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Asset Pricing Model with Heterogeneous Investment Horizons

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Asset Pricing Model with Heterogeneous Investment Horizons*

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Abstract

In this paper we study the dynamics of a simple asset pricing model describing the trading activity of heterogeneous agents in a "stylized" market. The economy in the model contains two assets: a bond with risk-less return and a dividend paying stock. The price of the stock is determined through market clearing condition. Traders are speculators described as expected utility maximizers with heterogeneous beliefs about future stock price and with heterogeneous estimation of risk. In particular, we consider traders who base their investment decision on different time horizons and we analyze the effect of these differences on the price dynamics.

Under suitable parameterization, the stock no-arbitrage "fundamental" price can emerge as a stable fixed point of the model dynamics. For different parameterizations, however, the market shows cyclical or chaotic price dynamics with speculative bubbles and crashes. We find that the sole heterogeneity of agents with respect to their time horizons is not enough to guarantee the instability of the fundamental price and the emergence of non-trivial price dynamics. However, if different groups of agents are characterized by different trading behaviors, the introduction of heterogeneous investment horizons can help to decrease the stability region of the "fundamental" fixed point. The role of time horizons turns out to be different for different trade behaviors and, in general, depends on the whole ecology of agents' beliefs. We demonstrate this effect discussing a case in which the increase of fundamentalists time horizons can lead to cyclical or chaotic price behavior, while the same increase for the chartists helps to stabilize the fundamental price.

JEL codes: C62, D84, G12

Keywords: Asset pricing; Heterogenous beliefs; Investment horizons

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1 Introduction

In this paper we present a simple dynamical model of speculative markets where mean-variance maximizing, boundedly rational agents trade in a Walrasian equilibrium framework. The present work extends the model in Bottazzi (2002) introducing heterogeneity with respect to the time horizons on which agents base their investment decisions and, consequently, their trading activity[1].

It is impossible to explain within the standard paradigm of Efficient Market Hypothesis (EMH) (see Fama (1970) for review) most of the stylized facts characterizing the dynamics of financial markets. This has been recognized by many authors (e.g. Brock (1997)). Indeed, this impossibility has been the driving force behind the emergence, in relatively recent times, of a new strand of models, generally referred to as "agent-based" models, founded on the idea that markets can be described as complex systems of interacting boundedly rational and heterogeneous agents. The analysis of these models has focused on the emerging properties of the generated time series of prices, quantities and returns (see LeBaron (2000) for a review of early investigations).

Inside the large and rapidly growing body of contributions to the "agent-based" literature on financial markets, one can roughly recognize two groups of models. The first group is contributed by models explicitly built to be analytically tractable. These models have the advantage that their dynamics can be studied using the powerful mathematical tools. At the same time, it was shown that quite simple deterministic models with a small degree of heterogeneity can generate time series of price with properties closed to those of real data and also phenomena similar to observed "excess volatility" or "volatility clustering" (see the models in Lux (1995); Brock and Hommes (1998); Gaunersdorfer (2000); Chiarella and He (2001, 2002); Hommes (2002), among others).

The second group contains models essentially built to be simulated on computers. These models in general allow much more flexibility to the researcher in analyzing the role of different assumptions about agents behavior (see extensive review in Levy et al. (2000) and, among others, Levy et al. (1994); Arthur et al. (1997); Lux and Marchesi (1999); Kirman and Teyssiere (2002)). The systematic study of such models is made practically impossible by the enormous number of their degrees of freedom, however the statistical properties of the time series that they generate can be analyzed.

Acknowledging the importance of complementarity between analytical and simulation models, in the first part of this paper we present a rather general agent-based pricing model that is intended as a basic framework inside which many diverse simulation exercises and analysis can be undertaken. In the rest of the paper, however, we restrict the model analysis to a few particular cases that can be studied analytically and whose understanding constitutes, in our opinion, a necessary steps for further, numerical, investigations.

The structure of the model is purposely kept extremely simple. We consider an economy

[1]We use here, equivalently, the terms "investment horizon" and "time horizon" in the sense in which financial institutions do. That is, we refer to the length of time for which an agent would like to keep his investments in order to get the return afterwards. "Investment horizon" with the same meaning was also used in classical academic literature, e.g. in the recent monography Campbell and Viceira (2002). Osler (1995) simply uses "speculators' horizon" for the same concept. On the other hand, some scholars used the term "horizon" with a different meaning. In LeBaron (2001) "time horizon" stands for the number of periods in the *past* which are taken into consideration by technical traders in choosing their investment strategy. To avoid any misunderstanding, we want to stress here that these two concepts of "horizons" are completely different and unrelated. In our paper, the "horizon" extends in the future, while in LeBaron (2001) "horizon" is something referring to the past.

with two assets: one risk-less bond and one risky equity whose price is determined via Walrasian auction. The market participants are described as mean-variance utility maximizers. They choose their portfolio composition at each period based on their forecast of the future price dynamics. Even if the model allows a high level of heterogeneity among traders, we restrict our analysis to the case in which agents are taken as representatives of two distinct classes intended to stylize two basic attitudes toward market participation. The first class, *"fundamentalists"*, represents traders who obtain future return predictions on the base of the fundamental value of the asset. The second class, *"chartists"*, represents traders whose forecasts are based on the past return history.

Even if the model shares the same spirit as some previous contributions (Brock and Hommes, 1998; Hommes, 2001), it differs from them in some important aspects. First, in our model the investors' expectations are built about the return and not about the price. Second, we consider traders who explicitly take into account not only expected return but also the *risk* involved in their market positions. Third, the relative shares of different trading behaviors (chartists and fundamentalists) are, in our model, kept fixed.

Bottazzi (2002) showed that the model with these three features can lead to non trivial aggregate price dynamics with sudden bubbles and crashes. In this paper we introduce a further source of heterogeneity in the model: we consider groups of agents characterized by different investment horizons. To our knowledge, the market interaction of traders who base their investment decision taking into consideration different time horizons has never been analyzed in the agent-based literature[2].

The analysis of the present model reveals two interesting aspects of the question. On the one hand, we find that heterogeneity in time horizons alone is not enough to generate interesting price dynamics. In other terms, if agents are homogeneous with respect to their preferences and the processes they use for expectations formation, the presence of different trading behaviors based on different investment horizons has a minimal impact on the market stability. On the other hand, when agents are characterized by heterogeneous expectations about future market behavior, the heterogeneity in time horizons has a strong effect on the aggregate model behavior. In particular, the price dynamics turns out to be very sensitive to the way in which investors extrapolate the estimation of risk over time. To discuss this effect, we study two reasonable, albeit very simple, specifications for risk extrapolation and compare their respective effect on model dynamics.

One unexpected finding which we got is that for a quite large region in parameters space the increase of the investment horizon of fundamentalists leads to a *decrease* of the stability of the system. We argue in this paper that the ultimate reason of this phenomenon lies in the fact that the instability of the system is related with the relative demand of fundamentalists. If this demand is low (as it happens if they overestimate risk, for instance), then the system becomes unstable.

The rest of the paper is organized as follows. In the next section, as a justification of the present model, we recall the importance of time horizon in investor's financial decision and briefly discuss previous contributions. In Sec. 3 the analytical model of market participation is introduced and various assumptions are discussed. In Sec. 4 we describe the classes of agents, fundamentalists and chartists, who will populate our model. We make some behavioral assumptions about these classes on the basis of empirical evidence. On the same basis we also distinguish between two types of fundamentalists: sophisticated and unsophisticated. In Sec. 5

[2]On the contrary, this issue has received the greatest attention inside the more classical family of models. For a critical review see Campbell and Viceira (2002). Another example is Osler (1995).

we discuss, as a first simple example, the model in the particular case in which there are no chartists on the market. We show that heterogeneity in time horizons itself is not enough to generate non-trivial price dynamics. In Sec. 6 we perform the study of the more general case when both fundamentalists and chartist are on the market. The discussion of implications of our findings with particular emphasis on the role of time horizons is performed in Sec. 7. Sec. 8 contains some final remarks and suggestions on possible future developments.

2 Time Horizons

Any financial advisor nowadays would start the formation of the portfolio for individual client with the question "What is the time horizon of your investment?" The individual time horizon, i.e. the plan for how long to maintain the market position, affects the investor's level of risk and, consequently, his portfolio choice. Moreover, different types of assets are not equally suitable for those who have long-term needs and for those who have short-term objectives.

