Laboratory of Economics and Management
Sant'Anna School of Advanced Studies

Piazza Martiri della Libertà, 33 - 56127 PISA (Italy)
Tel. +39-050-883-343 Fax +39-050-883-344
Email: lem@sssup.it Web Page: http://www.lem.sssup.it/

LEM
Working Paper Series

Price and Wealth Asymptotic Dynamics with CRRA Technical Trading Strategies

Mikhail Anufriev*
Giulio Bottazzi*
Francesca Pancotto*

*Laboratory of Economics and Management,
Sant'Anna School for Advanced Studies

2004/23 February 2005

ISSN (online) 2284-0400

Price and Wealth Asymptotic Dynamics with CRRA Technical Trading Strategies *

Mikhail Anufriev Giulio Bottazzi Francesca Pancotto

S.Anna School for Advanced Studies, Pisa, Italy

9/02/2005

Abstract

We consider a simple pure exchange economy with two assets, one riskless, yielding a constant return on investment, and one risky, paying a stochastic dividend. Trading takes place in discrete time and in each trading period the price of the risky asset is fixed by imposing market clearing condition on the sum of traders individual demand functions. We assume that agents' strategies are consistent with maximizing a CRRA utility, so that the individual demand for the risky asset is expressed as a fraction of the agent's wealth and the evolution of price and wealth distribution is described by means of a dynamical system. We consider agents whose individual demand function is based on future prices forecasts obtained on the basis of past market history and we analyze two cases.

First, we consider a large population of quasi-homogeneous agents, whose individual choices can be described as a random deviation from an underlying, common, behavior. We denote the particular dynamics that emerges as "Large Market Limit" and, with the help of numerical simulations, we provide some hints on the range of its applications. As an example of application of the Large Market Limit we show how the results in Chiarella and He (2001) concerning models with optimizing agents can be replicated in our framework.

Next, we analyze the case of two agents with distinct, but generic, investment strategies. We study the set of equilibria allowed by the two strategies and provide an asymptotic characterization of their relative performances. We find that the market endogenously selects the dominant trader among the participants following a quasi-optimal selection principle. The same principle, however, implies the impossibility of defining a dominance order relation on the space of strategies.

JEL codes: G12, D83.

Keywords: Asset pricing, Price and wealth dynamics, Large market limit, Optimal selection principle.

*We would like to thank Cees Diks, Pietro Dindo, Cars Hommes and the participants of the 10th International Conference on Computing in Economics and Finance for helpful comments on earlier drafts of the paper. Part of the paper was written while M.Anufriev was visiting the Center for Nonlinear Dynamics in Economics and Finance (CeNDEF) at the University of Amsterdam whose kind hospitality and stimulating environment are gratefully acknowledged. The usual disclaimers apply.

Contents

1 Introduction

The standard classical models of financial markets, based on the presumption of the existence of a representative agent with full information and rational expectations are regarded with an increasing skepticism by a growing number of scholars working in the field of finance and economic theory. This dissatisfaction led to the development of a new strand of literature essentially built around quite heterogeneous contributions, developed in the last two decades, which are often joined under the label of "agent-based models". One could roughly divide these contributions in two partially overlapping classes. The first class contains models where results come from a rigorous analytical investigation. Among many examples, let us mention the models discussed in Chiarella (1992),Kirman (1993), Lux (1995, 1997, 1998), Brock and Hommes (1998), Gaunersdorfer (2000), Chiarella et al. (2002) and Chiarella and He (2001, 2002b, 2003), which bear a closer relationship with the present work. The second class consists of models based on the presentation and discussion of extensive computer simulations. Such an approach was used, among many others, in Levy et al. (1994), Arthur et al. (1997), LeBaron et al. (1999), Lux and Marchesi (1999, 2000), Zschischang and Lux (2001), Farmer (2002), Farmer and Joshi (2002) and Bottazzi et al. (2004).

This paper is mainly intended as a contribution to the first class of models. We present a generic agent-based model and investigate its properties, under different specifications, using analytical tools from the theory of dynamical system. At the same time, however, our results shed new light upon various regularities that have been previously, and repeatedly, observed in simulation-based studies.

We consider a simple, pure exchange, two-asset economy. The first asset is a riskless security, yielding a constant return on investment. This security is chosen as the numéraire of the economy. The second asset is a risky equity, paying a stochastic dividend. Trading takes place in discrete time and in each trading period the price of the risky asset is fixed by imposing market clearing condition on the aggregate demand function. The economy is populated by a fixed number of agents acting as speculative traders. Agents participation to the market is described in terms of their individual demand for the risky asset. The individual demand functions are assumed proportional to traders wealth. That is, at each time step, each trader expresses its demand for the asset, relative to a certain notional price, as the share of its present wealth it is willing to invest in that asset. We refer to this particular choice for the description of agents behavior as CRRA (Constant Relative Risk Aversion) framework. Indeed, this behavior is consistent with the maximization of an expected constant *relative* risk aversion (CRRA) utility function. Inside this framework many different agent behavior can be considered and many different investment strategies devised. In the present paper, both for definiteness, and, to a lesser degree, for mathematical tractability, we will focus our analysis on the case in which the agents investment functions depend solely on the past aggregate market performances, that is, so to speak, to "technical trading" behaviors.

Our goal is to study the market equilibria, i.e. the asymptotic properties of the price and the wealth dynamics, when different heterogenous agents populate the economy. In general, when the number of agents become large, this task can become quite difficult. In what follows we will present the analysis of two particular, and complementary, cases: the case with an infinite number of quasi-homogeneous agents and the case with just two agents.

In the first case we discuss a simplifying assumption, called "Large Market Limit", which postulates that the aggregate outcome of stochastic individual choices can be described as a deterministic function of the state of the market. This simple situation can indeed be observed when the investment choices of different agents are assumed to be independent random deviations from a common behavior and the number of agents operating on the market tends to infinity. Under the hypothesis of the "Large Market Limit", the analysis of the system reduces to the analysis of an economy where a single, representative agent, operates. We characterize the market equilibria and derive their stability conditions under generic aggregate demand function.

Next, we describe the asymptotic dynamics of our model when two agents, with distinct but generic trading behavior, operate on the market. We derive the system describing the evolution of the model and study the emerging equilibria and their local stability conditions. We compare the two agents case with the case of homogeneous preferences and expectations and show how a quasi-optimal selection principle endogenously characterizes the market selfselection toward best performing strategies.

In both cases, we show how, even if one leaves the explicit description of the agent investment choices unspecified, the market dynamics can be characterized, at least locally, in terms of few simple parameters. Our results can be used to address some important questions raised in previous analytical and numerical contributions. As an example, we extend to "generic" agents behaviors the comparison performed in Chiarella and He (2001) between "fundamental", "trend-following" and "contrarian" agents. Following the suggestion in Zschischang and Lux (2001) we also investigate the interplay between heterogeneity in traders risk aversion and heterogeneity in the length of the temporal window of past prices they use to obtain prediction about future.

The rest of the paper is organized as follows. In the next Section we present a brief discussion of the motivations behind the choice of a CRRA framework for a financial agent based model, with some reference to previous works. In Section 3 the general description of our simple economy is provided. Under the assumption of CRRA agents behavior, we are able to derive the price and the wealth dynamics as a multi-dimensional dynamical system. In Section 4 we discuss the "Large Market Limit". This limit is applied to the study of models where "technical" agents forecast the future return distribution on the basis of past market history in Section 5. The validity of the "Large Market Limit" is further discussed, with the help of numerical analysis, in Section 6. Section 7 is devoted to the case of two distinct agents operating on the market. In Section 8 we summarize our findings and list the directions for further researches. The proofs of propositions are given in Appendix A. Some of them use classical results from the theory of dynamical systems which we collect in Appendix B.

2 Agent Based Trading Models in the CRRA Framework

The choice of a CRRA framework is somewhat unusual among agent-based analytical models, where the general preference seems to go for models in which the demand functions of agents is independent from the level of their wealth; a choice which is consistent with the maximization of a constant absolute risk aversion (CARA) expected utility. Examples of this approach are the Santa-Fe artificial market model described in Arthur et al. (1997) and LeBaron et al. (1999), and, among the analytical investigations, Brock and Hommes (1998) and its generalizations in Gaunersdorfer (2000), Brock et al. (2004) and Chiarella and He (2002a, 2003). The CARA framework is relatively more simple to handle, exactly because, in this case, the price dynamics is independent from the wealth distribution. This implies, as a direct economic consequence, that in these models all agents have the same impact on price formation, irrespectively of their wealth. This relatively awkward property hints to the possibility that these models miss some important features of the real markets. Moreover, there are empirical and experimental evidence (see e.g. Levy et al. (2000)) suggesting that the behavior of traders and investors is consistent with a decreasing (with wealth) absolute risk aversion and/or with a constant relative risk aversion.

For this reason, the simulation-based models, which are not constrained by the need of mathematical simplicity, often assume CRRA utility-maximization, as in Levy et al. (1994, 1995) and Levy and Levy (1996). There has been,however, some controversy about the robustness and the relevance of the CRRA results (see e.g. Zschischang and Lux (2001)), so that an analytical investigation of such a framework seems necessary. Few analytical attempts in this direction have been made. To our knowledge, they have been so far limited to the models presented in Chiarella and He (2001, 2002b). Moreover, the impossibility of deriving a closed analytical expression for the solution of the CRRA expected utility maximization problem forced Chiarella and He to rely upon approximated expressions for the agent demand function and, more hazardously, to introduce an ad-hoc assumption about the normality of the underlying returns distribution which is, in general, not consistent with the actual dynamics generated by their models.

Our approach is different. Instead of deriving the agent's individual demand from an approximated utility maximization principle, we model, in total generality, the investment choice of the agent as some smooth function of his expectations. This *investment function* can be agent-specific, partially due to the fact that its shape should somehow depend on the agent's attitude toward risk, and partially due to the different possible ways in which an agent can transform an available information set (public or private) into predictions about the future. Since we only require the proportionality of agents investment allocation to agents current wealth, without further assumptions about the way in which these allocations are determined, our model substantially extends previous analytical agent-based contributions with respect to the actual degree of heterogeneity allowed in the description of agents behavior.

3 Model Structure

We consider a simple pure exchange economy where trading activities are supposed to take place in discrete time. The economy is composed by a risk-less asset (bond) giving a constant interest rate $r_f > 0$ and a risky asset (equity) paying a random dividend D_t at the end of period t. The risk-less asset is assumed the numéraire of the economy and its price is fixed to 1. The price P_t of the risky asset is determined at each period on the base of its aggregate demand imposing a market-clearing condition.

Suppose that this simple economy is populated by a fixed number of N traders. The dynamics through each period proceeds as follows. At time t, before the trade starts, agent i ($i = 1 \dots N$) possesses $A_{t,i}$ shares of risky asset and $B_{t,i}$ shares of risk-less asset. Thus, the wealth of the agent can be computed for any notional price P as $W_{t,i}(P) = A_{t,i}\, P + B_{t,i}$.

Suppose that the agent i decides to invest a fraction $\pi_{t,i}$ of his wealth into the risky asset and a fraction $1 - \pi_{t,i}$ into the riskless asset. Then his individual demand for the risky asset becomes $\pi_{t,i}\, W_{t,i}(P)/P$. The aggregate demand is the sum of all individual demands, and the price P_t is defined as one where aggregate demand is equal to aggregate supply. Assuming a constant supply of risky asset, whose quantity can then be normalized to 1, the price P_t is determined as the solution of the equation

$$\sum_{i=1}^{N} \pi_{t,i}\, W_{t,i}(P) = P \quad . \tag{3.1}$$

At this moment also the new portfolio of any agent is determined. After the trade, at the end of period t, the random dividend D_t per risky asset and constant interest rate r_f per riskless asset are paid (in terms of numéraire). The economy is ready for the next round. A diagram describing the different steps composing a trading session is presented in Fig. 1.

To derive the dynamics of the wealth, notice that before the next trading session the agent i possesses $A_{t+1,i} = \pi_{t,i}\, W_{t,i}(P_t)/P_t$ shares of risky asset and also $B_{t+1,i} = (1 - \pi_{t,i})\, W_{t,i}\,(1 + r_f) + \pi_{t,i}\, W_{t,i}\, D_t/P_t$ of risk-less security. The first term in the latter expression reflects the

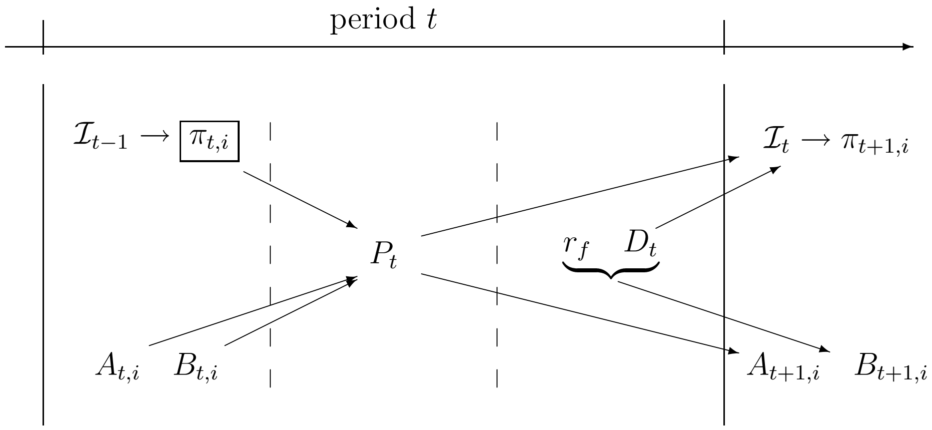


Figure 1: A diagram of the trading round at period t. The participation of the agent in the trade is limited to the choice of the investment share $\pi_{t,i}$ (shown inside the box). This share depends, in general, on the information set $\mathcal{I}_{t-1}$ available to the agent and, together with the current portfolio composition, determines the individual demand. From these individual demands, the clearing price P_t can be fixed, and, at the same time, the new agents' positions in the risky asset are determined. At the end of period t, the dividend and interest rate are paid. Their realized values, together with the fixed price, both enhance the information set and also define the new portfolio composition.

contribution of the risk-free interest payment and the second term is the dividend payments. The agent's wealth at time $t+1$ is $W_{t+1,i} = A_{t+1,i}\, P_{t+1} + B_{t+1,i}$, so that:

$$W_{t+1,i} = W_{t,i}\,(1 - \pi_{t,i})\,(1 + r_f) + \frac{W_{t,i}\,\pi_{t,i}}{P_t}\,(P_{t+1} + D_t) \quad . \tag{3.2}$$

Provided that the share $\pi_{t,i}$ does not depend on current wealth explicitly, the following system determines the value of $W_{t,i}$ and P_t

$$\begin{cases} P_t = \displaystyle\sum_{i=1}^{N} \pi_{t,i}\, W_{t,i} \\ W_{t,i} = W_{t-1,i}\,(1 - \pi_{t-1,i})\,(1 + r_f) + \dfrac{W_{t-1,i}\,\pi_{t-1,i}}{P_{t-1}}\,(P_t + D_{t-1}) \quad \forall i \end{cases} \quad , \tag{3.3}$$

where we rewrote (3.2) at time t. In the present framework, as it also happens on real markets, the wealth of each agent and the prevailing asset price are determined simultaneously. The investment fraction $\pi_{t,i}$ can be considered as a complete description of the agent's behavior.

