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A Simple Micro-Model of Market Dynamics. Part I: the "Large Market" Deterministic Limit

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A Simple Micro-Model of Market Dynamics. Part I: the "Large Market" Deterministic Limit

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Abstract

We present a simple agent based model aimed at the qualitative description of trading activity in a "stylized" financial market. A two assets economy is considered, with a bond providing a riskless constant return and a risky stock, paying constant dividends, whose price is fixed via Walrasian auction. The market participants are speculators described as myopic utility maximizers provided with limited forecasting ability. If one varies the parameters describing the market and the agents behavior, the model presents many distinct "phases". In particular, the no-arbitrage "fundamental" price can emerge as a stable fixed point, while for different parameterizations the market shows chaotic dynamics with speculative bubbles and crashes.

1 Introduction

This paper is devoted to the formulation and study of an agent-based model intended to describe the dynamics of a bare-bone financial market. We consider a two assets economy: one riskless bond and one risky equity whose price is determined via Walrasian auction.

The market participants are described as myopic utility maximizers acting as short-horizon speculators. They choose their portfolio composition taking in consideration the forecasted price dynamics. Inside this general framework we consider two stylized classes of trading behaviors: *trend followers* who obtain future prices predictions starting from past price history and *fundamentalist* traders whose forecasts are based on the asset fundamental value. Moreover, agent-specific idyosincratic components are introduced in terms of independent random shocks affecting personal demand for assets.

The present analysis extends previous contributions (Brock and Hommes, 1998, 2001; Hommes, 2001; Gaunersdorfer, 2000) by considering traders who explicitly take in consideration the risk involved in their market positions. One finds that, with this natural extension, the set of possible market dynamics is strongly enriched. The non-arbitrage price is recovered as a fixed point of the dynamics. Its possible instability however can lead to non-trivial (i.e. non constant or explosive) dynamics even in the limit of single representative agent. These findings seem to militate against the presumed necessity of having heterogeneous strategies and agents who switch with high-frequency between them in order to generate non trivial aggregate dynamics. As to this problem, notice that this presumption constitutes a common point that almost always appears in the discussion about the agent-based models in finance.

Particularly so in the models bred from the "complex system" paradigm (Arthur et al., 1997), where not only different trading strategies are competing, but also the number and structure of the competing strategies do change over time. These models display many interesting features, nevertheless their systematic study is made impossible by the enormous number of their degrees of freedoms.

More recently a different approach has emerged, focusing on more "sober" and treatable settings, in some sense reverting to the pioneering investigations dating back to the '70s (see LeBaron (2000) for a review of early contributions). The main novelty in this new stream of literature is constituted by the introduction of some form of "bounded rationality" (or better "inductive rationality" as in Arthur (1994)) that accounts for the agents decisions (see Levy et al. (2000) for a critical review). In these models, however, the heterogeneity in agents behavior and a strong dynamics in agents believes play again prominent roles in shaping the aggregate dynamics. The basic idea is that only an heterogeneous population of traders, characterized by a dynamics switching between different "trading strategies" (or, more generally, "visions of the world") induced by an (apparent) difference in their relative "rewardingness" (Brock and Hommes, 1998; Chiaromonte et al., 1999) or by imitative behavior (Lux, 1995; Kirman and Teyssiere, 2002) can actually lead to interesting market dynamics.

The approach followed in this paper is, in some sense, simpler and can be considered builded upon the recent investigations of the effect of generic "trading rules" on the market aggregate dynamics (Farmer, 1998; Farmer and Joshi , 1999; Farmer and Lo, 1998; Levy et al., 2000). After the introduction of an heterogeneous population of traders we take the "large market" limit, i.e. the limit in which the number of agents is sent to the infinity, to obtain a deterministic dynamical system that completely describes the behavior of the model. The analysis of this simpler framework leads to two interesting conclusions: first, under very natural assumptions, an infinitesimal deviation from perfect rationality can generate market dynamics that finitely and persistently deviate from the rational expectation benchmark, with a consequent enormous reduction in market "efficiency". Second, that the hypothesis of "herding behavior" and "evolving believes" are not always necessary to generate these effects.

The outline of the paper is as follows: in Sec. 2 the model of market participation is introduced and the various assumptions discussed. In Sec. 3 the "large market" limit is considered and the low-dimensional system of equations describing the dynamic of the model is obtained. The analytical and numerical study of this system is performed in Sec. 4 while Sec. 5 contains some final remarks and suggestion on possible future developments.

2 Model structure and trading behaviors

The present model depicts the market dynamics emerging from the interaction of speculative traders described as bounded rational utility maximizing agents. These agents shape their trading activity only on the basis of their possible wealth one step in the future. We consider a simple economy of two assets, a risky stock paying constant[1] dividend D and a riskless bond with return R. We begin the Section with the description of the market participation for a generic "prototypical" agent satisfying the mentioned requirements. The subsequent introduction of specific "trading attitudes" will allows us to use this prototype to implement different classes of traders.

At the beginning of each trading round agents form their personal demand functions deciding the amount of risky asset they want to buy or sell for any possible value of the transaction

[1]The effect of introducing random dividends in the model is briefly discussed in Appendix A

price. The decisions of each agent are based on the estimate of the wealth of his own post-trading portfolio. The procedure is straightforward: suppose that at the end of period t, after his participation to the market, the agent possesses $B(t)$ riskless assets and $A(t)$ risky assets. The agent wealth then reads

$$W(t) = B(t) + A(t)p(t) \tag{1}$$

where $p(t)$ is the stock price fixed by the market at time t.

Let x be the fraction of agent wealth invested in the risky asset. The future wealth of the agent portfolio (i.e. its wealth at the beginning of the next time step) depends on the future return on the stock price $h(t) = p(t+1)/p(t) - 1$ and reads

$$W(t+1; h(t)) = x\, W(t)\, (h(t) - R + D/p(t)) + W(t)\, (1 + R) \tag{2}$$

where the dividends D are paid after the payment of the riskless interest R at the end of time t.

The future value of the portfolio depends on the future price of the stock. Supposing that the agent possesses some forecasting ability concerning the future price return h he would be able to formulate expectations on his own future wealth. The problem of the agent becomes to maximize its utility U consistently with his expectations.

In this framework, a natural idea would be to refer to the "expected utility theory" EUT (see, for instance, Fama and Miller (1972)) and to pretend that the agent behavior is obtained by the maximization of his expected utility with respect to the forecast on the probability distribution of the portfolio future value. In principle different expressions can be devised for the exact form of the agent utility (For a recent discussion and critical review on the various choices found in literature see Levy et al. (2000)). However, a generic choice of the utility function would easily lead to difficult analytical expression and, more important, in its generality this theory requires the agent to forecast a whole probability distribution for future wealth, which is a rather strong requirement compared to the portfolio management techniques today applied by the majority of traders on the real financial markets.

Being aware of these difficulties we prefer to follow a different direction and (analogously to what done in e.g. Brock and Hommes (1998) and Kirman and Teyssiere (2002)), we suppose that the agent utility depends only on his forecast of expected price return and variance. This choice is consistent with the "mean-variance portfolio theory" that can be considered a "standard" procedure to compare different investment possibilities (Elton and Gruber, 1981). In contrast to a more general EUT approach this choice allows to model an agent who takes in consideration a "finite" (possibly small) amount of information in his decision processes and it guarantees the possibility of performing some analytical study of the resulting dynamics. It is interesting, however, to observe that, as some empirical investigations have shown (see Kroll et al. (1984) and Levy and Markowitz (1979)), in real applications the use of a mean-variance approach with respect to a more demanding utility maximization leads to a reduction of efficiency in the portfolio of less than 5%.