Investors who have thirty years or more to invest in the market are usually thought of as those who have long time horizon. These investors are typically young professionals or even high school or college students. There is quite strong evidence that financial planners encourage such young investors to have a portfolio with greater risk comparing with other investors, which tends also to produce higher market returns over time. On the contrary, those investors who are of pre-retiree and retiree status usually have short time horizon, less than five years. Investors who have short-term time horizons are generally the least tolerant of investment risk and are more concerned with preserving their existing capital and income. These two classes of investors judge risk very differently, and so it is not surprising that the optimal choice for these two groups can be different.

The necessity to distinguish between different types of traders' strategies is widely acknowledged. For example, pension or investment funds are usually thought as market participants with long investment horizons (at least, some years) who trade on the basis of fundamental information. On the contrary, speculators try to get profit over shorter periods and often use technical rules. For example, Frankel and Froot (1990) found that speculators on the foreign exchange market tend to use extrapolation of past trends to build the forecast on short horizons, while they revert to fundamental information to forecast long-run equilibrium. Thus, there are no doubts that on real markets traders with heterogeneous investment horizons interact. Consequently, it seems natural to investigate the influence of such interaction on the dynamics of market.

Since the classical financial asset pricing models of Markowitz and Sharpe-Linner, known as the Capital Asset Pricing Model (CAPM), did not capture the problem of different time-horizons in investment decisions, the natural question about its generalization arose in the academic community. However, the intertemporal generalizations both of CAPM (LeRoy (1973), Merton (1973)) and of other asset pricing models (Samuelson (1969), Lucas (1978)) had extremely strict assumptions both about the preference structure of agents and the nature of returns dynamics. Indeed, the theoretical results about investment horizons[3] have been obtained entirely within the representative agent tradition.

In this paper we extend the analysis of the time-horizon issue bringing it within the agent-based framework, where the price dynamics is endogenously generated by the interaction of

[3] See, for example, Samuelson (1969) who found the conditions under which long-horizon investors would make the same choice as short-horizon investors, and general discussion of other results in Campbell and Viceira (2002).

heterogenous boundedly rational agents. To our knowledge this issue has never been investigated within this framework. In these models, on the contrary, agents are typically modeled as having "myopic" behavior, i.e. they are assumed to participate to the market in order to maximize the wealth at the next period in time. Essentially, the only source of heterogeneity which is left once this assumption is made is the difference among agents in terms of their forecasting rules.

As already mentioned, however, the investment decision of a trader strongly depends on the time horizon on which his judgment about the profit and risk of a given investment is based. In this respect the following questions naturally arise: do the results of standard agent-based models change if one introduces additional heterogeneity in terms of investment time horizons? And, relatedly, is it enough to introduce heterogeneity at the level of investment time horizons to destroy the predictions of standard representative-agent models? This paper tries to provide an answer to these questions.

3 Model Structure

3.1 General Setup

We consider an asset-pricing model with two assets: a risk-less asset (bond) that gives a constant interest rate $0 < R < 1$ and a risky asset (equity) that pays a dividend[4] D_t at the end of each period t. The bond is assumed to be the numéraire of the economy and its price is fixed to 1. The price of the risky asset p_t is determined each period on the base of its total demand through market clearing condition. The price return $\rho_{t,t+\eta}$ of the risky asset between time t and time $t+\eta$, without taking into account the dividend payments, reads

$$\rho_{t,t+\eta} = \frac{p_{t+\eta} - p_t}{p_t}.$$

To choose their portfolio composition the agents maximize the expected utility. However, different groups of agents take into consideration their forecasted wealth at different times in future. In this way we are going to model the investment horizon heterogeneity.

As we mentioned in the introduction, Bottazzi (2002) generalized the Brock-Hommes approach allowing agents to be heterogeneous in terms of beliefs about the return and volatility of return. In this paper we go further and assume that agents are different also in terms of investment time horizons. Thus, solving the optimization problem to choose their portfolio composition, different groups of agents take in consideration their forecast wealth at different times in future.

It is well known that there are many different approaches to model an intertemporal portfolio optimization problem. One can assume, for instance, that the agent with a time horizon of $\eta > 0$ periods forward does not correct his portfolio between time t, when his decision about portfolio is made, and time $t+\eta$. This agent will participate in the market activity only once each η periods and will choose his portfolio composition maximizing the expected wealth utility at η period in the future. This assumption introduces a quite high level of "irrationality" in the agent behavior, since the agent is supposed to ignore, purposely,

[4] Indeed, in this paper we assume that the dividend is constant to concentrate on the issue of investment horizons in as simple situation as possible. This assumption can be satisfiable if one means a bill by the riskless and a bond by the risky asset. Then the fixed coupon payment plays the role of dividend.

the information revealed by the trading activity that takes place in the η trading sessions occurring between his consecutive market participations.

Another, rather extreme, possibility is to assume that each agent participates in the market activity at each period, continuously correcting his portfolio composition in order to maximize the utility as a function of future wealth, but taking into account the fact that the portfolio will be revised by him each period, on the base of new information. Such approach guarantees a high level of rationality in agents description, and it has been widely used (in continuous-time case) after the seminal paper of Merton (1973). However, the Merton model, in general, is not analytically solvable even in the limits of representative agent paradigm. The introduction of a certain degree of heterogeneity in agents behaviors would lead to even more complex dynamic programming problems that, in our opinion, would require a quite unrealistic degree of sophistication from the part of the agents.

Here we follow a different, in a sense intermediate, strategy. We assume that an agent with time horizon η maximizes at period t his expected wealth at period $t+\eta$ without taking into account the possibility of future portfolio revisions. However, each period the agent revises his portfolio if it is necessary, i.e. if the new market situation or his new expectations lead him to an optimal portfolio composition different from the one chosen in the last period.

This approach can be formalized as follows. Let W_t be the wealth of the agent at time t and let x_t be the share invested in the risky asset. Then the wealth of the agent in period $t+\eta$ as a function of return $\rho_{t,t+\eta}$ reads:

$$W_{t+\eta} = (1-x_t)W_t(1+R)^\eta + x_t W_t \Big(1+\rho_{t,t+\eta}+\frac{\hat{D}_\eta}{p_t}\Big) \ , \tag{1}$$

where $\hat{D}_\eta$ stands for the discounted stream of dividends paid to the agent from t to $t+\eta$. Thus, we assume that the agent fully reinvests the dividends in the risk-less bond. In the case of constant dividend D one has $\hat{D}_\eta = D((1+R)^\eta - 1)/R$.

At each period t the agent with time horizon η chooses the share of wealth to invest in the risky asset x_t in such a way to maximize the expected utility of the future wealth defined in (1), thus in principle avoiding participation in the market until period $t+\eta$ *if the market behavior, in terms of price and returns dynamics, would not change*. However, new information about realized prices may force (and usually forces) agent to change the composition of his portfolio in next period. Thus, on the contrary to the dynamical programming approach, agents in our model do not take into account their future actions. In this sense, agents are boundedly rational.

The procedure goes as follows. Assume that at the beginning of time t the agent i with time horizon η_i has $A_{i,t-1}$ risky and $B_{i,t-1}$ risk-less assets. The agent is assumed to be a price-taker, so for each notional value of price p he makes some assumptions about the distribution of wealth $W_{i,t+\eta_i}$, conditional on: (i) his wealth at the end of current period[5] $W_{i,t}(p) = pA_{i,t} + B_{i,t}$, and on (ii) the available information set $I_t = \{p_{t-1}, p_{t-2}, \ldots\}$. If we denote by $F_i(z)$ the belief of an agent i about the probability distribution of future wealth $W_{i,t+\eta_i}$, the expected utility becomes $E[U_i(W_{i,t+\eta_i})] = \int U_i(z)dF_i(z)$. We denote by $x^*_{i,t}$ the value for which this expected utility reaches the maximum value. This value, generally speaking, depends both on the assumptions on the utility function U_i and on the agent's beliefs about future wealth distribution F_i.

Given that both the utility function and beliefs about wealth distribution are well specified, the agent solves the optimization problem and computes the number of risky assets which he

[5] The budget constraint assures that this wealth is the same as the wealth immediately before the trade, i.e. $W_{i,t}(p) = pA_{i,t} + B_{i,t} = pA_{i,t-1} + D + (1+R)B_{i,t-1}$.

would want to have for a given notional price:

$$\tilde{A}_{i,t}(p) = \frac{x^*_{i,t}(W_{i,t}(p), p)\, W_{i,t}(p)}{p} \tag{2}$$

The demand function of the agent i is then $\Delta A_{i,t}(p) = -A_{i,t-1} + \tilde{A}_{i,t}(p)$. Given that all agents have formed their demand functions, the market clearing condition can be written as $\sum_{i=1}^{N} \Delta A_{i,t}(p) = 0$, where N is the total number of agents on the market.