3.1 Rescaling the Economy

The dynamics described by (3.3) represent an economy which is intrinsically growing. In order to see that let us sum the second equation in (3.3) over all the agents to obtain the dynamics of the total wealth

$$W_t = W_{t-1}\,(1 + r_f) + [P_t + D_{t-1} - P_{t-1}\,(1 + r_f)] \quad . \tag{3.4}$$

From (3.4) it is immediate to see that the presence of a constant positive riskless return r_f introduces an "exogenous" expansion of the economy, due to the continuous injections of new bonds, whose price remains, under the assumption of totally elastic supply, unchanged. This effect is obvious if one assumes that the market is perfectly efficient and no arbitrage is possible. Under the no-arbitrage hypothesis, indeed, the expected value at time $t-1$ of the second term in the right-hand side of (3.4) has to be equal to zero, so that $E_{t-1}[W_t] = W_{t-1}\,(1+r_f)$. Consequently, the total wealth is characterized by an unbounded steady increase.

Both for theoretical (i.e. analytical investigations) and for practical (i.e. computer simulations) reasons it is convenient to remove this exogenous economic expansion from the dynamics of the model. To this purpose it is useful to introduce rescaled variables

$$w_{t,i} = \frac{W_{t,i}}{(1+r_f)^t} \quad , \qquad p_t = \frac{P_t}{(1+r_f)^t} \quad , \qquad e_t = \frac{D_t}{P_t\,(1+r_f)} \quad , \tag{3.5}$$

denoted with lower case names. The relations between the different quantities expressed in terms of the un-rescaled, upper case, variables and the rescaled, lower-case, ones can be easily obtained using (3.5). In particular the return from the risky asset R_t reads

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = r_t(1+r_f) + r_f \quad , \tag{3.6}$$

where r_t stands for the price return in terms of the rescaled prices

$$r_t = \frac{p_t}{p_{t-1}} - 1$$

so that zero total return for the rescaled prices $r_t = 0$ corresponds to the risk-free actual return $R_t = r_f$.

We rewrite the market dynamics defined in (3.3) using the set of variables introduced in (3.5) to obtain

$$\begin{cases} p_t = \displaystyle\sum_{i=1}^{N} \pi_{t,i}\, w_{t,i} \\ w_{t,i} = w_{t-1,i} + w_{t-1,i}\,\pi_{t-1,i}\left(\dfrac{p_t}{p_{t-1}} - 1 + e_{t-1}\right) \quad \forall i \in \{1,\ldots,N\} \quad . \end{cases} \tag{3.7}$$

These equations give the evolution of the state variables $w_{t,i}$ and p_t over time, provided that the stochastic (due to random dividend payment D_t) process $\{e_t\}$ is given and the set of investment shares $\{\pi_{t,i}\}$ is specified. As expected, the dynamics defined in (3.7) imply a simultaneous determination of the equilibrium price p_t and of the agents' wealths $w_{t,i}$: the N variables $w_{t,i}$ defined in the second equation appear on the right-hand side of the first, and, at the same time, the variable p_t defined in the first equation appears in the right-hand side of the second. Due to that simultaneity, these $N+1$ equations provide only an implicit definition of the state of the system at time t. For analytical purposes, it is necessary to derive the explicit equations that govern the system dynamics.

3.2 Dynamical System for Wealth and Return

The transformation of the implicit dynamics of (3.7) into an explicit one is not generally possible and entails restrictions on the possible market positions available to agents. This can be easily seen in the case of a single agent. Suppose that only one agent operates on the market, and suppose that he possesses B bonds and 1 equity (i.e. the total supply). If the agent decides to invest a share π of capital in the risky asset, the price equation would read

$$\pi\ (B + P_t) = P_t \quad . \tag{3.8}$$

The left- and the right-hand sides of this equations are linear functions of the asset price P_t and the equilibrium price is thus obtained by the intersection of these two straight lines. It is immediate to see that if $\pi = 1$ the two lines never intersect and if $\pi > 1$ the intersection is in the third quadrant and gives a negative price. We are then led to assume $\pi < 1$, i.e. to forbid short positions in bonds B. In the dynamical setting such requirement transforms into the restriction on possible *sequences* of investment shares, since the current trader position depends on his past investment. Hence, in the general n-agents case we expect to have a restriction on the possible values of the N-tuple $(\pi_1, \ldots, \pi_N)$'s over time.

Let us introduce an useful notation that allows the formulation of the dynamics in a more compact form. Let a_i be an agent specific variable, dependent or independent from time t. We denote with $\langle a \rangle_t$ the *wealth weighted average* of this variable at time t on the population of agents, i.e.

$$\langle a \rangle_t = \frac{\sum_{i=1}^{N} a_i\, w_{t,i}}{w_t} \quad , \quad \text{where} \qquad w_t = \sum_{i=1}^{N} w_{t,i} \quad . \tag{3.9}$$

The next result gives the condition for which the dynamical system implicitly defined in (3.7) is well defined explicitly with positive prices.

Theorem 3.1. *From equations* (3.7) *it is possible to derive a map* $\mathbb{R}^{+N} \to \mathbb{R}^{+N}$ *that describes the evolution of wealth* $w_{t,i}$ $\forall i \in \{1, \ldots, N\}$ *with positive prices* $p_t \in \mathbb{R}^+$ $\forall t$ *provided that*

$$\Big(\langle \pi_t \rangle_t - \langle \pi_t\, \pi_{t+1} \rangle_t \Big) \Big(\langle \pi_{t+1} \rangle_t - (1 - e_t) \langle \pi_t\, \pi_{t+1} \rangle_t \Big) > 0 \qquad \forall t \quad . \tag{3.10}$$

If this is the case, the price growth rate $r_{t+1} = p_{t+1}/p_t - 1$ *reads*

$$r_{t+1} = \frac{\langle \pi_{t+1} - \pi_t \rangle_t + e_t \langle \pi_t\, \pi_{t+1} \rangle_t}{\langle \pi_t\, (1 - \pi_{t+1}) \rangle_t} \tag{3.11}$$

and the evolution of wealth, described by the wealth growth rates $\rho_{t+1,i} = w_{t+1,i}/w_{t,i} - 1$*, is given by*

$$\rho_{t+1,i} = \pi_{t,i}\, \big(r_{t+1} + e_t \big) = \pi_{t,i}\, \frac{\langle \pi_{t+1} - \pi_t \rangle_t + e_t \langle \pi_t \rangle_t}{\langle \pi_t\, (1 - \pi_{t+1}) \rangle_t} \qquad \forall i \in 1, \ldots, N \quad . \tag{3.12}$$

Proof. See appendix A.1. □

The explicit price dynamics can be now derived from (3.11) in a trivial way. Price will be positive if the condition (3.10) is satisfied. This inequality is nothing else than the intertemporal many-agents extension of the constraint $\pi < 1$ that we derived from (3.8) on the permitted values of the control variables. In general, it may be quite difficult to check whether the inequality in (3.10) holds at each time step. From the practical point of view, it is better to make this condition more binding but, at the same time, simpler. This is performed in the following

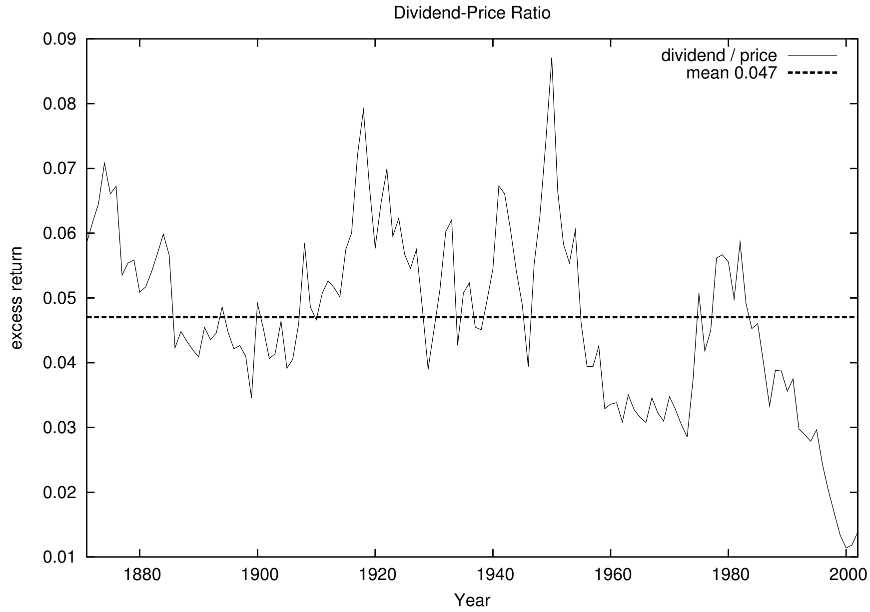


Figure 2: Dividend-price ratio. Source: data set on Standard&Poor 500 index by Robert Shiller available online `http://www.econ.yale.edu/~shiller/data.htm`.

Proposition 3.1. *Consider the system defined in Theorem 3.1. If there exist two real values* π_{min} *and* π_{max} *such that*

$$0 < \pi_{min} \leq \pi_{t,i} \leq \pi_{max} < 1 \quad \forall t\,,\ \forall i \in \{1, \ldots, N\} \tag{3.13}$$

and $e_t \geq 0$*, the condition* (3.10) *is always satisfied and the dynamics of* (3.12) *and* (3.11) *is bounded. Moreover, in this case there exist constants* ρ_{min}*,* ρ_{max}*,* r_{min} *and* r_{max}*, such that:*

$$\rho_{min} \leq \rho_{t,i} \leq \rho_{max} \quad \forall t \quad \forall i \qquad and \qquad r_{min} \leq r_t \leq r_{max} \quad \forall t \quad .$$

Proof. See appendix A.2. □

Thus, if all possible investment choices are confined on some compact subinterval in $(0, 1)$, then the equations (3.11) and (3.12) give well-defined dynamics in terms of price and wealths.

Having obtained the explicit dynamics for the evolution of price and wealth one would be interested in characterizing its fixed point. Unluckily, the dynamics defined by (3.11) and (3.12) does not possess any interesting fixed point. Indeed, if the price and the wealth are constant, one would have $r_{t+1} = \rho_{t+1,i} = 0$ for any t and i. This would imply, in periods when a positive dividend e_t is paid, $\pi_{t,i} = 0$ for any i. That is, the only possible fixed point is the one in which there is not demand for the risky asset. The reasons for the lack of interesting fixed points rests in the fact that, even if an exogenous expansion due to the risk-free interest rate has been removed with the rescaling of variables performed in Section 3.1, an expansion due to the dividend payments is still present. The presence of such an expansion would suggest to look for possible asymptotic states of steady growth. More so if Proposition 3.1 is assumed valid and all the returns are bounded. Notice, however, that this analysis in not possible in general: the possible dependence of the dividend yield e_t on price makes impossible to rewrite the dynamics in Theorem 3.1 is terms of the sole price and wealth returns. But, do we need to retain this dependence? The historical time series of the annual data of e_t for the Standard&Poor 500 index are reported in Fig. 3.2. It is apparent that the yield can be reasonably described as a bounded positive random variable whose behavior is roughly stationary. This empirical evidence suggests the following additional [1]

Assumption 1. The dividend yields e_t are i.i.d random variables obtained from a common distribution with positive support, mean $\bar{e}$ and standard deviation σ_e.

Under Assumption 1, the direct dependence on price disappears from (3.11) and (3.12) so that the dynamics of the economy is fully specified in terms of r_t and $\rho_{t,i}$.

4 Large Market Limit and Homogeneous Expectations

The general dynamics described by (3.11) and (3.12) are rather difficult to analyze. One reason being the necessity to track the wealth of each agent, and, consequently, of having a system whose dimension is, at least, as large as the number of agents.

Additional complication can come from the actual specification of the investment decisions π's, which should be considered, in general, endogenous variables. Let assume a common information scenario and let $\mathcal{I}_{t-1}$ be the information set publicly available to agents at the beginning of time t. The subscript here stresses the fact that the last available information is from period $t-1$. This set can contain past realizations of price $p_{t-1}, p_{t-2}, \ldots$, of dividend yield $e_{t-1}, e_{t-2}, \ldots$, and, in principle, also the entire state of the economy in the previous time steps, encompassing the choices of different agents and their portfolios. The investment decision $\pi_{t,i}$ is naturally based on the set $\mathcal{I}_{t-1}$ which, in turn, possesses an infinite dimension. We are then led to consider an infinite dimensional system in order to dynamically derive the investment decisions of agents.