We choose as the expression of the agent utility the simplest function of the expected return and variance (Brock and Hommes, 1998; Hommes, 2001; Kirman and Teyssiere, 2002)

$$U(t) = E_{t-1}[W(t+1)] - \frac{\beta}{2} V_{t-1}[W(t+1)] \tag{3}$$

where $E_{t-1}[.]$ and $V_{t-1}[.]$ stand respectively for the expected return and variance computed at the beginning of time t, i.e. with the information available at time $t-1$, and where β is the "risk-aversion" parameter[2].

Using the expression for W in (2) one obtains

$$E_{t-1}[W(t+1)] = x\,W(t)\,(E_{t-1}[h(t)] - R + D/p(t)) + W(t)\,(1+R) \tag{4}$$

and

$$V_{t-1}[W(t+1)] = x^2\,W(t)^2\,V_{t-1}[h(t)] \quad . \tag{5}$$

The portfolio position chosen by the agent is the one that maximizes its utility. Using (4) and (5) and remembering the definition of x from the first order condition one obtains

$$\Delta A(t) = -A(t-1) + \frac{E_{t-1}[h(t)] - R + D/p(t)}{\beta\,V_{t-1}[h(t)]\,p(t)} \tag{6}$$

where the quantity of stock $\Delta A(t)$ the agent is willing to trade (i.e. to buy if it is positive or to sell if it is negative) at time t is related to the stock price $p(t)$.

As said before, the demand curve in (6) represents the model of market participation for a single "prototypical" agent. Keeping constant the form of the utility function, one can describe different types of agents by implementing different methods to obtain the forecasted quantities $E_{t-1}[.]$ and $V_{t-1}[.]$ and by choosing different values for the risk aversion parameter β. For the present discussion we limit our specifications to two distinct classes of agents, intended to represent the most "basic" attitudes observed among real traders.

The first class represents **trend following** agents behaving as "naive econometricians" who obtain forecasted variables using EWMA (exponentially weighted moving averages) predictors. The expressions for their expected returns and variance become:

$$\begin{aligned} E_{t-1}[h(t)] &= (1-\lambda)\textstyle\sum_{\tau=2}\lambda^{\tau}h(t-\tau) \\ V_{t-1}[h(t)] &= (1-\lambda)\textstyle\sum_{\tau=2}\lambda^{\tau}h(t-\tau)^2 - E_{t-1}[h(t)]^2 \end{aligned} \tag{7}$$

where $\lambda \in [0,1]$ is a weighting coefficient setting the "time scale" on which the averaging procedure is performed. Notice that the expression for $V_{t-1}[h(t)]$ is analogous to the one proposed by the RiskMetrics group (see the RiskMetrix Technical Manual), and widely applied by the real operators in their forecasting activity[3].

The second class represents **fundamentalist** traders who, following Hommes (2001), we denote as "Efficient Market Believers" (EMB). We assume that the forecasted price for an EMB trader is "somewhere in between" the present price and a "fundamental" price $\bar{p}$ that they believe to be the correct price of the asset. Under the no arbitrage hypothesis, the future $p(t+1)$ and present $p(t)$ value of the risky asset must satisfy the relation

$$p(t+1) + D = p(t)(1+R) \tag{8}$$

Indeed the left hand side is the value, at time $t+1$, of a portfolio made up of a single asset bought at time t, while the right hand side is the value, always at time $t+1$ of an equivalent (equally valued) portfolio made up of riskless assets. Equation (8) defines the fundamental price[4] $\bar{p} = D/R$. We choose as the EMB recipe for the forecasted future price the simplest

[2] Note that, in this particular case, the same expression can be obtained from EUT with a negative exponential utility function $U(W) = -\exp(-\beta w)$ under the hypothesis of normally distributed returns.

[3] The RiskMetrics group actually proposes an EWMA estimator of the volatility, defined as the second moment of the returns distribution. The expression above represents its natural extension to central moment

[4] Of course (8) posses also a non-stationary divergent solution with an exponentially increasing price which is sometimes referred as "rational expectation bubble". Since, as we will discuss in the following, the dynamics of the present model is bounded, this solution can be safely ignored.

linear combination $p(t) + \theta(\bar{p} - p(t))$ where the parameter $\theta \in (0, 1)$ can be thought as the agent rough estimate of the market "reactivity" in recovering the equilibrium price when moved away from it. The associated expression for the forecasted return is

$$E_{t-1}[h(t)] = \theta(\frac{\bar{p}}{p(t)} - 1) \ . \tag{9}$$

Concerning volatility, we assume that the fundamentalist estimation is equal to the trend follower estimation[5] in (7).

Using the two definition provided above for the forecasted variables one can build to classes of agents. Moreover, one can add heterogeneity at the level of the single agent inside each class. Following Levy et al. (1994, 2000) we assume that the actual demand curve for a given agent i is a noisy perturbation around the demand curve defined in (6), namely

$$\Delta A_i(t) = \Delta A(t) + \epsilon_i(t) \tag{10}$$

where ϵ are independent (across agents population) stochastic terms with mean 0 and where the unperturbed part $\Delta A(t)$ is builded using one of the two "recipes" in (7) or (9).

The model derived from the agents market participation described above present an essential stochastic character which is builded in the noisy terms introduced in (10). The study of dynamical models based on stochastic process is in general difficult. Extensive Monte Carlo numerical investigations are indeed required to identify and understand the effect of the different realization of the noisy terms. The analytical investigation presents in general many difficulties. Particularly so if one introduces budget constraints on agent behavior, for instance requiring that no short positions in stock can be achieved. This kind of non-analytical constraints introduces many sources of non-linearity in the system dependence on parameters and is likely to make direct study of the system impossible.

In the rest of this paper we take a different direction and we restrict the analysis of the model to a particular case, that we call "large market limit". In this limit, the number of traders and of outstanding asset shares are both sent to the infinity. The effect is that the system behavior is now completely characterized by a low-dimensional deterministic system, the so called "deterministic skeleton", and its analysis is consequently greatly simplified.

Due to the abstractness of the model discussed, it's maybe pretentious to justify this limit simply recalling that the number of shares and of active traders are, in any main stock exchange, very large. For the present purpose it is sufficient to say that the study of the "deterministic skeleton" is enough to reveal the features we are interested in (see Introduction) and that these features do not in general disappear with the (re)introduction of the noisy components.

3 The "large market" dynamics

As said before, the present model assumes that the stock price is determined via a Walrasian auction. The individual demand curves $\Delta A_i(p)$ are "aggregated" in a global demand curve and the asset present price $p(t)$ is computed through the market clearing condition $\sum_i \Delta A_i(p(t)) = 0$.

[5]This assumptions is in accordance with the observed behavior of real traders among which the risk estimation based on historical data seems largely adopted even when "fundamentalist" approach is recommended for returns forecasting.

Consider a market composed of N_1 trend followers and N_2 fundamentalists for a total of $N = N_1 + N_2$ agents. Using the previous definitions for the personal demand curve in (6) and for the forecasted variables in (7) and (9) the market clearing condition $\sum_{i=1}^{N_1} \Delta A_i(t) + \sum_{i=N_1+1}^{N} \Delta A_i(t) = 0$ can be written:

$$\beta V_{t-1}[h(t)] \left(\frac{A_{\mathrm{TOT}}}{N} - \frac{\sum_i \epsilon_i(t)}{N} \right) p(t)^2 = D + f_1 \left(E_{t-1}[h(t)] - R \right) p(t) + f_2 \left(\theta \frac{D}{R} - (\theta + R)\, p(t) \right) \tag{11}$$

where A_{TOT} is the total amount of the asset outstanding shares and $f_1 = N_1/N$ and $f_2 = N_2/N$ are respectively the fraction of trend followers and fundamentalists operating on the market.