After the price has been determined, each agent i possesses $A_{i,t}$ shares of risky asset, obtained according to (2), where for p we substitute p_t, and $B_{i,t} = p_t(A_{i,t-1} - A_{i,t}) + D + B_{i,t-1}(1+R)$ of risk-less assets. At the very end of period t the dividend of risky asset is paid: agent i gets $D\, A_{i,t}$ of numéraire.

3.2 Expected Utility and Heterogeneous Beliefs

As mentioned before, to compute the demand function, agent i has to form expectations as to his wealth at time $t+\eta_i$. It follows from (1) that in order to do this, the agent has to make assumptions about the distribution of return $\rho_{t,t+\eta_i}$.

Let us assume that agent i believes that the return $\rho_{t,t+\eta_i}$ is normally distributed with expected value[6] $E^i_t[\rho_{t,t+\eta_i}]$ and variance $V^i_t[\rho_{t,t+\eta_i}]$ and that the dividends are normally distributed as well. Then, from (1), the conditional distribution of future wealth $W_{i,t+\eta_i}$ given today's price p_t and today's wealth $W_{i,t}$ is also normal with the following expected value and variance:

$$\begin{aligned} E^i_t[W_{i,t+\eta_i}] &= W_{i,t}(1+R)^{\eta_i} + x_{i,t}W_{i,t}\Big(E^i_t[\rho_{t,t+\eta_i}] + \Big((1+R)^{\eta_i} - 1\Big)\Big(\tfrac{D}{pR} - 1\Big)\Big) \\ V^i_t[W_{i,t+\eta_i}] &= (x_{i,t}W_{i,t})^2 V^i_t[\rho_{t,t+\eta_i}] \end{aligned} \tag{3}$$

It is easy to show[7] that if the agent with absolute risk aversion coefficient β_i maximizes the expected utility in the case with negative exponential utility function (so $U(W) = -e^{-\beta_i W}$) and future wealth is assumed to be normally distributed, then this maximization problem is equivalent to the maximization of the mean-variance utility $E^i_t[W_{i,t+\eta_i}] - \frac{\beta_i}{2} V^i_t[W_{i,t+\eta_i}]$. Maximization of that utility function (where the moments of wealth are computed according to (3)) with respect to $x_{i,t}$ gives the share of wealth which agent i wants to invest in risky asset. According to (2) the demand for the risky asset of agent i then reads:

$$\tilde{A}_{i,t}(p) = \frac{E^i_t[\rho_{t,t+\eta_i}] + ((1+R)^{\eta_i} - 1)(\frac{\bar{p}}{p} - 1)}{\beta_i p\, V^i_t[\rho_{t,t+\eta_i}]} \tag{4}$$

where $\bar{p} = D/R$ stands for the fundamental price (expected value of discounted stream of future dividends).

Let us now shortly[8] discuss the issue of the choice of utility function. First of all, note that the demand function (4) is the solution of the mean-variance utility maximization. As we mentioned before, this amounts to saying that agents maximize the exponential utility function assuming normal distribution of future wealth. More precisely, it means that the agent i believes that wealth $W_{i,t+\eta_i}$ is distributed normally.

[6]Our notation captures both sources of heterogeneity of agents which we focus on: both difference in time horizons for investment and difference in how beliefs about two first moments of return distribution are formed.

[7]See, for example, Grossman and Stiglitz (1980).

[8]More complete discussion can be found in Levy et al. (2000) and Campbell and Viceira (2002).

One might criticize this assumption arguing that then the same agent must expect that the distribution of the wealth for any future period different from $t+\eta_i$ is *not* normal. One way to avoid this problem would be to assume that tomorrow wealth is log-normally distributed (and then it will be log-normal for any future period) and maximize the power utility function, i.e. $U(W) = (W^\beta - 1)/\beta$, where $1-\beta > 0$ is the relative risk aversion coefficient. It is easy to check that maximization of such expected utility under the assumption of log-normality of wealth distribution also leads to the maximization of the mean-variance utility. Unfortunately, such an approach would be affected by two issues. First, the hypothesis of log-normality of wealth cannot be justified on the base of stylized facts about the distribution of returns: indeed if in (1) the return $\rho_{t,t+\eta}$ is assumed to be log-normal (as suggested by some empirical investigations), wealth can not be log-normally distributed. Second, even if the demand function were in this case analogous to (4), it will depend explicitly on the wealth, leading to an analytically intractable model[9].

In this paper we will not go further into these technical issues. For our purposes it is enough to assume that demand of the risky asset is simply proportional to the excess return and inversely proportional to the risk, as in (4). Anyway, demand can be considered either as the solution of the mean-variance utility maximization, or the result of maximization of the expected exponential utility function when $W_{i,t+\eta_i}$ is normally distributed.

The price p_t is given implicitly as a solution of the equilibrium equation $\sum_{i=1}^{N} \Delta A_{i,t}(p) = 0$. Assuming that the number of assets on the market is constant and equal to A_{TOT}, we rewrite equilibrium equation as follows:

$$p^2 \ \bar{A} - p \ \left\langle \frac{E_t^i[\rho_{t,t+\eta_i}] - \hat{R}_i}{\beta_i V_t^i[\rho_{t,t+\eta_i}]} \right\rangle_i - \left\langle \frac{\bar{p}\hat{R}_i}{\beta_i V_t^i[\rho_{t,t+\eta_i}]} \right\rangle_i = 0. \tag{5}$$

where $\hat{R}_i = (1+R)^{\eta_i} - 1 > 0$, $\bar{A} = A_{TOT}/N$ stands for the number of assets per person on the market and $\langle \cdot \rangle_i$ denotes the average of an argument on population of agents.

This equation is central in our model. In contrast to the approach in Brock and Hommes (1998) we obtain a non-linear price equation due to the fact that we do not assume a null total supply of asset, i.e. we do not put $\bar{A} = 0$. The term $\bar{A}$ is positive and (5) has two real roots, of which only one positive, that we assume as the price p_t. Thus, the assumed utility function generates demand which in equilibrium gives unique and positive price.

Moreover, note that if all agents expect that the price will be constant so that they predict zero mean and zero variance for futures returns, the only solution[10] of (5) is $p = \bar{p}$. This solution corresponds to the fixed point of the dynamics of the model, and, noticeably, does not depend on any agents behavioral parameter[11].

To close the model we only have to specify how beliefs are formed and evolve for each agent. In the next Section we will follow the standard approach of the agent-based literature of financial markets and consider various cases in which only few stylized types of traders participate to the market.

[9] Analytical tractability can be recovered at the expenses of further approximations, c.f. Chiarella and He (2001).

[10] This fact can be easily seen from (5) by taking the limit when both $E_t^i[\rho_{t,t+\eta_i}]$ and $V_t^i[\rho_{t,t+\eta_i}]$ go to 0 for each i.

[11] Actually, it is easy to show that it corresponds to the rational expectation equilibrium of the model.

4 Types of Agents: Chartists and Fundamentalists

Since in this paper we focus on an analytical study of the model, we have to simplify the pricing equation (5). First of all, we assume that $\beta_i = \beta$ for all i. Second, we confine our attention to the case when only two types of agents trade in the market, and we call them, respectively, *chartists* and *fundamentalists*. Both types assume the existence of some underlying stochastic process describing the dynamics of price. Moreover, we distinguish two types of fundamentalists according to the rule which they use to extrapolate the prediction of next-period return variance on longer periods. In this section we specify how agents form their expectation about the mean and variance of future returns starting from their respective "visions of the world".

