho}{1-\theta} + \delta_x)^{-\theta}$  and  $g(\gamma) = 0 \iff \gamma = \frac{1-\theta}{\theta} (\frac{\rho}{1-\theta} + \delta_y)^{-\theta}$ , this ensure the existence of unique positive  $\{\beta, \gamma\} \in [\frac{1-\theta}{\theta} (\frac{\rho}{1-\theta} + \delta_x)^{-\theta}, +\infty] \times [0, \frac{1-\theta}{\theta} (\frac{\rho}{1-\theta} + \delta_y)^{-\theta}]$ .