From (11) one has that when N increases the contribution of the independent random perturbations ϵ_i, as a result of the Law of Large Numbers, is progressively reduced. If one considers the "large market" case and takes the limit $N \to \infty$ and $A_{\mathrm{TOT}} \to \infty$ keeping the average quantity of stock per agent $\bar{A} = A_{\mathrm{TOT}}/N$ finite the random perturbations disappear and the market clearing condition becomes

$$\beta \bar{A} V_{t-1}[h(t)] p(t)^2 = D + f_1 \left(E_{t-1}[h(t)] - R \right) p(t) + f_2 \left(\theta \frac{D}{R} - (\theta + R)\, p(t) \right) . \tag{12}$$

This equation defines the deterministic price behavior obtained by considering the interaction of the two classes of agents defined in the previous Section when the number of agents belonging to each class is assumed extremely large.

Using the positive root of (12) and a recursive expression for the quantities in (7) we can finally write the dynamical equations governing the evolution of the market

$$\begin{array}{lcl} p(t) & = & \left(E_{t-1}[h(t)] - r + \sqrt{(E_{t-1}[h(t)] - r)^2 + 4\gamma V_{t-1}[h(t)] d} \right) / \left(2\gamma V_{t-1}[h(t)] \right) \\ E_t[h(t+1)] & = & \lambda E_{t-1}[h(t)] + (1-\lambda) h(t-1) \\ V_t[h(t+1)] & = & \lambda V_{t-1}[h(t)] + \lambda(1-\lambda)(h(t-1) - E_{t-1}[h(t)])^2 \end{array} \tag{13}$$

where

$$\begin{array}{lcl} \gamma & = & \beta \bar{A} / f_1 \\ r & = & R(1 + f_2 \theta / R)/f_1 \\ d & = & D(1 + f_2 \theta / R)/f_1 \end{array} \tag{14}$$

and where $h(t) = p(t+1)/p(t) - 1$ stands for the realized return at time t.

Notice that the analysis of the system for different mixtures of the two groups of traders, i.e. for different values of f_1 and f_2, can be easily performed since it simply maps in the redefinition of the three parameters in (14).

As expected, the system described in (13) possesses a fixed point in $(\bar{p}, 0, 0)$ where $\bar{p} = d/r = D/R$ corresponds to the no-arbitrage "equilibrium" price, i.e. the present value of the future stream of dividends. Quite interestingly, however, this fixed point is not always stable. In the next Section we will analyze the dependence of the stability of the "no-arbitrage" fixed point and, more in general, of the system trajectories on the values of the various parameters. For now, however, it is useful to mention a couple of qualitative features that generally shape the model behavior.

First of all, notice that the dynamic path of $p(t)$ described by (13) is bounded. Indeed if the forecasted return tends toward a constant value, the variance is progressively reduced and the price increases. This behavior rules out the possibility of an indefinite exponential increase

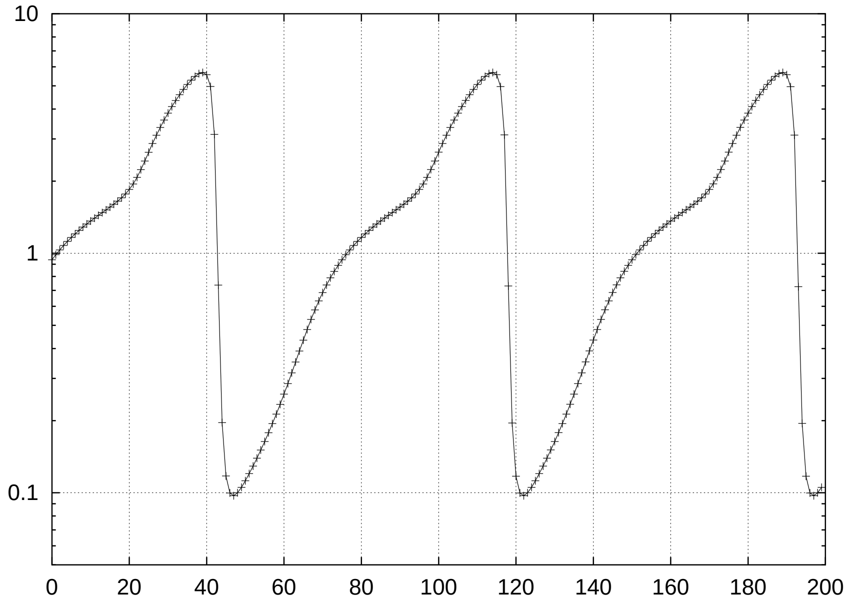

Figure 1: The price history computed with $\beta = 2.5$, $d = .01$, $\lambda = .95$, $r = .01$ after a transient of 1000 time steps. The initial conditions are $p(0) = 100$, $E_t[h(1)] = .01$ and $V_0[h(1)] = .0001$. Prices fluctuate in a bounded region around the fundamental price $\bar{p} = d/r = 1$.

of the price (as in the case of Rational Expectation Bubbles). Second, if the forecasted return $E_{t-1}[h]$, after a period of explosively (i.e. more then exponentially) increasing prices, scales of a factor a, the forecasted variance $V_{t-1}[h]$ scales of a factor a^2 but then the price scales down of a factor $1/a$. This implies that the system cannot display a sustained explosive dynamics.

A typical[6] price history is shown in Fig. 1: with the chosen parameters (see caption) the dynamics is stuck in a periodic cycle. The boundedness of the dynamics manifests itself as a relative slow rise in price followed by a sudden fall, that reminds the "crashes after speculative bubbles" dynamics found in financial markets.

To see how these "crashes" are generated, we can inspect the few steps that precede one of them. Figure 2 reports the price, forecasted return and forecasted variance of the same simulation as in Fig. 1 for the time interval $30 - 40$ that precedes one price crash at around $41 - 42$. As can be seen from the first steps, the constant increase in price comes both from an increase in forecasted returns and a decrease in the forecasted variance. Indeed in the computation of the forecasted variance the high contribution from the last price crash is progressively discounted. Nevertheless, the contribution from the progressively increasing returns keeps $V_t[h(t+1)]$ bounded away from zero so that, at a given point, the progressive decrease in the forecasted variance starts to slow down. This slowing down, in turn, decreases the price growth rate and consequently the value of $E_t[h(t+1)]$, generating a feedback effect on the same forecasted variance and strengthening its slowing trend. After few steps, the reversed trend in returns is so high that the same variance starts to increase. At this point, the price starts to go down. This generates a big jump in the forecasted return value and, consequently, on the forecasted variance and, thanks to the feedback mechanism, a sudden

[6]In the next Section we will see that in fact the model possesses many phases and, depending on the parameters values, displays quite different trajectories. In this respect, here "typical" has to be intended as both "not strange" and "not trivial".