4.1 Chartists

Chartists assume that the future return can be predicted on the base of past history using some consistent statistical estimator. More precisely, they use the exponentially weighted moving average of return R^{MA}_{t-1} to obtain the forecast for the next return, and the exponentially weighted sample variance of past returns V^{MA}_{t-1} for the prediction of variance. R^{MA}_{t-1} and V^{MA}_{t-1} are defined as follows[12]:

$$\begin{aligned} R^{\text{MA}}_{t-1} &= (1-\lambda)\textstyle\sum_{\tau=2}^{\infty} \lambda^{\tau-2}\rho_{t-\tau} \\ V^{\text{MA}}_{t-1} &= (1-\lambda)\textstyle\sum_{\tau=2}^{\infty} \lambda^{\tau-2}[\rho_{t-\tau} - R^{\text{MA}}_{t-1}]^2 \end{aligned}$$

where $\lambda \in [0,1)$ is the measure of relative importance which agent puts on the past observations: the less λ is, the less the relative weight of far past observations. Notice that these expressions are analogous to the one proposed by the RiskMetrics™ group (see RiskMetrics™ (1996)), and widely applied by real operators in their forecasting activity[13]. In what follows we will use the recursive form of the last two equations:

$$\begin{aligned} R^{\text{MA}}_{t-1} &= \lambda R^{\text{MA}}_{t-2} + (1-\lambda)\rho_{t-2} \\ V^{\text{MA}}_{t-1} &= \lambda V^{\text{MA}}_{t-2} + \lambda(1-\lambda)(\rho_{t-2} - R^{\text{MA}}_{t-2})^2 \end{aligned} \tag{6}$$

Equations (6) describe a one period ahead forecast. To build a long-term forecast, chartists assume that future price dynamics can be described as a (geometric), almost stationary, Brownian motion. Then the forecasts for both expected returns and variances for η periods forward can be obtained by simply multiplying the one-period forecast by a factor η. Thus:

$$E^c_t[\rho_{t,t+\eta}] = \eta E^c_t[\rho_{t,t+1}] = \eta\, R^{\text{MA}}_{t-1} \tag{7}$$

and

$$V^c_t[\rho_{t,t+\eta}] = \eta V^c_t[\rho_{t,t+1}] = \eta\, V^{\text{MA}}_{t-1} \tag{8}$$

4.2 Fundamentalists

Fundamentalists believe that the price is governed by dynamics which constantly revert towards the stock fundamental value. The future price p_{t+1} will be, on average, between the

[12] To avoid the overusing of indexes we denote as $\rho_{t-\tau}$ the return for one period ahead, i.e. $\rho_{t-\tau,t-\tau+1}$.

[13] The RiskMetrics™ group actually proposes an EWMA estimator of the volatility, defined as the second moment of the returns distribution. The expression above represents its natural extension to central moment.

current price and the fundamental price, so that:

$$E_t^f[p_{t+1}] = p_t + \theta(\bar{p} - p_t) \tag{9}$$

where $\theta \in [0, 1]$ is the agent's belief about how reactive the market is in recovering the fundamental price[14]. In the case when $\theta = 0$ equation (9) gives so-called "naïve" expectation $E_t^f[p_{t+1}] = p_t$, while $\theta = 1$ corresponds to the case where fundamentalists believe that the fundamental value will be attained in the next period.

The expression of the multi-steps expected return under the assumption of mean-reverting market dynamics described by (9) reads

$$E_t^f[\rho_{t,t+\eta}] = \Big(1 - (1-\theta)^\eta\Big)\Big(\frac{\bar{p}}{p_t} - 1\Big) \tag{10}$$

This result can be straightforwardly obtained by recursive use of equation (9).

Since the volatility of the asset is determined essentially by the opinion of the market, we assume that fundamentalists build their forecast for the return volatility one period ahead in the same way as chartists do, i.e. $V_t^f[\rho_{t,t+1}] = V_{t-1}^{\mathrm{MA}}$.

At this stage, when we have to specify the fundamentalists' forecast $V_t^f[\rho_{t,t+\eta}]$ for the variance of return after η periods, we will distinguish between two types of fundamentalists. We will refer to one type as *sophisticated* fundamentalists and to the other type as *unsophisticated* fundamentalists, according to the complexity of the analytical tool which they use to compute their forecast.

Unsophisticated Fundamentalists

These agents use relatively simple reasoning, assuming like chartists that the variance of return linearly increases with time and so:

$$V_t^f[\rho_{t,t+\eta}] = \eta V_t^f[\rho_{t,t+1}] = \eta V_{t-1}^{\mathrm{MA}} \tag{11}$$

This assumption about fundamentalists' forecast of variance does, in fact, correspond to behavior commonly found among financial investors who tend to use fundamental evaluation in judging different investment opportunities while preferring an econometric (technical) approach for the evaluation of the implied risk. Moreover, the Brownian scaling of the volatility described in (11) is qualitatively similar to the one actually found for empirical markets (Dacorogna et al., 2001).

Sophisticated Fundamentalists

These fundamentalists use a more sophisticated (and more consistent) way of forecasting the variance. They believe that (9) describes the price dynamics *in each period.* This assumption allows them to model the forecast of time series of future returns as a mean reverting stochastic process. In Appendix A we derive the Fokker-Planck equation that describes the evolution of the long-horizon forecast and we show that, first, the prediction for the long-term return

[14] Notice that equation (9) differs from the prediction rule as we understood it before. It contains the value p_t which does not belong to the information set I_t. Fundamentalists just make a prediction about the *behavior* of prices in the future with respect to the price today, namely they believe that if the asset is underevaluated then the price of it will increase and if it is overevaluated then the price will fall.

satisfies (10), and, second, that the long-term asset volatility forecast can be obtained from the one-period volatility forecast $V_t^f[\rho_{t,t+1}] = V_{t-1}^{\mathrm{MA}}$ according to

$$\tilde{V}_t^f[\rho_{t,t+\eta}] = \frac{1-(1-\theta)^{2\eta}}{\theta(2-\theta)}\, V_{t-1}^{\mathrm{MA}} \tag{12}$$

The crucial difference between (11) and (12) consists in the fact that the former infinitely increases with time horizon η_1 while the latter converges to the asymptotic value $V_{t-1}^{\mathrm{MA}}/(\theta(2-\theta))$. We will see later that this difference leads to crucial change in the dynamics of the model.

With the specifications introduced in this Section, we can completely describe the deterministic model of our artificial market. In the next section we address one of the central questions of this paper: Is the assumption of heterogeneity in time horizons enough to generate dynamics of price different from the convergence to the fundamental value? We will start by considering the situation in which market traders consist only of fundamentalists. In Sec. 6 we continue this analysis and extend it to the situation in which both chartists and fundamentalists participate in the market.

5 Market without Chartists

This Section is devoted to the case when the market is populated only by fundamentalists. We explicitly provide the analysis for the case when all fundamentalists are unsophisticated. One can easily check that our result does not change in the case when fundamentalists are sophisticated.

5.1 Homogeneous Time Horizons

Let us start with the simplest situation and assume that on the market there are only non-sophisticated fundamentalists with the same time horizon η. They use (10) and (11) as forecasting rules. Then the pricing equation (5) becomes:

$$\eta\beta\bar{A}\, p^2\, V_{t-1}^{\mathrm{MA}} + p\Big((1+R)^\eta - (1-\theta)^\eta\Big) - \bar{p}\Big((1+R)^\eta - (1-\theta)^\eta\Big) = 0 \tag{13}$$

The price p_t of the market in period t, i.e. the positive solution of this quadratic equation, reads:

$$p_t = \frac{-r + \sqrt{r^2 + 4d\gamma\, V_{t-1}^{\mathrm{MA}}}}{2\gamma\, V_{t-1}^{\mathrm{MA}}}$$

where

$$\begin{aligned} \gamma &= \eta\beta\bar{A} \\ r &= (1+R)^\eta - (1-\theta)^\eta \;=\; (R+\theta)B_\eta(R,\theta) \\ d &= r\bar{p} \;=\; D(1+\theta/R)B_\eta(R,\theta) \end{aligned}$$

and we have introduced the polynomial $B_\eta(R,\theta) = ((1+R)^\eta - (1-\theta)^\eta)/(R+\theta)$.

From (6) and the last equation, introducing the following notation $x(t) = \gamma p_t$, $y(t) = R_t^{\mathrm{MA}}$, $z(t) = V_t^{\mathrm{MA}}$ and $s = d\gamma$, we get the 3-dimensional system:

$$\begin{cases} x(t+1) & = & f(z(t)) = \frac{-r+\sqrt{r^2+4sz(t)}}{2z(t)} \\ y(t+1) & = & \lambda y(t) + (1-\lambda)\Big(\frac{f(z(t))}{x(t)} - 1\Big) \\ z(t+1) & = & \lambda z(t) + \lambda(1-\lambda)\Big(\frac{f(z(t))}{x(t)} - 1 - y(t)\Big)^2 \end{cases} \tag{14}$$

This system is discussed in the Appendix B. We just notice here that the system has only one fixed point, which we prove to be locally[15] stable for all $\lambda < 1$. Thus, the price converges to the fundamental value independently on the value η. In fact, as can be seen, changes in η lead to changes in the values of parameters s and r (both of them are increasing together with η), but not to modifications of the system itself. The speed of convergence to fixed point depends, however, on η.