In this Section we introduce a simplifying assumption that can be used to obtain a description of the economy in terms of a low dimensional stochastic dynamical system even when many agents are supposed to operate in the market. To this purpose we need a preliminary simple

Lemma 4.1. *The wealth-weighted average of market investment choices at time $t+1$, $\langle \pi_{t+1} \rangle$ can be computed using present wealth shares $w_{t,i}$ according to*

$$\langle \pi_{t+1} \rangle_{t+1} = \langle \pi_{t+1} \rangle_t + (r_{t+1} + e_t) \left(\langle \pi_{t+1}\, \pi_t \rangle_t - \langle \pi_{t+1} \rangle_t \langle \pi_t \rangle_t \right) \quad . \tag{4.1}$$

Proof. See Appendix A.3. □

Now let us make the following

Assumption 2. There exist a deterministic function F such that

$$\langle \pi_t \rangle_t = F(\mathcal{I}_{t-1}) \quad . \tag{4.2}$$

Moreover, the investment choices at successive time steps satisfy

$$\langle \pi_t\, \pi_{t+1} \rangle_t = \langle \pi_t \rangle_t \langle \pi_{t+1} \rangle_t \quad . \tag{4.3}$$

[1] This assumption is common to several works in literature, for instance Chiarella and He (2001). Consequently, a further reason to have this assumption introduced is to maintain comparability with previous investigations.

In Assumption 2 we require that the average investment choice is described by some time-invariant function of past information. In addition, we assume that the sample average of the product of the investment shares can be replaced by the product of their sample averages. Notice that the averages in (4.3) are wealth-weighted as defined in (3.9).

Let us now analyze the consequences of Assumption 2 on our model. Using (4.3) the price return in (3.11) reads

$$r_{t+1} = \frac{\left\langle \pi_{t+1} \right\rangle_t - \left\langle \pi_t \right\rangle_t + e_t \left\langle \pi_t \right\rangle_t \left\langle \pi_{t+1} \right\rangle_t}{\left\langle \pi_t \right\rangle_t - \left\langle \pi_t \right\rangle_t \left\langle \pi_{t+1} \right\rangle_t} \quad , \tag{4.4}$$

while, using (3.12), the total wealth growth rate ρ_{t+1} becomes

$$\rho_{t+1} = \left\langle \pi_t \right\rangle_t (r_{t+1} + e_t) = \frac{\left\langle \pi_{t+1} \right\rangle_t - \left\langle \pi_t \right\rangle_t + e_t \left\langle \pi_t \right\rangle_t}{1 - \left\langle \pi_{t+1} \right\rangle_t} \quad . \tag{4.5}$$

These two equations contain only "average" quantities. Moreover, when (4.3) is valid, (4.1) reduces to $\left\langle \pi_{t+1} \right\rangle_t = \left\langle \pi_{t+1} \right\rangle_{t+1}$. Substituting this relation in (4.4) and (4.5) and using (4.2) one can obtain a description of the dynamics of the model in terms of aggregate variables only

$$\begin{cases} \left\langle \pi_{t+1} \right\rangle_{t+1} = F(\mathcal{I}_t) \\ \qquad r_{t+1} = \dfrac{F(\mathcal{I}_t) - \left\langle \pi_t \right\rangle_t + e_t \left\langle \pi_t \right\rangle_t F(\mathcal{I}_t)}{\left\langle \pi_t \right\rangle_t - \left\langle \pi_t \right\rangle_t F(\mathcal{I}_t)} \\ \qquad \rho_{t+1} = \dfrac{F(\mathcal{I}_t) - \left\langle \pi_t \right\rangle_t + e_t \left\langle \pi_t \right\rangle_t}{1 - F(\mathcal{I}_t)} \\ \qquad \mathcal{I}_{t+1} = \mathcal{I}_t \cup \{r_{t+1}, \rho_{t+1}\} \end{cases} \quad , \tag{4.6}$$

where the last line is meant to represent the increase in the information set when the new price, and consequently the new return and aggregate wealth, is revealed.

Under Assumption 2 the dynamics of the system is strongly simplified. One question, however, remain to be answered: under which circumstances this assumption can be considered a reliable description of a multi-agent system? Obviously, Assumption 2 is satisfied in the homogeneous case, when all agents possess the same beliefs and preferences, so that at each time step $\pi_{t,i} = \left\langle \pi_t \right\rangle_t$, $\forall i$.

A more interesting example is constituted by the case of "purely noisy" agents. Suppose that at each time step the investment shares of the N agents are randomly and independently drawn from a common distribution with average value $\bar{\pi}$ and support in $(0, 1)$. In this case the informational set is irrelevant and in the $N \to \infty$ limit, if the share of wealth of each agent goes to zero, one has $\left\langle \pi_t \right\rangle_t = \bar{\pi}$. In this case, (4.3) is clearly fulfilled, so that the conditions of Assumption 2 are replicated, and the dynamics of returns in (4.4) reduces to

$$r_{t+1} = \frac{\bar{\pi}}{1 - \bar{\pi}} \, e_t \quad . \tag{4.7}$$

The return r_t fluctuates around some average value which depends positively both on the dividend yield and on the average investment share $\bar{\pi}$.

We can generalize the "pure noise" model and assume that the investment decision of each agent is a "noisy" version of a basic common choice; formally

$$\pi_{t,i} = F(\mathcal{I}_{t-1}) + \epsilon_{t,i} \quad , \tag{4.8}$$

where the ϵ's are independent (across time and across different agents) random variables with zero mean. When $N \to \infty$ the sample of independent random variables becomes large, the sample average converges to the average of the variable distribution, so that one can expect[2] that $\langle \pi_t \rangle_t \to F(\mathcal{I}_{t-1})$. If shocks are independent across time, one can also expect the fulfillment of (4.3) for the corresponding limits. Therefore, the dynamics are described by the system (4.4) and (4.5) with $\langle \pi_t \rangle_t$ replaced by $F(\mathcal{I}_{t-1})$. Since any "theoretical" relation for the averages of random variables is violated on finite samples, we consider this behavior as a sort of "limiting" situation and call it a *Large Market Limit (LML)*. Proposition 3.1 guarantees that when the range of F is a compact subinterval of $(0, 1)$, the price is always positive.

Since all the statements in the previous paragraph were only presumable, one should immediately pose the question about their validity. The answer, of course, could depend on the particular functional form chosen for F. In more general terms, one could ask the question: Under which specifications the LML theoretically exists, i.e. in which sense and under which conditions the limit of $\langle \pi_t \rangle_t$ is defined and when relation (4.3) is satisfied? We do not give a general answer to this non-trivial question in this paper, leaving it for future research. Instead, we use the following approach: in the next Section we analyze different models, obtained from different specifications of the agents behavior, *assuming* that the LML exists. In Section 6, with the help of computer simulations, we show to what extent the LML can be considered a satisfactory approximation of the actual multi-agents dynamics.

5 Technical Trading in the Large Market Limit

For the present analysis we assume that the agent investment choice π_{t+1} only depends on past realized price returns, so that the information set is reduced to the past returns history $\mathcal{I}_t = \{r_t, r_{t-1}, \dots\}$. In this way the past wealth return ρ_t does not affect future investment decisions and the wealth dynamics can be removed from (4.6). Notice that, in our framework, the behavior of agents who base their investment choices on the sole analysis of past prices returns cannot be considered inconsistent. Since we postulate, in Assumption 1, that the dividend process is stationary and proportional to price and that this information is publicly available, the notion of under- or over-evaluated asset cannot be applied and the price and wealth levels should be disregarded in agent's decisions. In such circumstances, deciding the present investment choice on the base of past market returns is a completely consistent behavior.

Under the previous assumptions, the dynamics of the model is defined by the following

$$\begin{cases} \langle \pi_{t+1} \rangle_{t+1} = f(r_t, r_{t-1}, \dots) \\ r_{t+1} = \dfrac{\langle \pi_{t+1} \rangle_{t+1} - \langle \pi_t \rangle_t + e_t \langle \pi_t \rangle_t \langle \pi_{t+1} \rangle_{t+1}}{\langle \pi_t \rangle_t - \langle \pi_t \rangle_t \langle \pi_{t+1} \rangle_{t+1}} \end{cases} . \tag{5.1}$$

The function f fully describes the way in which agents process information about past market performances to decide their future investment. It encompasses both the forecasting activity and the portfolio decision that agents make based on the outcome of this activity. The system (5.1) remains, in general, infinitely dimensional due to the dependence of f on the entire history of realized returns.

[2]Note that all averages which we consider are weighted with respect to variables changing over time. Therefore we cannot straightforwardly use any limit theorem to justify our reasoning.

In what follows we will consider examples, of increasing generality, in which agents decide future investments making use of only a small number of statistics obtained from the available information set. In this way, one is able to reduce the description of the model to a low-dimensional system. For the sake of clarity but with some abuse of notation, in the following analysis we will use π_t to denote the wealth-weighted average $\langle \pi_t \rangle_t$.

5.1 Naïve forecast

The simplest forecasting rule consists in taking the last realized return as a predictor for the next period return. We denote this rule as "naïve forecast". If one assumes that agents conform to this rule, the system (5.1) reduces to

$$\begin{cases} \pi_{t+1} = f(r_t) \\ r_{t+1} = \dfrac{f(r_t) - \pi_t + e_t\, \pi_t\, f(r_t)}{\pi_t\,(1 - f(r_t))} \end{cases} \tag{5.2}$$

where the function f is defined on $(-1, +\infty)$, i.e. for all possible price returns. In principle the image of f can be the whole real set. In general, however, for certain values of π_t the ensuing dynamics can violate (3.10) and lead to negative or infinite prices. As a solution of this problem, one can supplement the system above with the requirement that Proposition 3.1 has to be fulfilled. This is equivalent to consider a function f with values in $(0, 1)$.

To analyze the possible equilibria that characterize (5.2), we consider its dynamical skeleton, replacing the random dividend e_t with its mean value $\bar{e}$. We have the following:

Proposition 5.1. *Consider the deterministic skeleton of* (5.2)*. If* (π^*, r^*) *is a fixed point of the system, then*

(i) The equilibrium is feasible, i.e. the equilibrium prices are positive, if $\pi^* < 1$.

(ii) The equilibrium investment share π^* *satisfies*

$$\pi^* = f\left(\bar{e}\,\frac{\pi^*}{1-\pi^*}\right) \quad . \tag{5.3}$$

and the price return is defined by

$$r^* = \bar{e}\,\frac{\pi^*}{1-\pi^*} \quad . \tag{5.4}$$

(iii) The fixed point is (locally) asymptotically stable if

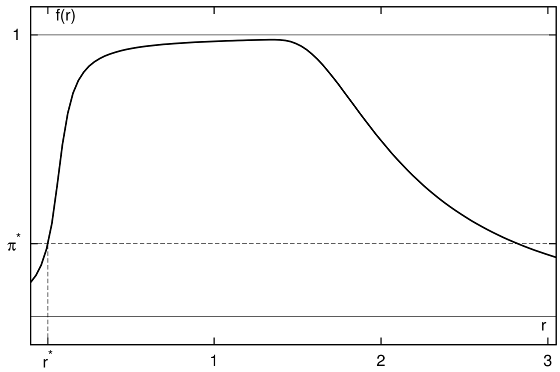
$$\frac{f'}{\pi^*\,(1-\pi^*)} < 1\,, \qquad \frac{r^*\, f'}{\pi^*\,(1-\pi^*)} < 1 \qquad \text{and} \qquad \frac{(2+r^*)\, f'}{\pi^*\,(1-\pi^*)} > -1 \quad , \tag{5.5}$$

where $f' = df(r^*)/dr$.

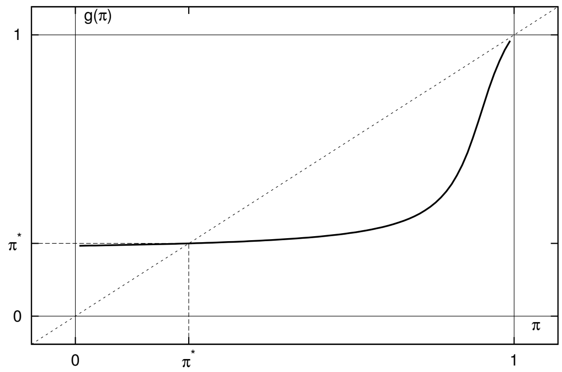
Proof. See Appendix A.4. □

The first condition states that economically meaningful equilibria[3] are characterized by values of the average investment share inside the open interval $(-\infty, 1)$. This condition derives

[3] In principle, also a feasible equilibria can lead to the appearance of negative prices when the average dividend $\bar{e}$ is substituted by a realization of the dividend process $\bar{e} \to e_t$.



(a)



(b)

Figure 3: An arbitrary function f is shown in panel (a), and the associated function g, defined according to (5.6), is reported on panel (b). Any fixed point of g, i.e. any intersection of its graph with the diagonal line gives the equilibrium value of the investment share π^*. The equilibrium value of the return r^* is defined according to (5.4)

from the specification of (3.10) to the case of homogeneous choices and from the further requirements that $r^* > -1$. According to (5.4), if π^* is positive, the equilibrium return is positive and the asymptotic dynamics is a steady increase of the risky asset price. If π^* is negative, the equilibrium price return is also negative and the price of the risky asset goes asymptotically to 0. Finally, if $\pi^* = 0$ the price of the asset is stationary[4]. Once the function f is specified, all the equilibria are characterize through (5.3). Notice that if one considers a choice function f with image in $(0, 1)$, so to fulfill Proposition 3.1, only equilibria associated with asymptotically increasing prices ($r^* > 0$) are possible. In general, however, any equilibrium investment share π^* can be seen as the fixed point of an auxiliary function g, defined on $(-\infty, 1)$ according to

$$g(\pi) = f\left(\bar{e}\,\frac{\pi}{1-\pi}\right) \quad . \tag{5.6}$$

The function g transforms monotonically the interval $(-\infty, 1)$ into $(-1, +\infty)$ and applies f to the result. See the example reported in Fig. 3.