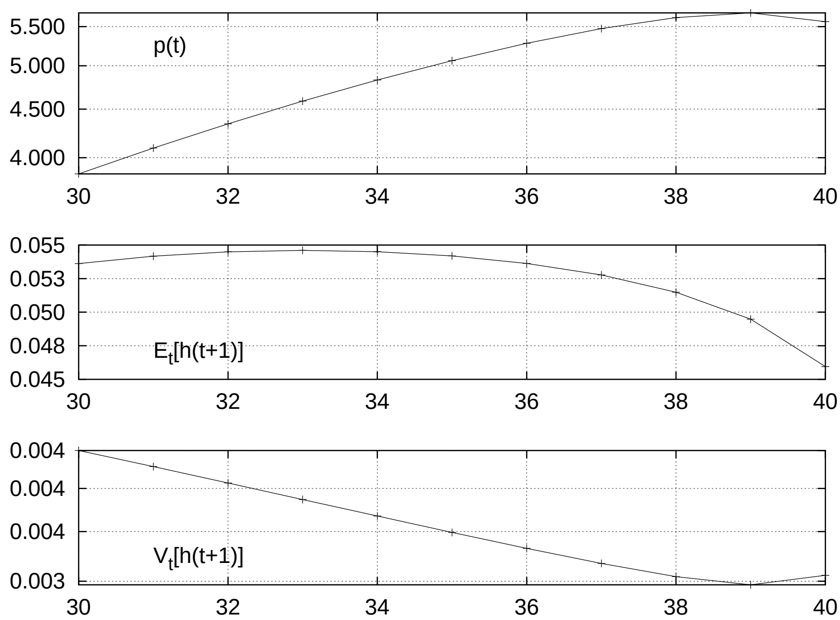

Figure 2: The same simulation as in Fig. 1. Here price (top), forecasted return (middle) and forecasted variance (bottom) are shown for few time steps preceding a sudden crash.

price change is generated in a very short time.

Finally, notice that previous works (Brock and Hommes, 1998; Hommes, 2001) used the same form (3) for the agents utility function but did not introduce an agent forecasting rule for the price variance, simply assuming that all the agents equated this variance to a given constant value. From the discussion above, it is clear that this approximation, apart from being inconsistent with the generated time series, which typically show strong volatility dynamics, does essentially change the nature of the model. For a discussion of this approximation and a comparison with (13) see Appendix B.

4 Studying the deterministic dynamical system

This Section is devoted to the study of the dynamical system defined by (13). The first part consists in the stability analysis of the fixed point associated with the no-arbitrage price of the stock. As we will see, depending on the values of the different parameters, this point can become unstable and, consequently, the model can display highly volatile dynamics. In the second part of the Section a qualitative description of the different "phases" of the model is presented, together with a characterization of the geometrical structure of the system steady states. The global analysis is mainly performed using numerical tools (c.f. Brock and Hommes (1998))[7].

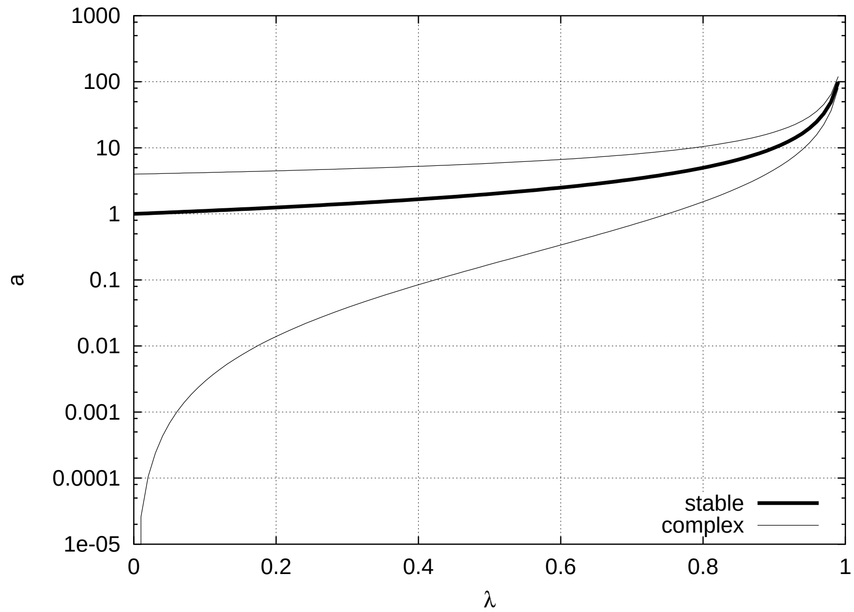

Figure 3: (λ, a) parameter space. The fixed point stable region is the bottom right region delimited by the thickest (denoted with "stable" in the legend) line. The region inside the two other lines (denoted with "complex" in the legend) is where the eigenvalues are complex. Notice that both these regions are unbounded from above.

4.1 Stability analysis of the fixed point

In order to simplify the analysis of the market dynamics it is convenient to rewrite the system in (13) as

$$\begin{aligned} x(t+1) &= f(y(t), z(t)) = \tfrac{y(t)-r+\sqrt{(y(t)-r)^2+4sz(t)}}{2z(t)} \\ y(t+1) &= \lambda y(t) + (1-\lambda)\left(\tfrac{f(y(t),z(t))}{x(t)} - 1\right) \\ z(t+1) &= \lambda z(t) + \lambda(1-\lambda)\left(\tfrac{f(y(t),z(t))}{x(t)} - 1 - y(t)\right)^2 \end{aligned} \quad (15)$$

where $x(t) = \gamma p(t)$, $y(t) = E_t[h(t+1)]$, $z(t) = V_t[h(t+1)]$ and $s = d\gamma$. This system is completely specified by only three positive parameters r, s and λ. Notice that the parameter γ, proportional to the agents risk aversion, has been absorbed in a rescaling of both the prices and the paid dividends.

With these new variables, the "fundamental" price becomes $\bar{x} = s/r$ and $(\bar{x}, 0, 0)$ is a fixed point for the dynamics. Notice that even if (15) is only defined for $z > 0$ and $y \geq 0$ it can be extended continuously to $z = 0$ when $y < r$. It is easy to check that the system does not posses any other fixed point apart from $(\bar{x}, 0, 0)$. But what about the stability of this point? Does there exist a region in the parameters space where the system evolves constantly towards this equilibrium price? This region would characterize a "rational" market constantly pricing the stock with its fundamental value. Since this situation would constitute an optimal outcome from the point of view of market efficiency it's interesting to check to want extent it can be realized by the model under study.

[7]The software used for the simulations is distributed as a part of a package called YAFiMM and can be directly downloaded from `http://www.sssup.it/~bottazzi/`

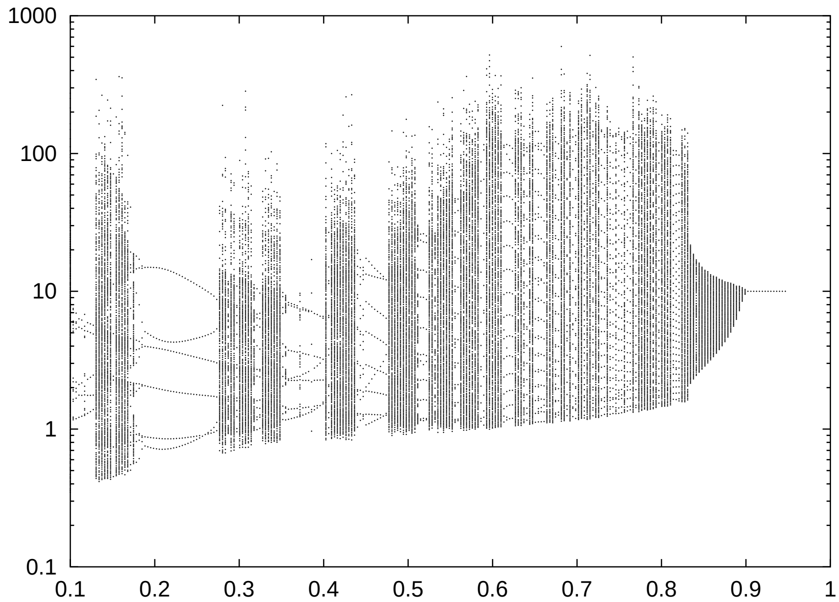

Figure 4: Bifurcation diagram. The x support of a 500 steps orbit (after a 1000 steps transient) is shown for 800 distinct values of λ in $[0,1]$ ($r = .1$ and $s = 1$). The initial condition is $(1, .01, .0001)$.