5.2 Heterogeneous Time Horizons

Consider now the market composed of N fundamentalists: N_1 of them have the horizon equal to 1, and N_2 have the horizon equal to 2.[16] If we denote the shares of different types of fundamentalists as f_1 and f_2 (so $f_i = N_i/N$ for $i = 1, 2$), the market clearing condition reads:

$$2\beta\bar{A}\,p^2\,V_{t-1}^{\mathrm{MA}} + (p - \bar{p})\Big(2f_1(R+\theta) + f_2(R+\theta)B_2(r+\theta)\Big) = 0$$

There is obvious similarity between (13) and the last equation. Indeed we have the same dynamical system (14) with the following values of parameters:

$$\begin{aligned} \gamma &= 2\beta\bar{A} \\ r &= (R+\theta)(2f_1 + f_2B_2) = (R+\theta)(2 - f_2\theta) \\ d &= r\bar{p} \end{aligned}$$

and the asymptotic behavior of the system remains the same.

5.3 Discussion

As we have shown above, the dynamical system does not change with the introduction of different groups of fundamentalists having different time horizons and the dynamics remains the same independently from the initial conditions. Thus, we can conclude that there is no difference in the dynamics between the case with homogeneous (with respect to time horizon) fundamentalists and the case with heterogeneous ones. In both cases the price converges to the fundamental value and expectation about variance goes to 0.

This result reminds a classical result of the financial literature, even if we obtained it using a completely different approach. Indeed the solution of the dynamic programming model of Merton (1973) in the case of homogeneous expectations and under specific assumption about preferences (power utility function) shows that the optimal choice of the investor, in

[15] The simulations with different parameters and initial values show that the point is, probably, even globally stable.

[16] The analysis can be straight-forwardly generalized to the case with any time horizons.

each period, does not depend on his horizon. This implies that the dynamics of the model remains the same even if agents have different time horizons. We reach the same qualitative conclusions, using a different utility function and in presence of heterogenous agents. The out-of-equilibrium dynamics of our model is, however, quantitatively different and, in general, dependent on the ecology of agents.

6 Complete Model: Fundamentalists vs. Chartists

The qualitative dynamics of the price in the previous model did not depend on the parameters: price always converged to the fundamental value. This result crucially depends on the assumption about how agents form the expectations. In this Section we consider complete model where both fundamentalists and chartists are present. We start with the case when all fundamentalists are nonsophisticated and in the last subsection consider the case with sophisticated fundamentalists. First of all, we consider the simplest benchmark case where all agents have the same (one period) time horizon and later go to the generalization on the case of different time horizons.

6.1 Benchmark: One Period Time Horizon

Consider the market composed of N_1 unsophisticated fundamentalists who make predictions according to (10) and (11), and N_2 chartists who make predictions according to (7) and (8). Denote as f_1 and f_2 the fractions of fundamentalists and chartists, correspondingly, and assume that all investors on the market have the same time horizon equal to 1. The same situation is studied at length in Bottazzi (2002).

Since the rule for the formation of the expected variance is the same for all agents, the market clearing condition reads:

$$\beta \bar{A}\, p^2\, V^{\mathrm{MA}}_{t-1} + p\left(R + f_1\theta - f_2\, R^{\mathrm{MA}}_{t-1}\right) - \bar{p}\left(R + f_1\theta\right) = 0.$$

The positive root of this equation, together with (6) gives the following 3 dimensional system:

$$\left\{\begin{array}{rcl} x(t+1) &=& g(y(t),z(t)) = \left(y(t) - r + \sqrt{(y(t)-r)^2 + 4sz(t)}\right)/2z(t) \\ y(t+1) &=& \lambda y(t) + (1-\lambda)\left(\frac{g(y(t),z(t))}{x(t)} - 1\right) \\ z(t+1) &=& \lambda z(t) + \lambda(1-\lambda)\left(\frac{g(y(t),z(t))}{x(t)} - 1 - y(t)\right)^2 \end{array}\right. \tag{15}$$

where as in (14) we have $x(t) = \gamma p_t$, $y(t) = R^{\mathrm{MA}}_t$, $z(t) = V^{\mathrm{MA}}_t$, $s = d\gamma$, where

$$\begin{array}{rcl} \gamma &=& \beta\bar{A}/f_2 \\ r &=& (R + f_1\theta)/f_2 \\ d &=& r\bar{p} \;=\; (D + f_1\theta\bar{p})/f_2 \end{array} \tag{16}$$

In Appendix C we show that the only fixed point of the system $(\gamma\bar{p}, 0, 0)$ is not always locally stable. The necessary and sufficient condition for the local stability of the system reads:

$$r + \lambda > 1 \tag{17}$$

where r was defined in (16). With decreasing of λ and/or r the system loses the stability when complex eigenvalues cross the unit circle. In other words, the system displays Hopf bifurcation.

6.2 Generalization: Different time horizons

Now we assume that all unsophisticated fundamentalists have the same time horizon η_1 and all chartists have the same time horizon η_2. This modification of setup does not change the system (15) but leads only to changes of the values of the parameters. To see it we again write the market clearing condition:

$$\begin{aligned}\eta_1\eta_2\beta\bar{A}\,p^2\,V_{t-1}^{\mathrm{MA}} \;&+\; p\left(\eta_2 f_1(R+\theta)B_{\eta_2}(R,\theta) - \eta_1\eta_2 f_2\,R_{t-1}^{\mathrm{MA}} + \eta_1 f_2 R B_{\eta_2}(R,0)\right)\\ &-\; \bar{p}\left(\eta_2 f_1(R+\theta)B_{\eta_2}(R,\theta) + \eta_1 f_2 R B_{\eta_2}(R,0)\right) = 0\end{aligned}$$

and one obtains the same system as (15) but with parameters

$$\begin{aligned}\gamma &= \tfrac{\beta\bar{A}}{f_2}\\ r &= \tfrac{1}{\eta_1}\tfrac{f_1}{f_2}(\theta+R)B_{\eta_1}(R,\theta) + \tfrac{1}{\eta_2}RB_{\eta_2}(R,0)\\ d &= r\bar{p}\end{aligned} \tag{18}$$

The system remains the same and still has only one fixed point $(\gamma\bar{p},0,0)$. The condition for the local stability of that point is still given by (17) but the parameter r is now defined in (18).

6.3 Case of sophisticated fundamentalists

Now we can turn to the case when fundamentalists on the market are sophisticated, i.e. when they use predictions for variance provided by expression (12) instead of (11). In both cases considered above the overall effect of this modification will amount to the redefinition of parameters values. It leads to the change in dynamics for given parameters value. Indeed, it is easy to check that with new forecasting rule the parameters of the system (15) become

$$\begin{aligned}\gamma &= \tfrac{\beta\bar{A}}{f_2}\\ r &= \tfrac{\theta(2-\theta)}{1-(1-\theta)^{2\eta_1}}\tfrac{f_1}{f_2}(\theta+R)B_{\eta_1}(R,\theta) + \tfrac{1}{\eta_2}RB_{\eta_2}(R,0)\\ d &= r\bar{p}\end{aligned} \tag{19}$$

7 Discussion

Unsophisticated Fundamentalists

Let us focus, first, on the case when fundamentalists are unsophisticated. The relevant set of parameters in this case given by (18). The analysis of the condition (17) and the dependence of r in (18) with respect to the different parameters shows that the stability region in the parameter space of the fixed point increases together with

- the "smoothness" of the agents' forecasting behavior λ (in other words, with the length of the agents' memory);
- the share of fundamentalists on the market f_1;
- the risk-less return R;

- the perception of fundamentalists about the efficiency of the market θ.

These results are quite intuitive and do not deserve further description. Their emergence can be easily traced back to the assumptions about agents behavior and market structure. Notice also that according to (17), parameter s does not play any role in the stability of the fixed point[17]. Let us now turn to the analysis of the role of different time horizons.

First of all, remember that the introduction of heterogeneity in time horizons did not change the qualitative dynamics of the model when only fundamentalists are present. One can easily see that the situation is the same when only trend-followers are considered. In fact, when $f_1 = 0$, the parameter r in (18) is a strictly increasing function of η_2 and so never less then the value r in (16). Thus, the presence of only technical traders on the market, even if characterized by different time horizons, does not increase the likelihood of observing the emergence of destabilizing dynamics.

Consider now the case of a mixture of fundamentalists and trend followers. Note, first, that r in (18) depends on η_1 in a non-monotonic way. Namely, r as a function of η_1 displays an U-shaped behavior (see Fig. 1) reaching its minimum in the solution of the following equation:

$$(1+R)^{\eta_1}(\eta_1 \ln(1+R)+1) = (1-\theta)^{\eta_1}(\eta_1 \ln(1-\theta)-1)$$

This fact is responsible for the non-monotonic way in which the increase of the value of time horizons of agents influences the dynamics of the system. We plot on the Fig. 2 the price dynamics obtained with a set of parameters (see caption) that in the benchmark case (when all agents have time horizons $\eta_1 = \eta_2 = 1$) generates a convergence toward the fundamental value. The only difference is the increasing of time horizon for fundamentalists, we chose $\eta_1 = 18$. This graph illustrates also one of the typical non-converging pattern of price generated by the system (15) – price follows a periodic behavior: after relatively slow rise in price a sudden fall happens. This behavior reminds the crashes after "speculative bubbles" found in real financial markets.