Finally, the third point of Proposition 5.1 provides the conditions for the asymptotical stability of the fixed point. The stability regions can be easily represented in the parameters space using the function g defined in (5.6). Since $g'(\pi^*) = \bar{e}\, f'(r^*)/(1 - \pi^*)^2$, the conditions (5.5) can be rewritten in a more compact form using g and read

$$\frac{g'}{r^*} < 1\,, \qquad g' < 1 \qquad \text{and} \qquad g'\,\frac{2 + r^*}{r^*} > -1 \quad , \tag{5.7}$$

where g' stands for the derivative of the function g computed in π^*. In Fig. 4 we draw the stability region for the fixed point in the coordinates r^* and $g'(\pi^*)$. As can be seen, if the absolute value of the slope of g, and, hence, of f, computed at the equilibrium return r^*, increases, the system tends to lose its stability. At the same time, fixed points characterized by smaller values of r^* are, in general, less stable.

[4]Remember, however, that we are describing the economy in terms of the rescaled variables defined in (3.5)

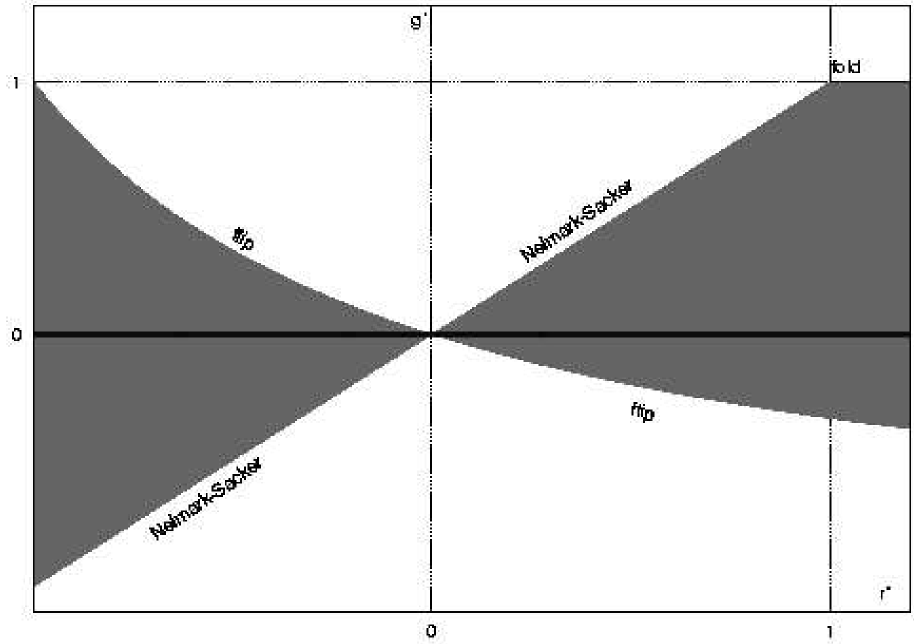


Figure 4: The stability region (gray) and the bifurcation types for the system (5.2) in parameter space with coordinates r^* and $g'(\pi^*)$. The fixed point is locally stable if it belongs to the gray area which is the union of two disjoint open sets. The type of bifurcation that the system undertakes when it leaves this area are denoted by labels on the boundaries that would be crossed.

To understand the effect of the slope of the function f on the fixed point stability, let us observe the impact on the system (5.2) of a sudden positive shock in the investment share. We will assume that the function f is increasing in the fixed point r^*, the case of the decreasing function being similar. Let us suppose that at time 0 the system is in the fixed point (π^*, r^*). At the beginning of time $t = 1$ the investment share suddenly increases of an amount $\delta\pi > 0$ so that $\pi_1 = \pi^* + \delta\pi$. Since the price return is an increasing function of the present investment choice, the price return at time $t = 1$ becomes larger than the equilibrium value r^* and, for small values of $\delta\pi$ one has $r_1 \sim r^* + \alpha\,\delta\pi$ where the coefficient α depends on the values of π^* and $\bar{e}$. At the next time step $t = 2$, the previous increase in the price return leads to an increase in the agent excecation about future return and, consequently, to an higher value of the agent investment shares pi_2, given by $\pi_2 = f(r^* + \alpha\,\delta\pi) \sim \pi^* + f'\,\alpha\,\delta\pi$. The slope of the investment function f' affects the strength with which agent reacts to variations in expectation level. The future dynamics of the system crucially depends on how sensitive the investment share is to these variations. Indeed, if $f' \ll 1$ it is $\pi^* < \pi_2 < \pi_1$ and the system moves again toward the fixed point. In general, the fixed point can be stable also when f' is large enough to generate a reinforcement effect, such that $\pi_2 > \pi_1$. Since the return r_2 depends on both π_2 and π_1, the reinforcement should be large enough to lead to an increase in the price return $r_2 > r_1$. If this is not the case, the system starts to revert toward the fixed point in the following step $t = 3$.

On Fig. 5 we show how the oscillations of π and r damp when the slope of the function f is small and how they are reinforced in the opposite case. On panel (b), the return falls down immediately at $t = 1$, while on panel (d) it increases further and create a quasi-periodical behavior with period 6.

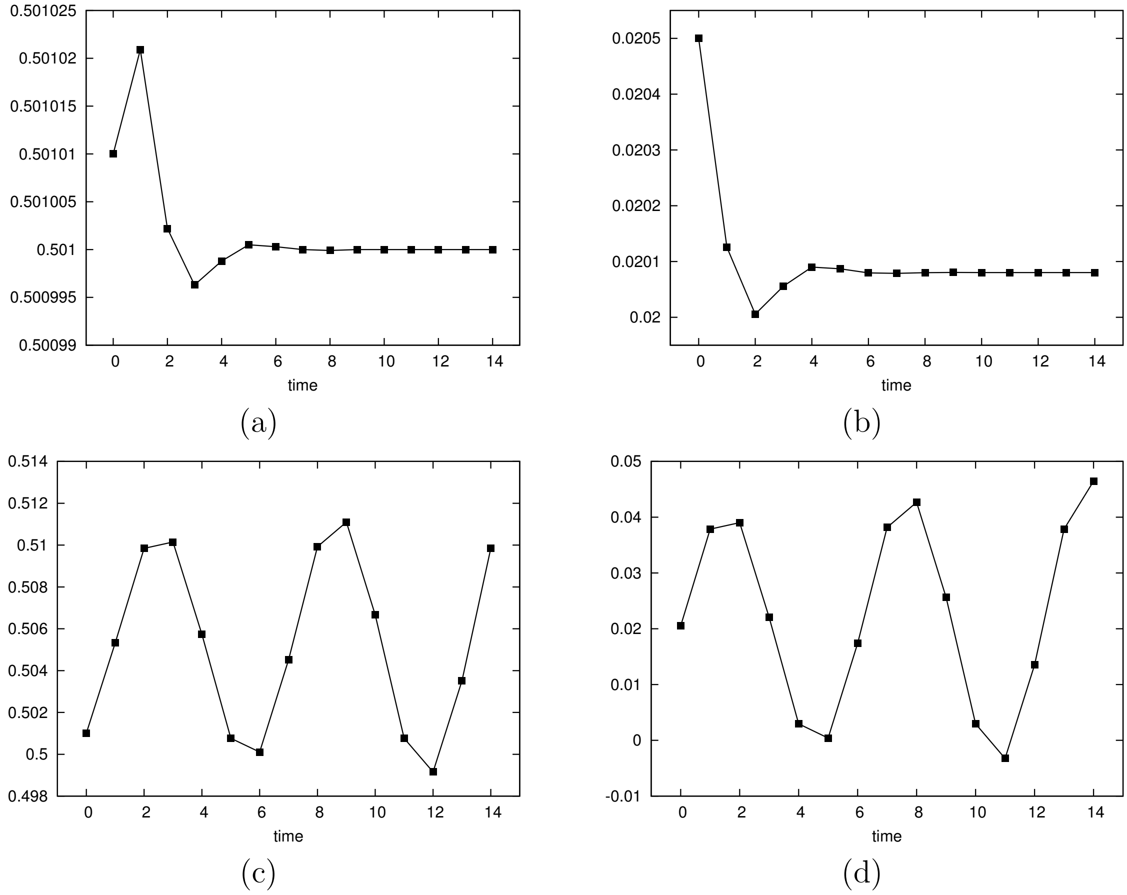


Figure 5: Trajectories for systems perturbed away from equilibrium. When the function $f(r) = 0.5 + 0.0498\,r$ and the fixed point $(0.501, 0.02)$ are considered, both π (panel a) and r (panel b) stabilize through the oscillations. For the function $f(r) = 0.5 + 0.26\,r$ the oscillations of π (panel c) and r (panel d) around the fixed point $(0.505, 0.02)$ are reinforcing and the system does not stabilize. In both cases is $\bar{e} = 0.02$.

One can get more insights by applying the Theorem B.2 from Appendix B to analyze the type of bifurcation occurring when one of the conditions in (5.5) is violated. The bifurcation analysis is provided by the following

Proposition 5.2. *Let us suppose that two of the three inequalities in* (5.5) *hold. Then, generally, the fixed point of the system* (5.2) *undertakes a*

(i) fold bifurcation, if $\quad r^*\, f' = \pi^*\,(1 - \pi^*)$;

(ii) flip bifurcation, if $\quad (2 + r^*)\, f' = -\pi^*\,(1 - \pi^*)$;

(iii) Neimark-Sacker (secondary Hopf) bifurcation, if $\quad f' = \pi^*(1 - \pi^*)$.

A summary of the previous results is reported in Fig. 4 where the labels along the boundaries of the stability set specify the type of bifurcation that the system undertakes when that particular curve is crossed. Notice that for relatively small values of the equilibrium price return, i.e. $|r^*| < 1$, the system can loose stability through a flip or Neimark-Sacker bifurcation. In this case, if one perturbs the system away from its equilibrium with a small shock on the investment share, the large absolute value of f' is responsible for an amplification of this perturbation, based on the feedback mechanism that links investment share and market return, which ultimately leads to an oscillatory system behavior. For larger values of the equilibrium return, namely $r^* > 1$, if the investment choice function is too steep at the equilibrium, i.e. if $f' > 1$, the equilibrium is lost by a fold bifurcation, which implies a local exponential growth of the price returns.

5.2 Forecast by extrapolation of past average return

Next we move from the "naive" forecast of the previous example to the case in which agents use a slightly more sophisticated forecasting techniques: they take, as predictor of future return, the average of past realized returns. Inside this framework, it seems reasonable to consider weighted averages, since one can think that the mood prevailing on the market may change over time and more recent values of the price return could contain more information about future prices than the older ones. This requirement is satisfied by the exponentially weighted moving average (EWMA) estimator y_t defined as

$$y_t = (1-\lambda)\sum_{\tau=0}^{\infty} \lambda^\tau \, r_{t-\tau} \tag{5.8}$$

where $\lambda \in [0, 1)$ is a parameter describing the weights attached to past realized returns. The weights are declining geometrically in the reverse time, so that the last available observation r_t has the highest weight. The value of λ determines how the relative weights are distributed among the recent and older observations. Intuitively, even if all past observations are taken into account in the definition of y_t, one can say that λ measures the length of the agent's memory. The extreme value $\lambda = 0$ corresponds to the the naïve forecast considered above.

The EWMA estimator admits the following recursive definition

$$y_t = \lambda y_{t-1} + (1-\lambda) r_t \tag{5.9}$$

so that (5.1) can be reduced to a finite dimensional system

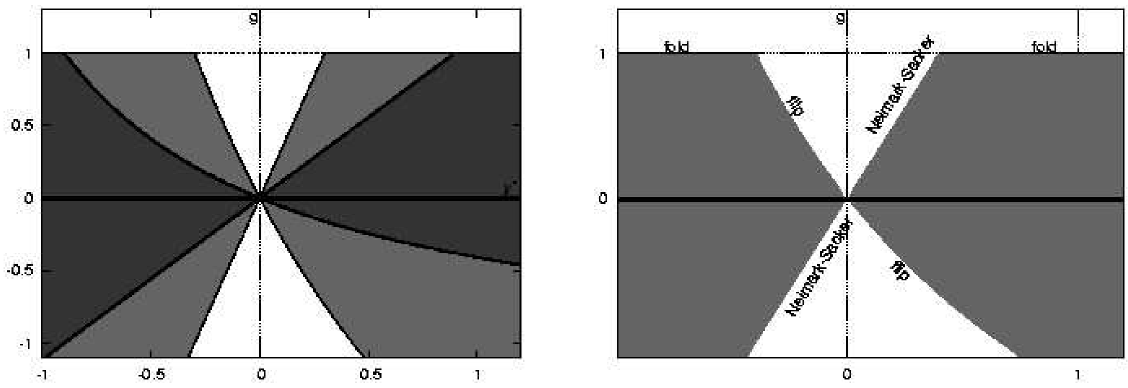
$$\begin{cases} \pi_{t+1} = f(y_t) \\ y_{t+1} = \lambda y_t + (1-\lambda)\, \dfrac{f(y_t) - \pi_t + e_t \, \pi_t \, f(y_t)}{\pi_t \,(1 - f(y_t))} \end{cases} . \tag{5.10}$$

Analogously to the naïive forecast case, the investment share is defined as some function of the forecasted return. In the present case, however, the forecasted return is provided by the EWMA estimator y_t. The function f is defined on the interval of possible returns $(-1, +\infty)$. As in the previous example, further constraints on the image of the function f might be required in order to keep the dynamics of the system inside the set of economically meaningful states.