In order to answer this question, notice that the partial derivatives

$$\begin{aligned} \partial_y f(y,z) = f_y(y,z) &= f(y,z)/\sqrt{(y-r)^2+4sz} \\ \partial_z f(y,z) = f_z(y,z) &= (s/\sqrt{(y-r)^2+4sz} - f(y,z))/z \end{aligned} \tag{16}$$

are continuous for the domain $\mathcal{D} = \{y \geq 0, z > 0\} \bigcup \{y < r, z = 0\}$. In particular $f_y(0,0) = s/r^2$ and $f_z(0,0) = -s^2/r^3$. Since the dynamics described in (15) is bounded in $\{x > 0, z > 0\}$ one can conclude that there is a neighborhood of the fixed point $(\bar{x}, 0, 0)$ such that in its intersection with the largest invariant set of the dynamics, the system is differentiable with continuous derivatives. This is enough to use the following theorem, which can be applied to slightly more general cases than the one at hand.

Theorem Suppose that a system dynamics is described by a set of equations analogous to (15) with a generic function f continuous in $(0,0)$ with continuous first derivatives. Then if $a = \partial_y \ln(f(0,0))$ the point $(f(0,0), 0, 0)$ is stable when

$$a < \frac{1}{1-\lambda} \tag{17}$$

Moreover, the stability of the fixed point is lost by an Hopf bifurcation (Katok and Hasselblatt, 1995) (i.e. by two complex conjugate eigenvalues that cross the unit circle) when $a = 1/(1-\lambda)$.

Proof See Appendix C.

The curve defined in (17) is plotted in Fig. 3 together with the region in which the Jacobian of the system computed in the fixed point possesses two conjugated complex eigenvalues.

The above result suggests some considerations:

- The validity of the theorem for a "generic" function f unties the obtained result from our choice for the utility function in (3). The existence of a stability region for the fixed point, when the agent evaluates future prices starting from forecasted returns and variances, is thus guaranteed whatever expression one chooses for the utility as long as $\partial_y f(0,0) > 0$. This is a rather general assumption in a speculative trading framework. Moreover, it's easy to check that when $f_1 \to 0$ in (11) one has $a \to 0$, since f no longer depend on y, and the fixed point is always stable. We recover the quite natural condition that, when only fundamentalist traders operate on the market, the price steadily converges to the non-arbitrage value whatever its initial value.

- A market can be perfectly stable, with asset priced at its fundamental value, even when only trend follower traders are present (i.e. $f_1 = 1$). This suggests that the general idea of technical trading as a destabilizing force of the market is not always true or , at least, is not enough to generate highly volatile dynamics, when risk evaluation is taken in account.

- The characteristic time of the EWMA procedure defined in (7) is $1/(1-\lambda)$. Then, the expression in (17) tells us that long memory agents, i.e. agents smoothing their forecasts on time scales that are large if compared to a, behave like fundamentalists, even if they base their choices only on the forecasted price movement.

- With the expression of f as in (15) and following (16), one has $a = 1/r$ and (17) becomes $\lambda > 1 - r$. Remembering (14) one can conclude that the market tends toward the equilibrium fixed point when the riskless return and/or the share of fundamentalists in traders population are relatively high and the agents forecasting behavior sufficiently "smooth".

- Quite surprisingly, for the expression of f defined in (15) , the s parameter does not play any role in the stability of the fixed point. This means that the existence of a stable fixed point does not depend either on the dividend d or on the value of the "aggregate" risk aversion γ.

4.2 Bifurcations and the structure of steady states

The stability analysis of the fixed point performed above revealed that in a large region of the parameters space the asymptotic behavior of the model does not converge to the fixed point steady state implied by the non-arbitrage hypothesis. In this respect two questions naturally emerge: when the fixed point $\bar{x}$ is stable, is it a global attractor or, on the contrary, does it possess a bounded basin of attraction? And when the point is no more stable, what is the structure of the model's steady state? We are not able to discuss these points in general terms and, in what follows, we will refer to the expression of f defined in (15).

Let us postpone the discussion of the fixed point attraction domain and proceed with a straightforward inspection of the system behavior when one leaves the stability region in the parameters space. Keeping fixed $r = .1$ and $s = 1$ we plot the support for the x values (after a suitable "transient" period) when the λ parameter is varied, to obtain a bifurcation plot. The result is reported in Fig. 4. As can be seen, for $\lambda > .9$ the system is stationary in the stable fixed point. This is in fact our expectation, following the previous analysis and the chosen value for r.

As the nature of the bifurcation suggests, when the level of λ crosses the .9 boundary, the system moves toward quasi-periodic, multi frequency orbits (this cannot be detected from

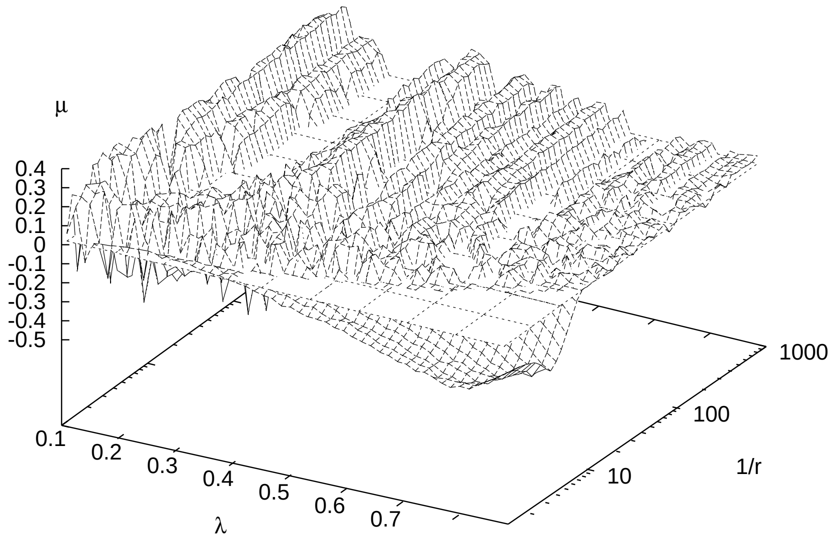

Figure 5: The system largest Lyapunov exponent as a function of λ and $a = 1/r$. The values are obtained with a simulation length of 3000 steps, after discarding the first 1000 as transient ($s = 1$).

Fig. 4 due to its coarse grain, but can be directly checked). Moreover, when λ keeps moving towards lower values, we see the subsequent appearance of regions in which the system shows clear periodic behavior intermixed with other regions where the density of the support suggests the presence of strange attractors (i.e. attractors whose dimension is not integer) and chaos.