The Fig. 3 is a bifurcation diagram for the price, when one changes the time horizon of fundamentalists η_1 (all other parameters have the same values as in Fig. 2). As we can see, with the increase of η_1 the system looses the property of convergence to the fundamental price and goes to the region with dynamics similar to Fig. 2. However, with further increasing of η_1 the fundamental value becomes stable again.

To better understand the dependence of the stability of fixed point on the time horizons we plot on the Fig. 4 the graph of the parameter r in (18) as a function of both η_1 and η_2. If now we fix the value of λ as, say, 0.9, the region of stability will be determined by intersection of the surface on the picture with horizontal plane with value 0.1. We project the curve in this intersection and show it on the same graph. The curve with parabolic shape on the Fig. 4 gives the boundary of fixed-point stability region in the space of parameters (η_1, η_2).

Thus, the role played by the time horizons of fundamentalists is bigger, in a certain sense, than the role played by the time horizons of chartists. Even if the increase of both parameters will eventually drive the system in the region of stability, for small enough values the increase of the time horizon of fundamentalists can break the stability of the fixed point while the increase of the time horizon of chartists never do it. To understand the roots of this striking result, let us consider the case when all fundamentalists are sophisticated.

[17]However, this parameter is crucial for the shape of the domain of attraction of the fixed point. Simulations show that if the fixed point is stable, the increase of the value of s leads to a decrease of its domain of attraction.

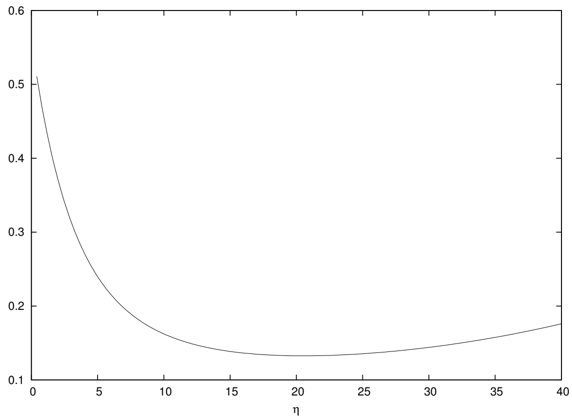

Figure 1: The typical shape of the function $((1+R)^\eta - (1-\theta)^\eta)/\eta$. (Computed with $R = 0.05$ and $\theta = 0.4$.) With η increasing the function increases exponentially to the infinity.

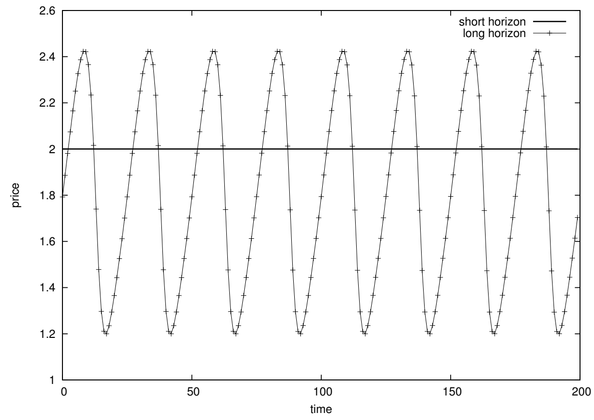

Figure 2: The price dynamics generated by the system (15) after 1000 transition periods. Parameters are $R = 0.05$, $D = 0.1$, $\bar{A} = 2$, $\beta = 1$, $\lambda = 0.9$, $f_1 = 0.2$, $\theta = 0.4$, $\eta_2 = 1$. The initial conditions are $p = 1$, $R^{\mathrm{MA}} = 0.01$, $V^{\mathrm{MA}} = 0.0001$. With $\eta_1 = 1$ the price converges to the fundamental value (solid line), but with $\eta_1 = 18$ the price fluctuates around that value (thin line).

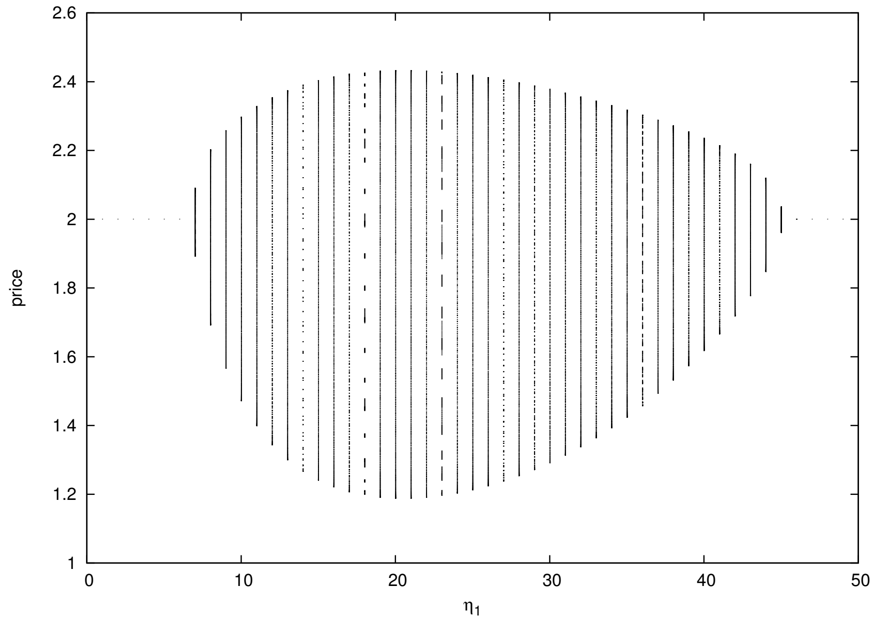

Figure 3: Bifurcation diagram. The price support of a 1000 steps orbit (after a 1000 steps transient) is shown for 50 different values of η_1 from 1 to 50. (The values of parameters for simulation: $R = 0.05$, $D = 0.1$, $\bar{A} = 2$, $\beta = 1$, $\lambda = 0.9$, $f_1 = 0.2$, $\theta = 0.4$, $\eta_2 = 1$). The initial conditions are $p = 1$, $R^{\mathrm{MA}} = 0.01$, $V^{\mathrm{MA}} = 0.0001$X.

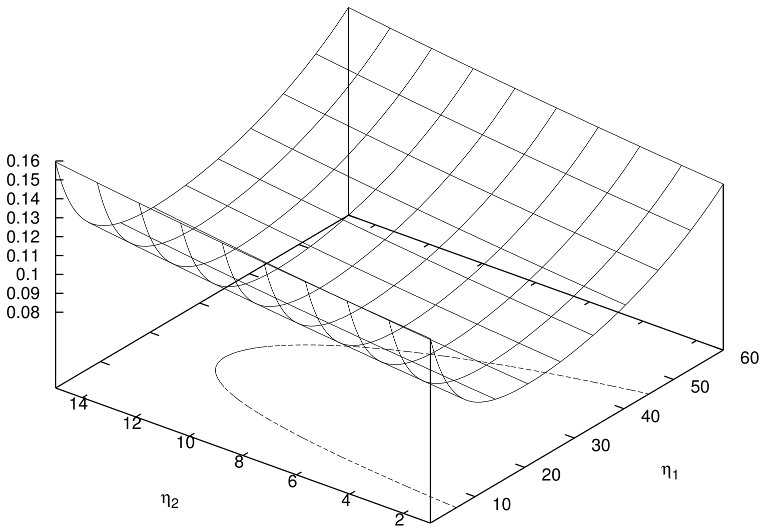

Figure 4: Parameter r as a function of η_1 and η_2. The values of other parameters in (18) are $R = 0.05$, $f_1 = 0.2$, $\theta = 0.3$. The curve on the horizontal space confines the closed region in space (η_1, η_2) where the fundamental price is unstable fixed point for $\lambda = 0.9$.

Sophisticated Fundamentalists

In order to analyze the stability of the system in this case we have to look at the definition of r given in (19). Notice that the first term for r does no longer have the U-shape behavior as a function of η_1. This implies that, on the contrary to the previous situation, the increasing of time horizons of both fundamentalists and chartists lead to more stable dynamics. In other words, the situation shown on the bifurcation diagram in Fig. 3, where the increase of the time horizon η_1 brings the system outside the stable region, cannot be replicated.