The following result characterizes the existence and the local stability of the fixed points of system (5.10)

Proposition 5.3. *Consider the deterministic skeleton of system* (5.10)*. If* (π^*, y^*) *denotes a fixed point of the system then*

(i) The equilibrium is feasible, i.e. the equilibrium prices are positive, if $\pi^* < 1$.



(a) (b)

Figure 6: In the coordinates y^* and $g'(\pi^*)$ for the system (5.10), it is shown: (a). The enlargement of the stability regions with increase of λ. For small value $\lambda = 0.1$, the fixed point is locally stable in one of two black sets. When λ increases to the value 0.7, the stability area in addition contains two grey areas. (b). Types of the bifurcations.

(ii) The equilibrium investment share π^* satisfies

$$\pi^* = f\left(\bar{e}\,\frac{\pi^*}{1-\pi^*}\right) \quad , \tag{5.11}$$

the equilibrium price return is equal to the equilibrium value of the predictor, $r^* = y^*$, and is given by

$$r^* = \bar{e}\,\frac{\pi^*}{1-\pi^*} \quad . \tag{5.12}$$

(ii) The fixed point is (locally) asymptotically stable if

$$\frac{f'}{\pi^*\,(1-\pi^*)} < \frac{1}{1-\lambda}\,, \quad \frac{r^*\,f'}{\pi^*\,(1-\pi^*)} < 1 \quad \textit{and} \quad \frac{(2+r^*)\,f'}{\pi^*\,(1-\pi^*)} > -\frac{1+\lambda}{1-\lambda}\,, \tag{5.13}$$

where f' denotes the derivative of the function f in the point y^*.

Proof. See Appendix A.5. □

The dynamics of system (5.10) is very similar (at least locally, near the fixed points) to the dynamics in the case of naïve forecast. Indeed, the possible values of economically meaningful equilibria can be still characterized as fixed points of the auxiliary function g defined in (5.6).

According to item *(ii)*, at equilibrium the realized price return r^* coincides with the prediction of the EWMA estimator y^*. This is an important consistency result. In general, any meaningful economic dynamics should avoid equilibria based on systematic mistakes by the side of traders.

The stability conditions (5.13) are similar to the conditions obtained for naïve forecast and reduce to them when $\lambda = 0$. With the help of the auxiliary function g the stability region can be defined by a simple set of inequalities

$$\frac{g'}{r^*} < \frac{1}{1-\lambda}\,, \qquad g' < 1 \qquad \text{and} \qquad g'\,\frac{2+r^*}{r^*} > -\frac{1+\lambda}{1-\lambda} \quad . \tag{5.14}$$

Two different regions defined by (5.14) with two different values of the parameter λ are shown in panel (a) of Fig. 6. As can be seen, the increase in the value of the parameter λ, that is, loosely speaking, in the agent's memory length, brings stability into the system [5]. This can be easily understood comparing the EWMA forecast (5.8) with the naïve forecast: any shock in the present investment behavior leads to smaller changes in the expected future return when the former is used instead of the latter.

Since the value of λ does not affect the position of the fixed point, the enlargement of the stability region with the increase of the value of λ implies that the system can be stabilized by means of this sole parameter. Any fixed point (except, possibly, the ones with $y^* = 0$) becomes stable when λ takes a sufficiently large value (see Fig. 6).

The possible ways in which a system fixed point can loose stability when the values of the different parameters are varied is provided by the following

Proposition 5.4. *Let us suppose that two of the three inequalities in* (5.13) *hold. Then, generally, the fixed point of the system* (5.10) *undertakes*

(i) fold bifurcation, if $r^* f' = \pi^* (1 - \pi^*)$*;*

(ii) flip bifurcation, if $(2 + r^*) f' = -\pi^* (1 - \pi^*) (1 + \lambda) / (1 - \lambda)$*;*

(iii) Neimark-Sacker (secondary Hopf) bifurcation, if $f' = \pi^*(1 - \pi^*)/(1 - \lambda)$*.*

Proof. This result is a straight-forward application of the Theorem B.2 from Appendix B. □

An illustration of the last Proposition is given in panel (b) of Fig. 6.

5.3 Forecast by extrapolation of past average return and variance

Our last example of the application of the LML consists in the extension of the previous model to agents who decide their future investment based not only on the estimated value of future return, but also on the estimated variance. The latter can indeed be thought as a measure of expected risk. We introduce the exponentially weighted moving average (EWMA) estimator for the variance

$$z_t = (1 - \lambda) \sum_{\tau=0}^{\infty} \lambda^\tau \big(r_{t-\tau} - y_t\big)^2 \quad , \tag{5.15}$$

where y_t is an EWMA of past returns as given in (5.8) and $\lambda \in [0, 1)$ is the usual "memory" parameter. Notice that the estimators (5.8) and (5.15) together admit the following recursive definition

$$\begin{aligned} y_t &= \lambda y_{t-1} + (1 - \lambda) r_t \\ z_t &= \lambda z_{t-1} + \lambda\, (1 - \lambda) \big(r_t - y_{t-1}\big)^2 \quad , \end{aligned} \tag{5.16}$$

which allows the reduction of (5.1) to a 3-dimensional dynamical system

$$\begin{cases} \pi_{t+1} = f(y_t, z_t) \\ y_{t+1} = \lambda y_t + (1 - \lambda)\, \dfrac{f(y_t, z_t) - \pi_t + e_t\, \pi_t\, f(y_t, z_t)}{\pi_t\, (1 - f(y_t, z_t))} \\ z_{t+1} = \lambda z_t + \lambda(1 - \lambda) \left[\dfrac{f(y_t, z_t) - \pi_t + e_t\, \pi_t\, f(y_t, z_t)}{\pi_t\, (1 - f(y_t, z_t))} - y_t \right]^2 \end{cases} . \tag{5.17}$$

[5] This is a quite intuitive and robust result that in Bottazzi (2002) has been found to characterize the LML also in CARA framework

The investment function f depends now on two variables, y_t and z_t, and is generally defined on $(-1, +\infty) \times [0, +\infty)$. Again, further requirements on its functional form might be needed in order to obtain meaningful economic dynamics. The discussion of the previous examples directly applies.

The following Proposition characterizes the fixed points of (5.17) and gives the sufficient conditions for their local stability.

Proposition 5.5. *Consider the deterministic skeleton of system* (5.17)*. If* (π^*, y^*, z^*) *denotes a fixed point of the system than*

(i) The equilibrium is feasible, i.e. the equilibrium prices are positive, if $\pi^* < 1$*.*

(ii) The equilibrium investment share π^* *satisfies*

$$\pi^* = f\left(\bar{e}\,\frac{\pi^*}{1-\pi^*}, 0\right) \quad . \tag{5.18}$$

the equilibrium price return is equal to the equilibrium value of its predictor, $r^* = y^*$*, and is given by*

$$r^* = \bar{e}\,\frac{\pi^*}{1-\pi^*} \tag{5.19}$$

while for the predictor of variance it is $z^* = 0$*.*

(iii) The fixed point is (locally) asymptotically stable if

$$\frac{f'_y}{\pi^*\,(1-\pi^*)} < \frac{1}{1-\lambda}\,, \qquad \frac{r^*\, f'_y}{\pi^*\,(1-\pi^*)} < 1 \quad \text{and} \quad \frac{(2+r^*)\, f'_y}{\pi^*\,(1-\pi^*)} > -\frac{1+\lambda}{1-\lambda}\,, \tag{5.20}$$

where f'_y *denotes the partial derivative of the function* f *with respect to the first variable computed in the point* $(y^*, 0)$*.*

Proof. See Appendix A.6. □

There are only minor differences between the last statement and Proposition 5.3. In item *(ii)* the condition characterizing the equilibrium average investment share π^* has slightly changed, because the function f now depends on two variables. However, the geometrical characterization of equilibria as fixed points of an auxiliary function g is still valid, provided one uses the function $g(\pi) = f(\bar{e}\,\pi/(1-\pi), 0)$. Accordingly, the illustration in Figure 3 is still valid. The consistency result for the equilibrium return estimator $y^* = r^*$ is confirmed and is extended to the variance estimator z^*, whose value at equilibrium becomes zero, as expected for a geometrically increasing price dynamics.

The stability conditions given in item *(iii)* are the same as in (5.13), with the only difference that now is the partial derivative of function f that enters the expression. With the help of the auxiliary function g, (5.20) can be reduced to the set of conditions (5.14). As a consequence, the stability regions looks the same as in Fig. 6 and the bifurcation analysis remains unchanged. Proposition 5.4 holds for the system (5.17) as well.

In conclusion, even if the introduction of a measure of risk could change the global behavior of the system, the local dynamics in a neighborhood of the fixed points remain essentially the same.

5.4 An Example: Extending the Chiarella and He model

The framework discussed in the previous sections can be used to model a large class of trading behaviors. Different behaviors are described by different possible specifications of the investment function f.

Here, as a simple exercise, we show how the general theorems provided above can be used to replicate and extend some of the results presented in Chiarella and He (2001). In the next Section we briefly review the assumptions of this paper and write down the relevant equations using the original notations. In the Section that follows, we show how to translate these equations in our framework and obtain the generalization of the original results.

5.4.1 Definition of the model

This paper describe a two asset economy identical to the one introduced in Sec. 3. The risky security pays a dividend D_t so that the ex-dividend return on the risky asset at period t reads

$$r^c_{t+1} = \frac{P_{t+1} + D_t - P_t}{P_t} \quad . \tag{5.21}$$

The dividend yield $\alpha_t = D_t/P_t$ is assumed $i.i.d$ and the investment choice $\pi_{t,i}$ of agent i at time t is obtained by maximizing the expected utility of future wealth

$$\max_{\pi_{t,i}} \mathrm{E}\left[U(W_{t+1,i})\right] \quad , \tag{5.22}$$

where U stands for a CRRA power-like utility function

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma} \quad , \tag{5.23}$$

and the parameter γ represents the degree of relative risk aversion. Due to the impossibility of finding an explicit expression of the solution of (5.22), Chiarella and He propose a mean-variance approximation obtained considering a continuous time analogue to this optimization problem[6].

Under this approximation each agent i is assumed to form expectation at time t about future return $\mathrm{E}_{t,i}[r^c_{t+1}]$ and variance $\mathrm{V}_{t,i}[r^c_{t+1}]$ and his individual demand curve reads

$$\pi_{t,i} = \frac{1}{\gamma} \frac{\mathrm{E}_{t,i}\left[r^c_{t+1}\right] - r_f}{\mathrm{V}_{t,i}\left[r^c_{t+1}\right]} \quad . \tag{5.24}$$

The precise definition of this demand function depends on how agents form expectations with respect to future returns and returns variance. Chiarella and He consider the following specifications

$$\mathrm{E}_t\left[r^c_{t+1}\right] = r_f + \delta + d\,\bar{r}^c_t \tag{5.25}$$

$$\mathrm{V}_t\left[r^c_{t+1}\right] = \sigma^2\left[1 + b\left(1 - (1+v_t)^{-\xi}\right)\right] \tag{5.26}$$

[6]See Appendix A.1 in Chiarella and He (2001). Notice that no optimization of the error incurred in considering the mean-variance approximation is provided so that one does not know how much of the final result depends on it.

where $\bar{r}^c_{t-1}$ and v_{t-1} stand for the sample estimates of the average return and variance computed as equally weighted averages of the previous L observations

$$\bar{r}^c_t = \frac{1}{L}\sum_{k=1}^{L} r^c_{t-k} \qquad \text{and} \qquad v_t = \frac{1}{L}\sum_{k=1}^{L}[r^c_{t-k} - \bar{r}^c_{t-1}]^2 \quad . \tag{5.27}$$

The expected conditional return (5.25) is defined as the risk free rate r_f plus the excess return. The latter is made of a constant component, a sort of risk premium, $\delta > 0$, and a variable component, $d\,\bar{r}^c_t$, proportional to the sample average of past realized returns. The parameter d is assumed to represent the way in which agents react to variations in the history of realized returns and can be used to distinguish between different classes of investors. A trader with $d = 0$ will ignore past realized returns and, consequently, can be thought as a *fundamentalist*. If $d > 0$ the agent can be considered a *trend follower*, and a *contrarian* if $d < 0$. Analogously, the expression for the conditional variance (5.26) contains a constant component σ^2 and depends on the realized variance through two parameters, b and ξ.

5.4.2 Equilibrium analysis in the homogeneous case

In Chiarella and He (2001) an analytical result is presented which characterizes the existence and the stability of the fixed points in the case of homogeneous expectations when $L = 1$. For larger values of L, the authors rely only on numerical simulations to show that an increase in the value of L tends to stabilize the system.

In what follows we introduce a slight modification of (5.27) that leaves, however, the spirit of the original assumptions unchanged. In this way we are able to present the analytical investigation of equilibrium stability for longer time horizons.

As a first step, however, it is convenient to rewrite the agent demand function according to the notations introduced in Sec. 3.1.