This can be confirmed studying the values of the system Lyapunov exponents for different parameterizations. In Fig. 5 the largest Lyapunov exponent is shown as a function of both λ and r. This plot confirms again the presence of "periodic" regions (with below 0 largest exponent) and "chaotic" regions, heavily intermixed. Even if the Jacobian is a smooth function of λ, the Lyapunov exponents show, at least as a first inspection, non-smooth behavior with respect to this parameter (this fact is reported as typical in Eckmann and Ruelle (1985))[8].

Another noticeable fact is that the "mountains landscape" of Fig. 5 seems to show rather stable valleys or hills along the r direction. This would suggest that the central role in the determination of the attractor structure is played by λ much more than by r.

As an extensive numerical investigation shows, this is actually the case. The two parameters mostly shaping the global structure of the system are s and λ. It turns out that even if the parameter s does not play any role in the stability of the fixed point, its role is of major relevance in the characterization of the domain of attraction of the latter. Let us start by investigating this role in the region where the fixed point is stable, i.e. for $\lambda > 1 - r$. In general, if one takes sufficiently small values for s, the stable fixed point is a global attractor. When the parameter s increases, however, a new attractor constituted by a periodic orbit appears and the domain of attraction of the fixed point shrinks rapidly to a small neighborhood of $\bar{x}$. This can be directly checked considering simulations with different initial conditions $(\bar{x}, y(0), z(0))$ and plotting the trajectory average distance from $(\bar{x}, 0, 0)$ after a sufficiently high number of steps. The results of this analysis for $r = .1$ and $\lambda = .91$ have been reported

[8] A typical shape of a strange attractor is shown in Fig. 9.

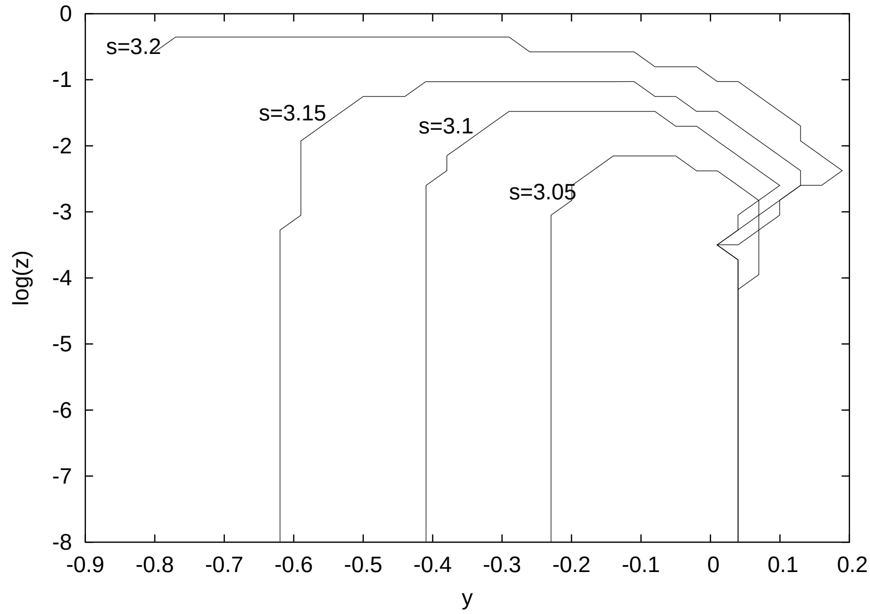

Figure 6: The boundaries of the fixed point domain of attraction in the $y - z$ plane. The domains lays "outside" the lines depicted for the different values of s (a tiny strip of width about .01 around the vertical axis at $y = 0$ has been omitted for clarity). Each point represent an initial condition for the y and z values. The initial condition for x is chosen equal to $\bar{x}$. The system is then iterated for $20,0000$ steps and the initial condition is assumed to belong to the domain of attraction if $|x - \bar{x}| + |y| + |z| < .00001$. The chosen values for the parameters are $r = .1$ and $\lambda = .91$. Notice that the choice of the threshold value and the form of the distance function are asymptotically irrelevant but introduce noticeable effect at finite time lengths. Thus, the lines in the present plot must be read as a qualitative guess of the real boundaries. No attempt has been made to obtain any estimate of the error.

in Fig. 6. The boundaries reported there delimit the fixed point attracting region for different values of s. As can be seen, when s increases above a given threshold, the attraction domain rapidly shrinks. For $s > 3.3$ it becomes a small neighborhood of the fixed point while for $s < 3.03$ the fixed point is a global attractor. This threshold value is an increasing function of λ and diverges for $\lambda \to 1$ (where the system dynamics is definitely frozen). Two attractors coexist also for low values of s, if λ is slightly higher then r. We can conclude that a quite complex picture emerges. However, using extensive numerical simulations it is possible to let a rough qualitative picture emerge.

In what follows will refer generically to "orbits" for the various structures appearing in the analysis since the actual topological nature of these objects, i.e. periodic orbits, quasi periodic orbits or strange sets, depends generally in a non smooth way on the parameters values as suggested by Fig. 5.

The qualitative behavior of the system for $\lambda \approx 1 - r$ is depicted in Fig. 7. For $\lambda > 1 - r$ and moderate values of s, only the global attractor constituted by the stable fixed point exists (region E in the plot). When s is relatively high two attractors coexist: the fixed point and an orbit (region D). The fixed point, in the $x - y$ plane, is external to the orbit (characterized by prices constantly lower than the equilibrium one). When s is low and λ near to the threshold value, a new orbit appears containing the fixed point in its interior, so that along this orbit

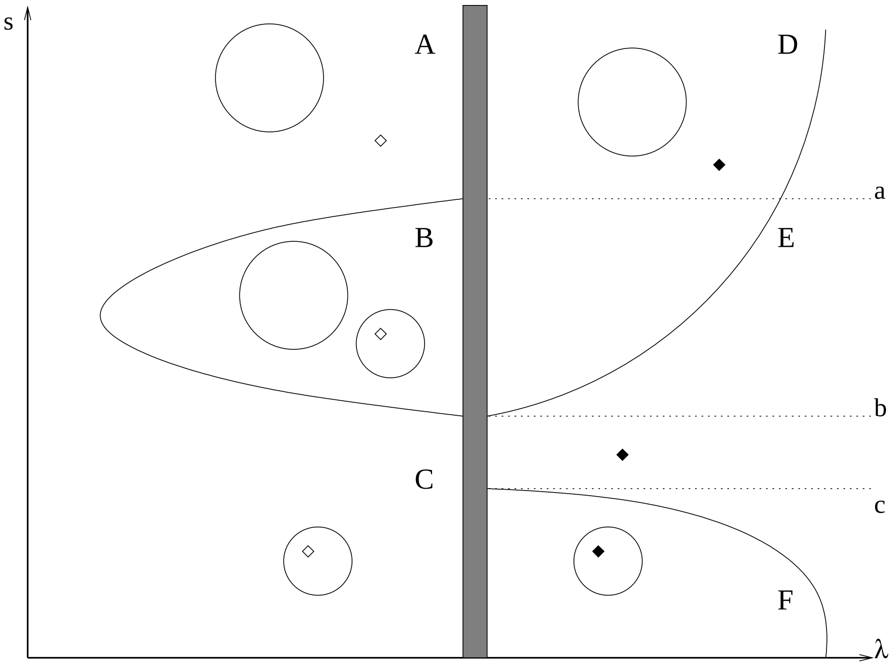

Figure 7: A summary description of the system behavior. See the text for comment. The various regions are indicated with capital letters and separated by continuous lines. The dot represent the fixed point (full dot if stable, white dot if unstable). The location of orbits (circles) with respect to the fixed point are obtained from a projection on the $x-y$ plane. This picture has been obtained with numerical simulations. In particular we have chosen $r = .1$ and we have studied the system near the bifurcation point $\lambda = .9$ along the lines $\lambda = .8991$ and $\lambda = .9001$ (being nearer to the bifurcation generally implies waiting for longer transients). With these choices the values for the boundaries of the various regions are $a = 2.39$, $b = 1.88$ for higher λ, $b = 1.758$ for lower λ and $c = .33$. The region B disappears for $\lambda \sim .89$ while region F does it for $\lambda \sim .905$.

the price oscillates around the equilibrium price (region F). For large enough values of λ this region disappears.