Comparing prediction rule for the long-term variance given by (11) with prediction according to (12), we can conclude that the source of unstable effect of fundamentalists' time horizon increasing is the overestimation by unsophisticated fundamentalists of the risk of asset in medium run (near 10 periods). Let us discuss this paradoxical result in further details.

Two effects of change in investment horizon

To find a source of destabilizing nature of the fundamentalists' investment horizons in the case when they overestimate the risk, let us, first, write the whole demand for the risky asset at time t $\tilde{A}_t(p)$ as

$$N_1 \tilde{A}_{f,t}(p) + N_2 \tilde{A}_{c,t}(p) \;=\; s_{f,t} \tilde{A}_t(p) + s_{c,t} \tilde{A}_t(p) \tag{20}$$

where $\tilde{A}_{f,t}$ stands for the demand of fundamentalist, $\tilde{A}_{c,t}$ denotes the demand of chartist, and $s_{f,t}$ and $s_{c,t}$ are relative shares of demand of fundamentalists and chartists, respectively, in total demand. (So, for example, $s_{f,t} = N_1 \tilde{A}_{f,t}(p)/\tilde{A}_t(p)$.) This trivial equality shows that the demand for the risky asset (and so its price) depends both on the change in the demand of two classes of agents (which eventually affect the total demand) and on the change in the relative shares of each class in the total demand.

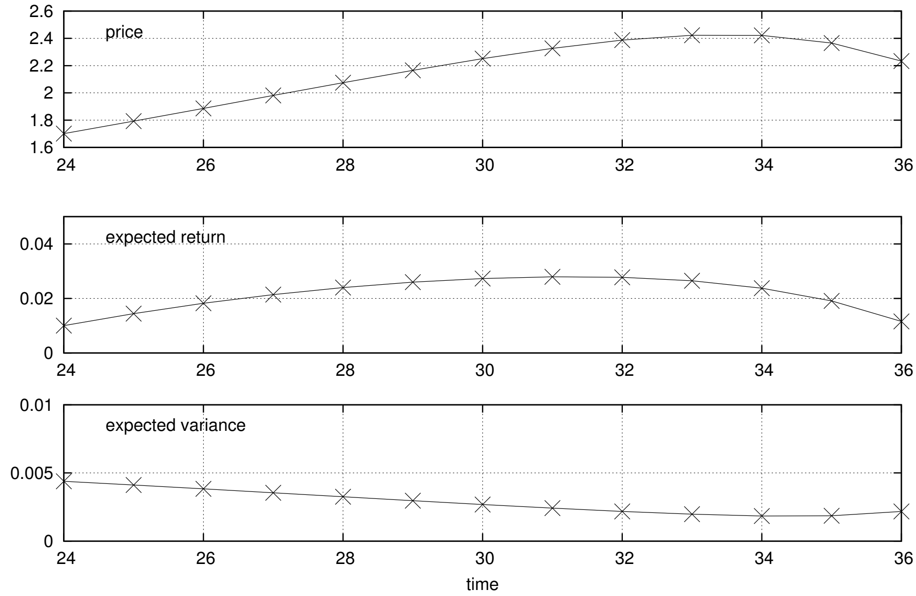

Figure 5: Time series of price (upper panel), expected return (middle panel) and expected variance (lower panel) computed according to (6) during 12 consequent periods. The value of parameters are the same as on the Fig. 2 with time horizon of fundamentalists equal to 18. The labels of time on horizontal panel corresponds to labels on the Fig. 2.

Second, comparing the behavioral rules according to which agents form the expected return, it is clear that independently on time horizons, fundamentalists at each time affect the price in such way that it has to move towards the fundamental value. On the contrary, the influence of chartists on price dynamics is such that it keeps the trend.

Having these two remarks in mind, let us consider the situation when the bubble is created. On the Fig. 5 we reproduce the dynamics of price, expected return and expected variance for the same value of parameters as on the Fig. 2 in the case when the time horizon of fundamentalists is $\eta_1 = 18$ starting from the time moment when price increases. It is easy to see from the demand function (4), that in this situation the rate of price increasing reduces due to the fact that expected variance approaches zero with decreasing rate. Therefore, the expected return decreases as well and so (from (4)) the demand decreases. Eventually, price starts to decrease and bubble explodes.

It is important at that point that since demand functions are different for different classes of agents, the mechanism described above works only for special values of parameters. Thus, to understand how the change in time horizons influence the dynamics, one has to explore how this change influence the demand functions $\tilde{A}_{f,t}$ and $\tilde{A}_{c,t}$. It turns out that both these functions are decreasing with respect to investment horizons. It means that both in the case of increase of time horizon of chartists and in the case of increase of time horizon of fundamentalists (sophisticated and unsophisticated), the corresponding demand decreases. Such decrease of demand means that in situation like described on Fig. 5 the price will go towards fundamental value faster in the situation with greater investment horizon. This effect of increase in the investment horizon on the price can be called "stabilizing".

However, this is not unique effect which changing in time horizons has on the demand structure. From (20) one can see that in the situation when the time horizon of fundamentalists increases the relative share of them in total demand $s_{f,t}$ *decreases*. It creates the opposite (destabilizing) effect due to increase of relative share of chartists' demand in total demand. Such substitution of one class of agents by another class is responsible for the shape of region depicted on Fig. 4.

The size of substitution effect depends on the way how fundamentalists scale the expected variance. For example, if they form the long-term forecast for the variance according to (12) which increases with η_1 slower then in (11), this substitution effect turns out to be not strong enough to destabilize price dynamics.

Summarizing, contrary to the cases when market is populated by the agents from one single group (fundamentalists or chartists), the time horizon of investors does influence the dynamics of price in the situation with mixture of two groups. Moreover, the expression for r in (18) show that the change in fundamentalists' time horizons leads to qualitatively different consequences than the changes in chartists' time horizons. In other words, the effect of changes in time horizons depends on the whole ecology of agents. Finally, comparative analysis of the situations when one of the groups increases their investment horizon shows that there are two effects relevant for the stability. In the case when fundamentalists increase investment horizons, these effects are opposite. In this case if the substitution effect of relative demand is strong enough, nonstable dynamics can be generated.

8 Conclusions

In this paper we analyzed an agent-based model with two sources of heterogeneity. First, agents can possess different horizons for their investments, and, second, they might have different expectations about future returns.

We found that the sole introduction of heterogeneity in time horizons is not enough to generate non-trivial price dynamics. However, when the expectations of agents are heterogeneous, the resulting dynamics can be strongly affected by their time horizons.

As already noted in Bottazzi (2002), the apparently "harmless" hypothesis of describing traders as utility-maximizing agents updating their expectations on the past market history can lead to huge movements in price and to an high degree of "inefficiency". This shows that the notion of equilibrium expressed by the Efficient Market Hypothesis is, in fact, extremely weak and can be made unstable with very mild assumption about the agents behavior.

The main conclusion of the present work is the emergence of non-obvious effects of different time horizons on the market price. First of all, we find that minor changes in the time horizons of some subpopulation of the agents may lead to large qualitative changes in the dynamics of price. Second, we see that whether the dynamics changes and the way how it happens strongly depends on the kind of this subpopulation. Third, the effect of changes in time horizons depends on the whole ecology of agents. And, finally, the dynamics does depend on the way in which the agents estimate risk.

This paper represents only a first step in the study of influence of heterogeneity in time horizons on the price dynamics. One of possible further directions could be the simulation of the model with large number of different classes of agents within the general framework that we presented in the first part of this paper. Also interesting, can be the investigation of the same behavioral models in a different, non Walrasian, market architecture as it has been done, for example, in Bottazzi et al. (2002). The modification of the trading protocol might indeed drastically modify the dynamics of the model.

APPENDIX

A Forecasts of sophisticated fundamentalists

In this section we find the time dynamics of the first two moments of the distribution of return. These moments are used by sophisticated fundamentalists as the forecast for the expected return and variance of the return. In order to do it, we remember that these agents think that the market always tends to correct the current price in the direction to the fundamental value. We can, first, describe such price behavior as mean-reverting stochastic process, then find the first two moments of the price and, finally, go back to the returns.

The following Fokker-Plank equation describes the dynamics of the density of price $f(p, t+\tau)$. Here we denote the time argument as τ to reserve the symbol t for the starting point of the process.

$$\dot{f}(p,t+\tau) = -\frac{\partial}{\partial p}[a_1(p,t+\tau)] + \frac{1}{2}\frac{\partial^2}{\partial p^2}[a_2(p,t+\tau)] \tag{21}$$

where dot denotes the derivative with respect to time, coefficient of diffusion a_2 is assumed to be constant (we denote it as σ^2), and the coefficient of the drift is assumed to be $\tilde{\theta}(\bar{p}-p)$, where $\tilde{\theta}$ is some positive parameter.