Separating the contribution of the capital gain from the dividend yield (5.27) becomes

$$\bar{r}^c_t = \frac{1}{L}\sum_{k=1}^{L} R_{t-k} + \frac{1}{L}\sum_{k=1}^{L} \alpha_{t-k-1} \tag{5.28}$$

$$v_t = \frac{1}{L}\sum_{k=1}^{L}\Big[R_{t-k} - \frac{1}{L}\sum_{k=1}^{L} R_{t-k} + \big(\alpha_{t-k-1} - \frac{1}{L}\sum_{k=1}^{L}\alpha_{t-k-1}\big)\Big]^2 \quad . \tag{5.29}$$

Since we are interested in the analysis of the deterministic skeleton of the system we can replace the yield process $\{\alpha_t\}$ with its average $\bar{\alpha}$ so that the second term in (5.29) disappears and, using the definitions in (3.5) one has

$$\bar{r}^c_t = r_f + \bar{e}\,(1 + r_f) + \frac{1}{L}\,(1 + r_f)\,\sum_{k=1}^{L} r_{t-k} \tag{5.30}$$

$$v_t = \frac{1}{L}(1 + r_f)^2\sum_{k=1}^{L}\big(r_{t-k} - \frac{1}{L}\sum_{k=1}^{L} r_{t-k}\big)^2 \tag{5.31}$$

where $\bar{e} = \bar{\alpha}/(1 + r_f)$.

In order to obtain a more tractable expression we replace the equally weighted averages that appears in the previous equations with the respective EWMA estimator defined in (5.8) and (5.15). With the substitution

$$\frac{1}{L}\sum_{k=1}^{L} r_{t-k} \rightarrow y_t \tag{5.32}$$

$$\frac{1}{L}\sum_{k=1}^{L}\left(r_{t-k} - \frac{1}{L}\sum_{k=1}^{L} r_{t-k}\right)^2 \rightarrow z_t \tag{5.33}$$

one obtains

$$\bar{r}_t^c = r_f + \bar{e}\,(1+r_f) + y_t\,(1+r_f) \tag{5.34}$$

$$v_t = \;(1+r_f)^2\, z_t \quad . \tag{5.35}$$

The estimators y_t and z_t depend on the parameter λ. For $\lambda = 0$ they reduce to the original equally weighted estimators for $L = 1$. With the increase of λ, the weight on the past realizations is higher so that the effect is equivalent to an increase in the parameter L.

With this choice for the estimators the agent investment function reduces to a particular case of the general form discussed in Sec. 5.3. Indeed, considering (5.34) and (5.35) together with the expression for the individual demand function (5.24) one has

$$f(y_t, z_t) = \frac{\bar{\delta} + \bar{d}\,\alpha + \bar{d}\,r_f + \bar{d}\,y_t\,(1+r_f)}{1 + b\,[1 - (1 + (1+r_f)^2\, z_t)^{-\xi}]} \tag{5.36}$$

where $\bar{\delta} = \delta/(\gamma\,\sigma^2)$, $\bar{d} = d/(\gamma\,\sigma^2)$ and $\alpha = (1+r_f)\,\bar{e}$. Applying Proposition 5.5 one gets

Proposition 5.6. *Consider the deterministic skeleton of system* (5.17)*, where the function f is given in* (5.36)*. Then:*

(i) The equilibrium investment share π^* satisfies

$$\pi^* = \bar{\delta} + \bar{d}\,\alpha + \bar{d}\,r_f + \bar{d}\,y^*\,(1+r_f) \quad . \tag{5.37}$$

(ii) The equilibrium price return is characterized as follows:

(ii.a) if $\bar{d} = 0$

$$y^* = r^* = \bar{e}\,\frac{\bar{\delta}}{1-\bar{\delta}} \quad , \tag{5.38}$$

(ii.b) if $\bar{d} \neq 0$

$$y^*_{\pm} = r^*_{\pm} = -\bar{e} - \frac{1}{2\,(1+r_f)}\left[r_f + \frac{\bar{\delta}-1}{\bar{d}} \pm \sqrt{\left(r_f - \frac{1-\bar{\delta}}{\bar{d}}\right)^2 - \frac{4\,\bar{e}\,(1+r_f)}{\bar{d}}}\,\right] \;, \tag{5.39}$$

when the expression under the squared root in the right-hand side is non-negative.

(iii) The following conditions are sufficient for the local asymptotical stability of the fixed point:

$$\bar{d}\,C < 1 \quad , \qquad -\bar{d}\,A < \frac{1}{1-\lambda} \qquad \textit{and} \qquad \bar{d}\,(2\,A - C) < \frac{1+\lambda}{1-\lambda} \quad , \tag{5.40}$$

where A and C read

$$A = \frac{1+r_f}{\pi^*\,(\pi^*-1)} \qquad \textit{and} \qquad C = \frac{\alpha}{(\pi^*-1)^2} \quad . \tag{5.41}$$

Proof. Consider the definition of f as in (5.36) and substitute it in (5.18) to obtain (5.37). Substituting this last equation in (5.19) one has

$$(y^* + \bar{e})\,(1 - \bar{\delta} - \bar{d}\bar{e}\,(1 + r_f) - \bar{d}r_f - \bar{d}y^*(1 + r_f)) = \bar{e} \quad . \tag{5.42}$$

If $\bar{d} = 0$, then $r^* = \bar{e}\,\bar{\delta}/(1 - \bar{\delta})$, and *(ii.a)* is proven. If $\bar{d} \neq 0$, you get *(ii.b)* by solving (5.42) as a second degree equation in y^*. From (5.36), it is $f_y^{'} = \bar{d}/[1 + b[1 - (1 + v_t)^{-\xi}]]$ and from (5.37) and (5.39) substituted into (5.20), *(iii)* immediately follows. □

First of all notice that the equilibrium conditions (5.38) and (5.39) in our model correspond to equations (3.4) and (3.5) in Proposition 3.1 of Chiarella and He (2001) respectively. The expression of the fixed points does not depend on the value of λ (or L).

For $\lambda > 0$ (i.e. $L = 1$), the conditions in (5.40) exactly replicate those in Corollary 3.3 in Chiarella and He (2001). As can be seen from (5.40), with the increase in the memory length, the system becomes more stable, coherently with the idea that the more cautious the choice of the agent with respect to recent returns, the higher the stability of the return dynamics.

6 Numerical Analysis of the Large Market Limit

In the previous analysis we have presented analytical results on the existence and stability of equilibria for a class of different deterministic dynamical systems. However, our main interest lies in the analysis of agent-based models characterized by, at least, two different sources of randomness: first, the idiosyncratic noise components $\epsilon_{t,i}$ appearing in the definition (4.8) of the agent investment choice and, second, the exogenous stochastic dividend process $\{e_t\}$ defined in Assumption 1. In order to reduce the analysis of stochastic models to deterministic ones, we have made two distinct simplifications: we postulated that, when a large number of agents is present on the market, the aggregate effect of the choices of the different agents can be reduced, under the Large Market Limit, to a deterministic function of the state of the market so that one can rely on Assumption 2 to obtain the dynamics of the model. Next, we replaced the dividends process e_t with its mean $\bar{e}$ in the low-dimensional system obtained under Assumption 2 in order to derive the deterministic dynamics that has been studied using the standard tools of the theory of dynamical systems.

Obviously, the analysis of the derived deterministic systems is worthwhile only as far as the two simplifications described above represent a meaningful approximation of the original model. In other terms, one should prove that the actual differences between the deterministic and the original stochastic model can be considered, in the cases of interest, "small". To this purpose, first, it is necessary to understand how an to what extent the dynamics generated by a finite population of agents differ from the ones obtained under Assumption 2. Second, one has to compare the behavior of the deterministic case to the behavior of the system when different realizations of the stochastic dividend process are considered.

With the help of computer simulations we try to explore both these issues here. We are not able to provide general results, however, and we limit our analysis to few examples that can, nonetheless, be considered representative of the cases discussed in the previous examples. As we illustrate below, it turned out that, even if differences between the limiting deterministic behavior and the actual stochastic implementation of the model do exist, they become smaller when the size of the population of agents increases and the variance of the dividend process decreases.

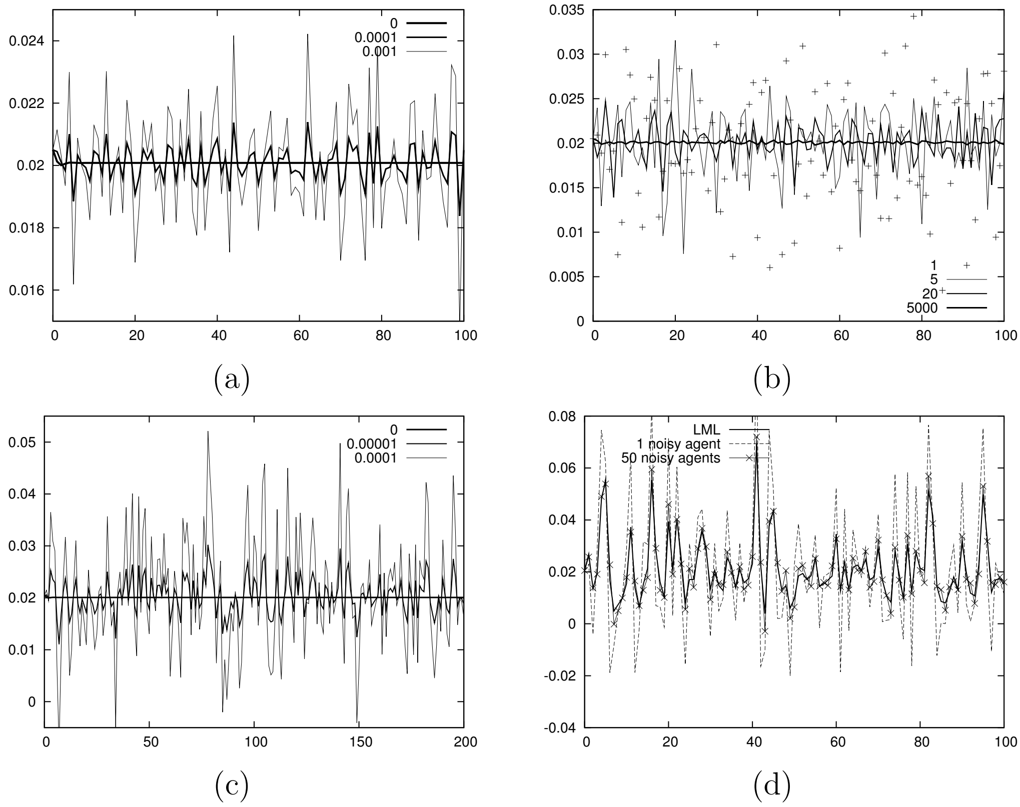
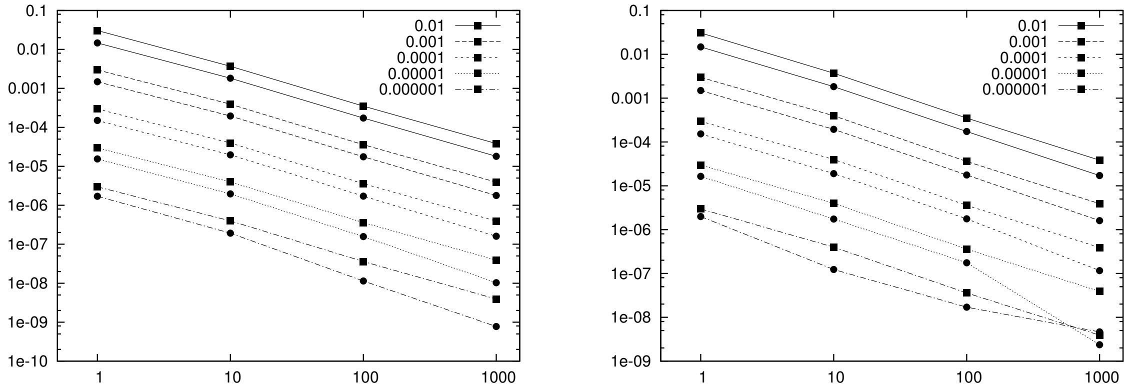


Figure 7: Trajectories of the system for naive forecasters with linear investment function $f_1(r) = 0.5 + 0.0498r$. Panel (a): the LML with constant dividend yield (the thickest line) is compared with simulations obtained with a population of $N = 1000$ idiosyncratic agents and two different values of the variance of the idiosyncratic shocks $\epsilon_{t,i}$. Panel (b): simulations with different number of agents and a fixed variance of the idiosyncratic components in investment functions $\sigma_\epsilon^2 = 0.00002$. Panel (c): the dynamics in the LML with constant dividend yield (the thickest line) is compared with the LML with yield drawn from the normal distribution. Results with two different values for the yield variance are shown. Panel (d): the LML with random dividend yield (the thickest line) is compared with the simulation with 1 and 50 agents.

6.1 Effect of agent-specific noise

Let us start considering a constant dividend yield $\bar{e}$ and focus on the effect of the noisy components in agents investment function. Let us consider the simple case of a linear investment function. For definiteness, consider the "naive" (see Section 5.1) investment function $f(r) = 0.5 + 0.0498\,r$. Under the LML the system converges to the stable equilibrium with price return $r^* \simeq 0.02008016$. If noise is added to the individual demand functions, the observed returns fluctuates, as shown in Fig. 7. However, when the number of agents is fixed, the variance of these fluctuations decreases with the variance of $\epsilon_{t,i}$ (panel a). At the same time, if the variance of the noise is kept fixed, the variance of the fluctuations decreases with the increase of the population size N (panel b). Both these effects are presented in Table 1 where the sample mean and variance of simulated trajectories are reported for different values of the number of agents N and of the variance of the idiosyncratic noise σ_ϵ^2. When $N \to \infty$ or $\sigma_\epsilon^2 \to 0$, the average of the return converges to r^* and the variance goes to 0. In order to show that these results are not simulation-specific, we compute the mean absolute deviation from r^*



(a) (b)

Figure 8: Average over 100 simulations of the sample mean (boxes) and variance (circles) of the deviation of returns from the LML deterministic system as function of the number of agents. Different lines correspond to different values of the variance σ_ϵ^2. Panel (a): constant dividend yield. Panel (b): random dividend yield.