When λ crosses the $1-r$ boundary, the fixed point looses its stability and for large (region A) or small (region C) values of s the orbits keep the same characteristics. Interestingly, for moderate values of s and for λ near the boundary (region B) two stable orbits coexist.

In order to understand the nature of the different phases it is interesting to look at the average price generated by the dynamics. In Fig. 8 we report the average price computed after a suitable transient as a function of λ and s for $r = .1$ and for values of λ respectively above and below the fixed point stability threshold. In both these plots the prices are rescaled by the equilibrium value. In the left plot both regions D and F of Fig. 7 clearly show up and are associated respectively to lower and higher average (rescaled) prices. In the second case, even if the price moves "around" the equilibrium price as mentioned above, its value is on average much higher. Another interesting feature is the appearance, in the right plot, for quite low values of λ and moderate values of s, of a region in which the price dynamics becomes "extreme": the big mountains in the average prices signal the presence of very large cycles in the $x - y$ plane. The typical trajectory is analogous to the one in Fig. 1 but with prices varying over several orders of magnitude.

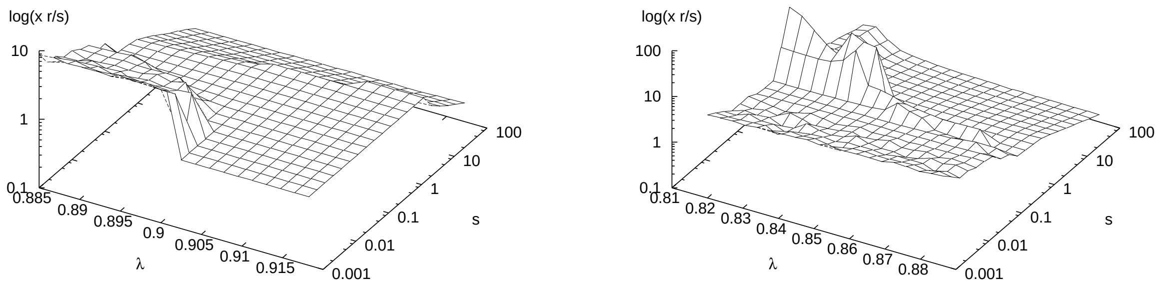

Figure 8: Rescaled average price $x/\bar{x}$ as a function of λ and s with $r = .1$ and initial condition $(\bar{x}, -.1, 0)$. The interior (left) and exterior (right) of the fixed point stability region are shown. Averages are computed for $1,000$ steps after a transient of $50,000$.

For different values of r the boundary of the stability region (the vertical line) moves. As a consequence all the regions in Fig. 7 undertake a shift, but their shapes remain qualitatively unchanged.

Finally let us plot in Fig. 9 a "typical" strange attractor. It is from the C region of Fig. 7 and, if plotted on the $x - y$ planes, the fixed point $(.1, 0, 0)$ clearly appears in its "interior", while the average price is almost 10 times larger.

5 Conclusions and Outlook

The most interesting feature emerging from the foregoing analysis is constituted by the richness of the dynamic scenarios one is able to generate starting from very simple assumptions about the agents behaviors and the structure of the market. Concerning the latter, even if a two assets economy with price fixed via Walrasian auction sounds admittedly rather simplistic, we think that the intrinsic difficulties in the implementation of multi-asset trading protocols and behaviors does not pay back in term of an increased richness in the model emerging features and, more importantly, does not reduce the emergence of market instabilities (some recent investigations seem indeed to strength this general belief, see e.g. Brock and Hommes (2001)).

The apparently "harmless" hypothesis of describing traders as utility-maximizing agents updating their expectations on the past market history leads to huge movements in price and to an high degree of "inefficiency". We can draw two lessons from this discovery:

- first, that the notion of equilibrium expressed by the Efficient Market Hypothesis is in fact extremely weak and can be easily made unstable with very mild assumption about the agents behavior (in some sense, this conclusion is analogous to Akerlof and Yellen (1985) where the more general idea of economic equilibrium is analyzed)
- second, that in order to destroy EMH stability is not necessary to suppose the existence of a complex ecology of strategies together with an high-frequency switching dynamics of agents behaviors.

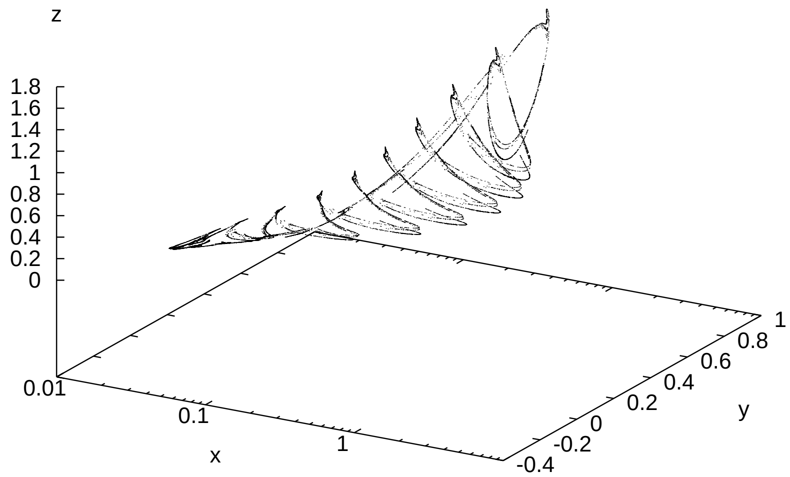

Figure 9: The shape of the strange attractor reconstructed with 20.000 points for $\lambda = .8$,$r = .1$ and $s = .01$. The associated Lyapunov exponents are $2.1e - 02$ $-6.3e - 02$ $-2.8e - 01$.

The present paper represents, by itself, just a first step in the study of the market model presented. An even more interesting part of this kind of studies resides in the analysis of the effect of heterogeneity on the dynamics of the system[9]. Nevertheless the investigation of the "large market" limit is almost mandatory if one wants to disentangle the contributions to the market dynamics that are generated by the assumptions on agents trading strategies and market structure and the ones that are instead introduced by the presence of heterogeneity in the agents population.