Multiplying both parts of the last equality on p and integrating it with respect to the price, one obtain the differential equation for the first moment of price $m_1(t+\tau) = \int pf(p,t+\tau)dp$:

$$\dot{m}_1(t+\tau) = \tilde{\theta}\bar{p} - \tilde{\theta}m_1(\tau)$$

Solving this equation with the initial condition (when $\tau = 0$) $m_1(t) = p_t$, we get the solution

$$m_1(t+\tau) = \bar{p} + (p_t - \bar{p})e^{-\tilde{\theta}\tau}$$

The equation (9) gives the condition for the correct definition of $\tilde{\theta}$, namely we have to impose the restriction $m_1(t+1) = E_t^f[p_{t+1}]$. It reads:

$$\bar{p} + (p_t - \bar{p})e^{-\tilde{\theta}} = p_t + \theta(\bar{p} - p_t)$$

and so

$$\tilde{\theta} = -\ln(1-\theta) \tag{22}$$

If we compute now the first moment m_1 for natural $\tau = \eta$, we get the belief of the agent about the average price. Since $E_t[\rho_{t+\eta}] = E_t[p_{t+\eta}]/p_t - 1$ we can obtain the belief about the average of the return: $E_t[\rho_{t+\eta}] = (\bar{p}/p_t - 1)(1 - e^{-\tilde{\theta}\eta})$. Finally changing $\tilde{\theta}$ according to the equation (22) we get the equation (10).

Multiplying both parts of (21) on p^2 and integrating the result with respect to the price, we get the differential equation for the second moment of price $m_2(t+\tau) = \int p^2 f(p,t+\tau)dp$:

$$\dot{m}_2(t+\tau) = \sigma^2 + 2\tilde{\theta}\bar{p}m_1(t+\tau) - 2\tilde{\theta}m_2(t+\tau)$$

The solution of that equation with the initial value $m_2(t) = p_t^2$ (which corresponds to zero variance for $\tau = 0$) reads:

$$m_2(t+\tau) = \frac{\sigma^2}{2\tilde{\theta}} + \bar{p}^2 + 2\bar{p}(p_t - \bar{p})e^{-\tilde{\theta}\tau} + \Big(p_t^2 - \frac{\sigma^2}{2\tilde{\theta}} - \bar{p}^2 - 2\bar{p}(p_t - \bar{p})\Big)e^{-2\tilde{\theta}\tau}$$

Now using the expression for the first two moments of price distribution, we can compute the belief of the fundamentalists about the variance $V_t^f[\rho_{t,t+\eta}]$ as a function of parameters:

$$V_t^f[\rho_{t,t+\eta}] = \frac{1}{p_t^2}\Big(m_2(t+\eta) - m_1(t+\eta)^2\Big) = \frac{\sigma^2}{2\tilde{\theta}p_t^2}\Big(1 - e^{-2\tilde{\theta}\eta}\Big)$$

To get rid of the parameter σ^2 we impose the condition that making one period forecast fundamentalists behave like chartists, i.e. that for $\eta = 1$ the last expression coincides with the forecast V_{t-1}^{MA}. Then

$$\sigma^2 = \frac{2\tilde{\theta} p_t^2}{1 - e^{-2\tilde{\theta}}} V_{t-1}^{\mathrm{MA}}$$

and, finally,

$$V_t^f[\rho_{t,t+\eta}] = \frac{1 - e^{-2\tilde{\theta}\eta}}{1 - e^{-2\tilde{\theta}}} V_{t-1}^{\mathrm{MA}}$$

Using now the equation (22) to express $\tilde{\theta}$ through θ, we get (12).

Finally notice, that if we start with Fokker-Plank equation (21) where both diffusion and drift coefficients are constant, then using the same technique we get the forecasted rules (7) and (8) for chartists.

B Analysis of the System (14)

Function f in (14) is defined only for positive arguments but can be extended continuously for $z = 0$. In fact,

$$\lim_{z \to 0} f(z) = \frac{s}{r} = \gamma \frac{d}{r} = \gamma \bar{p}$$

Thus, the system (14) becomes to be defined for any y and for $z \geq 0$. With such extension the system has only one fixed point: $(\gamma\bar{p}, 0, 0)$, which corresponds to the fundamental price with zero forecasted variance.

The local stability of the point can be checked computing the Jacobian matrix. First, note that the derivative of function f:

$$f'(z) = \frac{1}{z}\left(\frac{s}{\sqrt{r^2 + 4sz}} - f(z) \right)$$

can be extended to the point $z = 0$ continuously, so that $f'(0) = -s^2/r^3$. Second, we write the Jacobian matrix in $(\gamma\bar{p}, 0, 0)$:

$$\mathbf{J}(x, y, z)\Big|_{(\gamma\bar{p},0,0)} = \begin{bmatrix} 0 & 0 & f'(0) \\ -(1-\lambda)\frac{f(0)}{(\gamma\bar{p})^2} & \lambda & (1-\lambda)\frac{f'(0)}{\gamma\bar{p}} \\ -2\lambda(1-\lambda)\frac{f(0)}{(\gamma\bar{p})^2}\tilde{h} & -2\lambda(1-\lambda)\tilde{h} & \lambda + 2\lambda(1-\lambda)\tilde{h}\frac{f'(0)}{(\gamma\bar{p})} \end{bmatrix}$$

where $\tilde{h}$ now stands for the value of the function $h(x, y, z) = \frac{f(z)}{x} - 1 - y$ in point $(\gamma\bar{p}, 0, 0)$. Since $\tilde{h} = 0$:

$$\mathbf{J}(x, y, z)\Big|_{(\gamma\bar{p},0,0)} = \begin{bmatrix} 0 & 0 & -s^2/r^3 \\ -(1-\lambda)r/s & \lambda & -(1-\lambda)s/r^2 \\ 0 & 0 & 0 \end{bmatrix}$$

It is obvious that eigenvalues of $\mathbf{J}(x, y, z)$ are λ and 0 (with multiplicity 2, since the trace is λ), which implies that the point $(\gamma\bar{p}, 0, 0)$ of the system (14) is locally stable as far as $0 \leq \lambda < 1$.

The simulation shows that the system is also globally stable.

C Analysis of the System (15)

To analyze the stability of the system, we, first, note that function g can be extended continuously to $z = 0$ when $y < r$. The same is true for its derivatives. Namely, since

$$\begin{aligned} g'_y(y,z) &= g(y,z)/\sqrt{(y-r)^2+4sz} \\ g'_z(y,z) &= (s/\sqrt{(y-r)^2+4sz} - g(y,z))/z \end{aligned}$$

we can define

$$\begin{aligned} g(0,0) &= s/r \\ g'_y(0,0) &= s/r^2 \\ g'_z(0,0) &= -s^2/r^3 \end{aligned}$$

Then the Jacobian of the system reads:

$$\mathbf{J}(x,y,z) = \begin{bmatrix} 0 & g'_y & g'_z \\ -(1-\lambda)\frac{g}{x^2} & \lambda + (1-\lambda)\frac{g'_y}{x} & (1-\lambda)\frac{g'_z}{x} \\ -2\lambda(1-\lambda)h\frac{g}{x^2} & 2\lambda(1-\lambda)h\left(\frac{g'_z}{x} - 1\right) & \lambda + 2\lambda(1-\lambda)h\frac{g'_z}{x} \end{bmatrix}$$

where $h(x,y,z) = g(y,z)/x - 1 - y$.

Bottazzi (2002) showed that for generic function g the eigenvalues of this Jacobian (and so of the system (15)) computed in the fixed point depend only on two parameters: λ and $a = \partial_y ln(g(0,0))$. In the case of our function $g(y,z)$, we have $a = 1/r$ and three eigenvalues read:

$$\begin{aligned} \mu_0 &= \lambda \\ \mu_1 &= \tfrac{1}{2}\left(\lambda + (1-\lambda)\tfrac{1}{r} + \sqrt{(\lambda + (1-\lambda)\tfrac{1}{r})^2 - 4(1-\lambda)\tfrac{1}{r}}\right) \\ \mu_2 &= \tfrac{1}{2}\left(\lambda + (1-\lambda)\tfrac{1}{r} - \sqrt{(\lambda + (1-\lambda)\tfrac{1}{r})^2 - 4(1-\lambda)\tfrac{1}{r}}\right) \end{aligned}$$

It is easy to show that μ_1 and μ_2 have modulus less than 1 *iff* $\lambda > 1 - r$, and that these eigenvalues cross the unit circle being complex.

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