σ_ϵ^2	0.000001	0.00001	0.0001	0.001	0.01
$N = 1$	2.008205e-02	2.009684e-02	2.023999e-02	2.165633e-02	3.573289e-02
	1.773948e-03	5.609896e-03	1.774379e-02	5.621061e-02	1.807372e-01
$N = 10$	2.008040e-02	2.008231e-02	2.010091e-02	2.028336e-02	2.193753e-02
	6.410769e-04	2.027306e-03	6.409194e-03	2.018357e-02	6.131534e-02
$N = 100$	2.008017e-02	2.008032e-02	2.008195e-02	2.009849e-02	2.026070e-02
	1.943986e-04	6.147246e-04	1.943614e-03	6.138590e-03	1.920147e-02
$N = 1000$	2.008022e-02	2.008036e-02	2.008093e-02	2.008390e-02	2.010480e-02
	6.359172e-05	2.011009e-04	6.359529e-04	2.010011e-03	6.309014e-03
$N = 10000$	2.008018e-02	2.008023e-02	2.008040e-02	2.008099e-02	2.008099e-02
	2.034083e-05	6.432284e-05	2.033877e-04	6.426056e-04	6.426056e-04

Table 1: Sample means and standard deviations of the price return for different number of agents N and different values of the idiosyncratic component variance σ_ϵ^2.

and the return variance over 100 independent simulations. The results are presented on panel (a) of Fig. 8. as can be seen, both variables converge to 0 when the size of the population increases.

6.2 Randomness of the yield

Now let us explore the dynamic effect of considering random dividends. On panel (c) of Fig. 7 we compare the return dynamics of the stochastic system (5.2) with the dynamics of its deterministic skeleton. The presence of random dividends distorts the returns trajectory. When the variance of the yield decreases, however, these distortions become smaller and smaller.

Finally, it is interesting to investigate the interplay between the two sources of randomness. In particular, one can ask whether the low dimensional stochastic system obtained under Assumption 2 is a good approximation of the "typical" dynamics in the case of many agents with noisy investment functions. In panel (d) of Fig. 7 we compare the returns dynamics obtained from system (5.2) (with random dividends) with the dynamics generated by the multi-agents model for different population sizes N. As can be seen, the observed differences decrease when the number of agents increases. This is confirmed by the statistics obtained from repeated independent simulations reported in panel (b) of Fig. 8.

In conclusion, we can say that, for what concerns the framework considered in Section 5, the LML provides, at least locally, a reasonable approximation of the original models also for a moderately small ($N \sim 50$) population of agents.

7 Heterogeneous Agents

Until now the heterogeneity in agents' demand came into our analysis only through idiosyncratic small random components, that we averaged out using the Large Market Limit. Indeed, Assumption 2 played a major role in the previous analysis.

In this Section we will move to a "truly" heterogeneous model in which the individual investment functions of different agents are structurally different. We still assume that these functions are based on EWMA estimation of future price dynamics. For the sake of simplicity, we confine our analysis on the case of 2 agents, but the results can be easily generalized for larger population.

Let us suppose that 2 agents operate on the market. At the beginning of time $t+1$ they consider previous market performances to compute: (i) the exponentially weighted moving averages of realized returns $y_{t,1}$ and $y_{t,2}$ according to (5.8), and (ii) the exponentially weighted moving averages of realized variance $z_{t,1}$ and $z_{t,2}$ according to (5.15). We allow different weights in agents' estimates, which we denote as λ_1 and λ_2, respectively. Agent i ($i \in \{1,2\}$) chooses his investment share $\pi_{t+1,i}$ according to

$$\pi_{t+1,i} = f_i(y_{t,i}, z_{t,i}) \quad , \tag{7.1}$$

where f_1 and f_2 are two arbitrary continuous functions with values in $[\pi_{\min}, \pi_{\max}] \subset (0,1)$. According to the Proposition 3.1, the resulting dynamics is guaranteed to generate positive prices and can be described using (3.11) and (3.12).

As discussed in Section 3, the system cannot possess any fixed point in terms of wealths and prices. One has, instead, to analyze the system in terms of price and portfolios returns. To this purpose it is convenient to write the 2 agents system using the relative agents wealths $\varphi_{t,i} = w_{t,i}/w_t$ ($i \in \{1,2\}$), where $w_t = w_{t,1} + w_{t,2}$ stands for the total wealth. In this way, the dynamics of the total wealth can be removed from the system to obtain a 7-dimensional system with the following state variables: the agents investment choices $\pi_{t,1}$ and $\pi_{t,2}$ defined in (7.1), the agents forecast about future return $y_{t,1}$ and $y_{t,2}$, the agents forecast about future variance $z_{t,1}$ and $z_{t,2}$, and, finally, the relative wealth of the first agent $\varphi_{t,1}$ (of course, $\varphi_{t,2} = 1 - \varphi_{t,1}$).

$$\begin{cases} \pi_{t+1,1} = f_1(y_{t,1}, z_{t,1}) \\ \pi_{t+1,2} = f_2(y_{t,2}, z_{t,2}) \\ y_{t+1,1} = \lambda_1\, y_{t,1} + (1-\lambda_1)\, r_{t+1} \\ y_{t+1,2} = \lambda_2\, y_{t,2} + (1-\lambda_2)\, r_{t+1} \\ z_{t+1,1} = \lambda_1\, z_{t,1} + \lambda_1\,(1-\lambda_1)\,\big(r_{t+1} - y_{t,1}\big)^2 \\ z_{t+1,2} = \lambda_2\, z_{t,2} + \lambda_2\,(1-\lambda_2)\,\big(r_{t+1} - y_{t,2}\big)^2 \\ \varphi_{t+1,1} = \dfrac{\varphi_{t,1}\,\left(\pi_{t,1}\,(r_{t+1}+\bar{e}) + 1\right)}{1 + \left(\pi_{t,2} + \varphi_{t,1}\,(\pi_{t,1} - \pi_{t,2})\right)\,(r_{t+1}+\bar{e})} \end{cases} \quad , \tag{7.2}$$

where the price return r_{t+1} is given by

$$r_{t+1} = \frac{\varphi_{t,1}\;\left(f_1(y_{t,1},z_{t,1})\,(1+\bar{e}\,\pi_{t,1})-\pi_{t,1}\right)+(1-\varphi_{t,1})\;\left(f_2(y_{t,2},z_{t,2})\,(1+\bar{e}\,\pi_{t,2})-\pi_{t,2}\right)}{\varphi_{t,1}\,\pi_{t,1}\;\left(1-f_1(y_{t,1},z_{t,1})\right)+(1-\varphi_{t,1})\,\pi_{t,2}\;\left(1-f_2(y_{t,2},z_{t,2})\right)} \quad . \tag{7.3}$$

The following result (which can be easily proven with a bit of algebra) characterizes the fixed points of (7.2)

Proposition 7.1. *Let* $(\pi_1^*, \pi_2^*, y_1^*, y_2^*, z_1^*, z_2^*, \varphi_1^*)$ *be a fixed point of the deterministic skeleton of system* (7.2)*, then it is*

$$r^* = y_1^* = y_2^* \qquad \textit{and} \qquad z_1^* = z_2^* = 0 \quad . \tag{7.4}$$

and the following three cases are possible

(i) Dominance of agent 1*. In this case* $\varphi_1^* = 1$ *and it is*

$$\pi_1^* = f_1\left(\bar{e}\,\frac{\pi_1^*}{1-\pi_1^*},\,0\right) \quad , \qquad r^* = \bar{e}\,\pi_1^*/(1-\pi_1^*) \quad , \qquad \pi_2^* = f_2\left(r^*,0\right) \quad . \tag{7.5}$$

(ii) Dominance of agent 2*. In this case* $\varphi_1^* = 0$ *and it is*

$$\pi_2^* = f_2\left(\bar{e}\,\frac{\pi_2^*}{1-\pi_2^*},\,0\right) \quad , \qquad r^* = \bar{e}\,\pi_2^*/(1-\pi_2^*) \quad , \qquad \pi_1^* = f_1\left(r^*,0\right) \quad . \tag{7.6}$$

(iii) Coexistence of the two agents. In this case it is $\varphi_1^* \in (0,1)$ *and the investment shares of both agents must coincide and be equal to* π^*

$$\pi^* = f_1\left(\bar{e}\,\frac{\pi^*}{1-\pi^*},0\right) = f_2\left(\bar{e}\,\frac{\pi^*}{1-\pi^*},0\right) \quad . \tag{7.7}$$

Moreover, $r^* = \bar{e}\,\pi^*/(1-\pi^*)$*.*

Notice that the situations described in the first two items are symmetric under the interchange of the two agents. Thus, this Propositions actually distinguishes between two different cases. The first case is where one of the agents dominates the other, so that he conquers the market and asymptotically possesses all the wealth. The second case is where both agents survive and coexist on the market (item *(iii)*).

Let us see how the previous Proposition can help in understanding the question of survival of different strategies (i.e. agents) on the market. Assume that, initially, only one agent, let say agent 1, operates on the market[7]. In this single agent situation, the analysis developed in the Section 5 provides the conditions for the existence of a fixed point. Assume that the system is initially in a fixed point so that the investment share π_1^* satisfies (7.5) and the return is defined as $r^* = \bar{e}\,\pi_1^*/(1-\pi_1^*)$.

Now, imagine that the second agent, agent 2, enters the market. According to Proposition 7.1 different things can happen. It is possible that the investment choice of the first agent π_1 does not change, the price return r^* remains the same and the second agent adapts, accordingly, his investment strategy. In this case, the second agent remains marginal, and the

[7]Of course, with more realism, one can think about the average agent of some cluster of "similar" agents, "similar" in the Large Market Limit sense.

first agent maintains his dominant position. On the contrary, it is possible that the second agent takes a preponderant position on the market, displacing the first agent who, being unable to lead the market, adapts his investment choice to the newcomer strategy and maintain a marginal position. This is the fixed point described in item *(ii)*. In principle, it is also possible that after the entry of agent 2, the two agents coexist on the market, and both retain a finite share of wealth (item *(iii)*). This third situation is, however, non-generic. Indeed, it requires the fulfillment of equality (7.7) by the two functions f_1 and f_2. In order to see it, let us introduce the auxiliary functions g_i ($i \in \{1, 2\}$) defined, in analogy with the single agent case, as $g_i(\pi) = f_i(\bar{e}\,\pi/(1-\pi), 0)$. The equality (7.7) imposes that these two functions crosses the main diagonal exactly at the same point, as in panel (a) of the Fig. 9. A minimal variation in one of the two functions leads to the impossibility of this equilibrium.

Among the different effects that the entry of a new agent can possibly generate, which is the one actually chosen by the market? To answer this question, one has, first of all, to perform the stability analysis of the different possible fixed points. The results of this analysis are collected in the following

Proposition 7.2. *Let* $\boldsymbol{x}^* = (\pi_1^*, \pi_2^*, r^*, r^*, 0, 0, \varphi_1^*)$ *be a fixed point of the system in Proposition 7.1. Then*

(i) If $\varphi_1^* = 1$ *the fixed point is (locally) asymptotically stable if*

$$\pi_2^* < \pi_1^* \tag{7.8}$$

and

$$\frac{f'_{1,y}}{\pi_1^*\,(1-\pi_1^*)} < \frac{1}{1-\lambda_1}\,, \qquad \frac{r^*\,f'_{1,y}}{\pi_1^*\,(1-\pi_1^*)} < 1 \quad \text{and} \quad \frac{(2+r^*)\,f'_{1,y}}{\pi_1^*\,(1-\pi_1^*)} > -\frac{1+\lambda_1}{1-\lambda_1}\,, \tag{7.9}$$

where $f'_{1,y}$ *stands for the partial derivative of the function* f_1 *with respect to the first variable.*

(ii) If $\varphi_1^* = 0$ *the fixed point is (locally) asymptotically stable if* (7.8) *and* (7.9) *hold with reversed indices* 1 *and* 2.

(iii) The fixed point with $\varphi_1^* \in (0, 1)$ *is non hyperbolic.*

Proof. See Appendix A.7. □

The stability conditions (7.9) coincide with the single-agent conditions (5.20) as if the dominant strategy would operate alone on the market. However, we have the important additional condition (7.8). Remember that the investment share of the second agent is given by $\pi_2^* = f_2(r^*, 0)$. Continuing our example, we can conclude that the entry of agent 2 in the market will not reduce the dominant position of agent 1 if the share of wealth that the entrant agent would invest at the equilibrium point of the incumbent is lower that the share of wealth invested by the latter. In other terms, in order to displace the incumbent from his dominant position, the entrant must possess a more aggressive strategy, according to which an higher amount of wealth is invested in the risky asset.

In general, we found that a two-agents equilibrium is stable if (i) the corresponding one-agent equilibrium (i.e. the equilibrium obtained when the dominant strategy operates alone in the market) is stable and (ii) the dominant strategy is the one characterized by the highest investment in the risky asset. Following Chiarella and He (2001) we call this result a *quasi-optimal selection principle*. Indeed, the market dynamics follows an *optimal selection principle* since condition (7.8) states that it is the strategy with the highest investment levels in the risky asset which survives and, according to Proposition 7.1, this implies that the market selects the strategy generating the highest price returns. Due to the further requirement of (7.9), however, this optimal strategy may be unable to generate a stable steady growth, from which the *quasi* specifier.

In their cited contributions, Chiarella and He analyzed different particular models, with different typologies of agents. Common to all cases was the presence of multiple equilibria and the fact that, asymptotically, one of the two agents possessed the total market wealth while the other was displaced out of the market. Now, it should be clear that the Proposition 7.2 is the general formulation of this observation. It also clarifies that the total number of fixed points is the sum of the number of fixed points when only the first agent is present on the market plus the number of fixed points when only the second is present.