6 Acknowledgments

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APPENDIX

A Random dividends

If one wants to repeat the previous analysis in presence of a non constant stream of dividends, things become more complicated. In order to evaluate its portfolio, the agent must posses forecasts not only for the future stock returns, but also for the stock dividend and for the covariance between dividend and returns. To be clearer, consider expression (2) where the dividend D is replaced by a random variable $D(t)$. The expression for the portfolio value

[9]We will try to pursue this analysis in a following paper

expectation and variance reported in (4) and (5) now become respectively

$$E_{t-1}[W(t+1)] = x\,W(t)\,(E_{t-1}[h(t)] - R + E_{t-1}[D(t)]/p(t)) + W(t)\,(1+R) \tag{18}$$

and

$$V_{t-1}[W(t+1)] = x^2\,W(t)^2\,(V_{t-1}[h(t)] + V_{t-1}[D(t)]/p(t)^2 + 2C_{t-1}[h(t),D(t)]/p(t)) \tag{19}$$

where $C_{t-1}[.,.]$ stands for the covariance of the two variables. The individual demand curve then reads

$$\Delta A(t) = -A(t-1) + \frac{E_{t-1}[D(t)] + (E_{t-1}[h(t)] - R)p(t)}{V_{t-1}[D(t)] + 2p(t)C_{t-1}[h(t),D(t)] + p(t)^2 V_{t-1}[h(t)]} \quad . \tag{20}$$

Notice that now the agent demand for stock is bounded even when $p(t) \to 0$. Moreover the demand curve is not necessarily monotonic everywhere inside the agent budget constraints. The aggregate price dynamics obtained from the previous equation is obtained from

$$p(t)^2\,\gamma V_{t-1}[h(t)] + p(t)\;(\gamma C_{t-1}[h(t),D(t)] - E_{t-1}[h(t)] + R) + \gamma V_{t-1}[D(t)] - E_{t-1}[D(t)] = 0 \tag{21}$$

which is written in implicit form since its explicit form depends on the sign of the various coefficients. This expression can be strongly simplified if one assumes that $D(t)$ is a random variable independently extracted from a constant distribution at each time step. In this case $C_{t-1}[h(t),D(t)] = 0$, since $D(t)$ is by definition independent from any previous realization, while $E_{t-1}[D(t)]$ and $V_{t-1}[D(t)]$ are constant values. The agent forecasting is a noisy prediction of these constants but now one is able to obtain an expression similar to (11) where D is replaced by

$$D_1 = \gamma V_{t-1}[D(t)] - E_{t-1}[D(t)] \tag{22}$$

This quantity must be positive in order to ensure the existence of a real price for any value of R and of the forecasted return $E_{t-1}[h(t)]$

As the previous equations show, the inclusion in our model of a dividend dynamics can pose some problems, since now the ability of the market to express a price, and then the existence of a transaction, depends on the agent forecasts. As a first approximation, we can assume that agents are actually able to perfectly forecast dividends, i.e. they have perfect knowledge about dividends distribution so that D_1 becomes a constant parameter in the model. If one want to follow this approach, one can consider the D parameter in Sec. 3 to represent not a constant dividend but an expression as in (22).

B About the constant volatility assumption

The framework described in Sec. 2 shares many aspects with a series of works (see e.g. Brock and Hommes (1998); Gaunersdorfer (2000); Hommes (2001) and the home page of the Center for Nonlinear Dynamics in Economic and Finance, University of Amsterdam, http://www.fee.uva.nl/cendef/) where the market dynamics is generated from the aggregate outcome of a population of heterogeneous agents dynamically changing their trading strategies. In these works, one further approximation is however made with respect to the model described in Section 3: the agents forecasted volatility is assumed constant and homogeneous. In other words, in the cited models the dynamic of volatility is ignored by the agents when they choose their trading behavior. Given the strong similarity between the present model and the ones referred above, it is interesting to check what happen to our model when the same assumption is made.

Let v be the constant and common value of the forecasted stock return variance. This value will replace $V_{t-1}[h(t)]$ in (6) for any agent. One can repeat the same analysis performed in Sec. 3, and will obtain for the aggregate dynamics a two dimensional system

$$\begin{array}{rcl} x(t+1) & = & f(y(t),z(t)) = \frac{y(t)-r+\sqrt{(y(t)-r)^2+4sv}}{2v} \\ y(t+1) & = & \lambda y(t) + (1-\lambda)\left(\frac{f(y(t),z(t))}{x(t)} - 1\right) \end{array} . \tag{23}$$

It is immediate to see that this system possesses a single fixed point

$$\begin{array}{rcl} x^\star & = & \frac{\sqrt{r^2+4sv}-r}{2v} \\ y^\star & = & 0 \end{array} \tag{24}$$

which is notably different from the equilibrium price (even if the latter is recovered in the $v \to 0$ limit). This can be easily understood, since the agents discount the asset price by an amount proportional to their evaluation of risk, which is constant.

The deviation of (24) from the non-arbitrage price seems to suggest that some consistent evaluation of the risk is required in order for a group of agents characterized by a speculative behavior as modeled in Sec. 2 to stabilize the market around the equilibrium price. From a modeling point of view, the assumption of a constant variance forecast generates an "exogenous" differentiation between speculative and "fundamental" behavior, since the two groups evaluation of price, even when the market is stable, fluctuate around two distinct points.

C About the stability of the fixed point

In what follows the proof of the Theorem in Section 4 is outlined. It is a straightforward application of the stability theorem for dynamical system (see e.g. Hirsh and Smale (1974)).

Let us consider the general expression for the Jacobian matrix $J(x, y, z)$ of the dynamical system defined in (15). It reads:

$$\begin{bmatrix} 0 & f_y & f_z \\ -(1-\lambda)f/x^2 & \lambda + (1-\lambda)f_y/x & (1-\lambda)f_z/x \\ -2\lambda(1-\lambda)hf/x^2 & 2\lambda(1-\lambda)h\left(-1+f_y/x\right) & \lambda + 2\lambda(1-\lambda)hf_z/x \end{bmatrix} \tag{25}$$

where $h(x, y, z) = -1 - y + f(y, z)/x$. Computing it in the fixed point $(\bar{x}, 0, 0)$ one obtains

$$\begin{bmatrix} 0 & f_y(0,0) & f_z(0,0) \\ -(1-\lambda)/\bar{x} & \lambda + (1-\lambda)f_y(0,0)/\bar{x} & (1-\lambda)f_z(0,0)/\bar{x} \\ 0 & 0 & \lambda \end{bmatrix} \tag{26}$$

From this expression it's clear that the eigenvalues of $J(\bar{x}, 0, 0)$ do not depend on $f_z(0, 0)$. Setting $a = \partial_y \ln(f(0, 0))$ the three eigenvalues read

$$\begin{array}{rcl} \mu_0 & = & \lambda \\ \mu_+ & = & (\lambda + (1-\lambda)a + \sqrt{(\lambda + (1-\lambda)a)^2 - 4(1-\lambda)a})/2 \\ \mu_- & = & (\lambda + (1-\lambda)a - \sqrt{(\lambda + (1-\lambda)a)^2 - 4(1-\lambda)a})/2 \end{array} \tag{27}$$

The fixed point $(\bar{x}, 0, 0)$ is stable when $||\mu_i|| < 1$ for $i \in \{0, +, -\}$.

After a little algebra[10] one obtains the boundary of the "stable" domain in the parameters space as an explicit equation of the form $a = a(\lambda)$. Its simple expression reads

$$a(\lambda) = \frac{1}{1-\lambda} \tag{28}$$

Moreover, it is immediate to check that the stability of the fixed point is lost when two complex eigenvalues cross the unit circle (see Fig. 3) so that the system displays an Hopf bifurcation (Katok and Hasselblatt, 1995).

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[10] It is easy to show that purely real values for the μ_+, μ_- eigenvalues cannot have modulus equal to one and then it's immediate to obtain the explicit equation for the boundaries equating the modulus of the second (or third) eigenvalue in (27) to 1.

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