On the Fig. 9 we show some possible cases which can occur, according to our quasi-optimal selection principle, for different couples of investment functions. On panel (b) the situation of a single stable equilibrium is shown. Here the first agent dominates the market. On panel (c) the system has two stable equilibria: one where the first agent dominates and one where the second agent dominates. The domains of attraction of these equilibria depends on the exact specification of the demand functions. Finally, on the panel (d) we present a case without any stable fixed point. Notice that this situation may happen even when the corresponding 1-agent models, when only one of the two agents is present on the market, have stable equilibria.

Another useful aspect of our generalized selection principle is its applicability to the case of agents with different "memories", i.e. with different values of the parameter λ. Our result allows to compare the relative relevance of the memory parameter λ and of the shape of the individual demand functions in deciding the stability of a given equilibrium. As in one-agent case, the location of equilibria does not depend on the value of λ. Again, conditions (7.9) implies that the stability domain increases when the value of λ_1 increases. When $\lambda_1 \to 1$ these conditions eventually hold for almost any equilibrium (except those where $r^* = 0$). However, the additional condition (7.8) is independent from the value of λ, but only depends on the form of the agents' investment functions. Therefore, the role of λ for the stability may not be as important as the role of the demand functions: if f_1 and f_2 are such that (7.8) is not satisfied, the equilibrium is unstable irrespectively of the value of λ.

This result suggests a simple explanation of the following suggestive statement, concerning the result of a simulation model, appeared in Zschischang and Lux (2001) (p.568):

> Looking more systematically at the interplay of risk aversion and memory span, it seems to us that the former is the more relevant factor, as with different [risk aversion coefficients] we frequently found a reversal in the dominance pattern: groups which were fading away before became dominant when we reduced their degree of risk aversion.

In the model by Zschischang and Lux agents have a limited memory span and forecast the next return as the average of past realized returns. In all other respects the agents' behavior is the same as in our framework. Since different lengths of the memory span can be approximated by different values of λ, the result described in the above quotation is exactly what expected according to our principle.

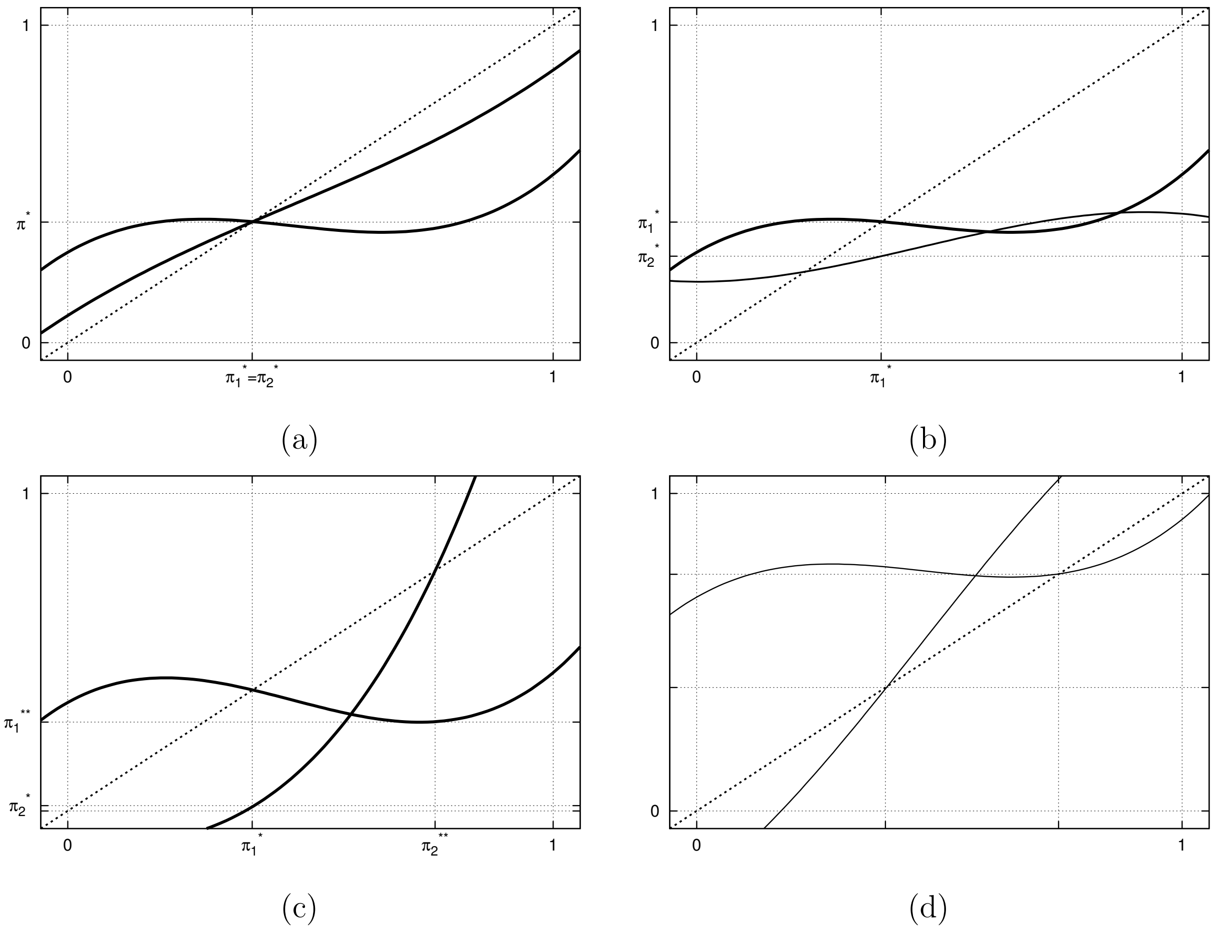


Figure 9: The equilibria for the dynamics with 2 agents in terms of the auxiliary functions g_1 and g_2. (a). Non-generic equilibrium with agents investing the same amount π^*. (b). One agent dominates another. The investment share of the first agent π_1^* is determined in a standard way, while the investment share of the second agent π_2^* is given as $g_2(\pi_1^*)$. (c). Two stable equilibria with investment shares (π_1^*, π_2^*) and (π_1^{**}, π_2^{**}). (d). Two unstable equilibria.

8 Conclusions

This paper extends previous contributions and presents novel results concerning the characterization and stability of equilibria in speculative pure exchange economies with heterogeneous traders. Our results also provide some analytical background to the growing literature about numerical simulations of artificial agent-based financial market.

Let us shortly review the assumption we made and our achievements in order to sketch the possible future lines of research. We considered a simple analytical framework using a minimal number of assumptions (2 assets and Walrasian price formation). We modeled agents as speculative traders and we imposed the sole constraint that their participation to the trading activity be described by an individual demand function proportional to the their wealth. We found that with prescribed but arbitrary specification about the agents' behavior, the feasible dynamics of the economy (i.e. the dynamics for which prices stay always positive) can be described as a multi-dimensional dynamical system.

For definiteness and in order to provide stricter characterization of the properties of the model, following a common approach in agent-based literature, we considered agents who form their individual demand function on the predictions about future price returns obtained from the publicly available past prices history.

Inside this framework, we analyzed models in which different agents possess demand functions that are random deviations from a common, underlying, behavior. We showed that, when the number of agents becomes large, the Large Market Limits can be applied to reduce the description of the economy to a low-dimensional stochastic system. This allowed us to interpret the Large Market Limit as the situation where the representative agent paradigm is valid. We thoroughly explored this situation, finding the local stability conditions for the market equilibria and showing the bifurcation types generated by the violation of these conditions. We also discussed the effect on equilibria stability of the agents "memory', i.e. of the length of the past market history that agents take in consideration to build they predictions about future prices.

Finally, we moved to the case of 2 heterogeneous agents and presented the fixed point stability analysis of this system. We showed that the conditions for the existence of fixed points and the conditions for their stability are very similar to the corresponding conditions in the situation with one single agent. Notwithstanding this similarity, however, we found that different scenarios are possible: in a stable fixed point, one can have a non-generic case where two traders coexist on the market and a generic case in which one of the agents dominates the other and ultimately capture the entire market. However, it can also be the case that two agents, whose strategies, when present alone on the market, lead to systems possessing stable fixed points, lead, when present together, to a system that does not possess any stable fixed point. Or, more interesting, they can lead to system with multiple stable fixed points. This last possibility implies that the final dominance of one strategy on the other depends on the market initial conditions. Ultimately, this is a proof of the impossibility (at least inside our framework) to build any dominance order relation on the space of trading strategies.

The analysis presented in this paper can be extended in many directions. First of all, inside our general framework, numerous specifications of the traders strategies are possible, in addition to the ones already considered. They range from the evaluation of the "fundamental" value of the asset, maybe obtained from a private source of information, to a strategic behavior that try to keep in consideration the reaction of other market participants to the revealed individual choices. The same can be said about the dividend yield process, that we assumed randomly and independently drawn from a stationary distribution. Actually, this assumption implies that the investors are not aware whether the price is growing because of some fundamental reasons, or because of a non-fundamental speculation-driven price bubble. It would be interesting to relax our assumptions on the yield process and consider agents who try to distinguish between these two different situations and behave correspondingly.

APPENDIX

A Proofs of Theorems, Propositions and Lemmas

A.1 Proof of Theorem 3.1

Plugging the expression for $w_{t,i}$ from the second equation in (3.7), into the right hand-side of the first equation and assuming $p_{t-1} > 0$ and, consistently with (3.10), $p_{t-1} \neq \sum \pi_{t,i}\, \pi_{t-1,i}\, w_{t-1,i}$ one gets:

$$p_t = \left(1 - \frac{1}{p_{t-1}} \sum \pi_{t,i}\, \pi_{t-1,i}\, w_{t-1,i}\right)^{-1} \left(\sum \pi_{t,i}\, w_{t-1,i} + (e_{t-1} - 1) \sum \pi_{t,i}\, w_{t-1,i}\, \pi_{t-1,i}\right) =$$

$$= p_{t-1} \frac{\sum \pi_{t,i}\, w_{t-1,i} + (e_{t-1} - 1) \sum \pi_{t,i}\, w_{t-1,i}\, \pi_{t-1,i}}{\sum \pi_{t-1,i}\, w_{t-1,i} - \sum \pi_{t,i}\, \pi_{t-1,i}\, w_{t-1,i}} =$$

$$= p_{t-1} \frac{\left\langle \pi_t \right\rangle_{t-1} - \left\langle \pi_{t-1}\, \pi_t \right\rangle_{t-1} + e_{t-1} \left\langle \pi_{t-1}\, \pi_t \right\rangle_{t-1}}{\left\langle \pi_{t-1} \right\rangle_{t-1} - \left\langle \pi_{t-1}\, \pi_t \right\rangle_{t-1}} \quad ,$$

where we used the first equation in (3.7) rewritten for time $t-1$ to get the second equality. Condition (3.10) is immediately obtained imposing $p_t > 0$. The price return and wealth return for the agent i at time t can now be derived straightforwardly.

A.2 Proof of Proposition 3.1

If (3.13) is valid for all the π's it is also valid for the wealth-weighted averages, i.e. we have: $\pi_{\min} \leq \left\langle \pi_t \right\rangle \leq \pi_{\max}$, $\quad \pi^2_{\min} \leq \left\langle \pi_t\, \pi_{t+1} \right\rangle \leq \pi^2_{\max}$ and $\pi_{\min}\, (1 - \pi_{\max}) \leq \left\langle \pi_t\, (1 - \pi_{t+1}) \right\rangle \leq \pi_{\max}\, (1 - \pi_{\min})$.

Since $e_t > 0$, both factors on the left hand side of (3.10) are positive and so the constraint is satisfied. At the same time, since the denominator of the expressions in (3.12) and (3.11) is strictly greater than zero, and numerators are bounded, these two expressions remain bounded.

A.3 Proof of Lemma 4.1

As we know $\rho_{t+1,i} = \pi_{t,i}\, (r_{t+1} + e_t)$, so that:

$$\begin{aligned}
\left\langle \pi_{t+1} \right\rangle_{t+1} &= \sum_i \pi_{t+1,i} \frac{w_{t+1,i}}{\sum_j w_{t+1,j}} = \sum_i \pi_{t+1,i} \frac{w_{t,i}\, (1 + \rho_{t+1,i})}{\sum_j w_{t,j}\, (1 + \rho_{t+1,j})} = \\
&= \frac{\sum_j \pi_{t+1,j}\, w_{t,j}\, (1 + \pi_{t,j}\, (r_{t+1} + e_t))}{\sum_j w_{t,j}\, (1 + \pi_{t,j}\, (r_{t+1} + e_t))} = \\
&= \frac{\sum_j \pi_{t+1,j}\, w_{t,j} + (r_{t+1} + e_t) \sum_j \pi_{t+1,j}\, \pi_{t,j}\, w_{t,j}}{\sum_j w_{t,j} + (r_{t+1} + e_t) \sum_j \pi_{t,j} w_{t,j}}
\end{aligned}$$

Dividing both numerator and denominator in the last expression on $\sum_j w_{t,j}$, one get

$$\left\langle \pi_{t+1} \right\rangle_{t+1} = \frac{\left\langle \pi_{t+1} \right\rangle_t + (r_{t+1} + e_t) \left\langle \pi_{t+1}\, \pi_t \right\rangle_t}{1 + (r_{t+1} + e_t) \left\langle \pi_t \right\rangle_t} \quad .$$

Thus

$$\left\langle \pi_{t+1} \right\rangle_{t+1} - \left\langle \pi_{t+1} \right\rangle_t = (r_{t+1} + e_t) \left(\left\langle \pi_{t+1}\, \pi_t \right\rangle_t - \left\langle \pi_{t+1} \right\rangle_{t+1} \left\langle \pi_t \right\rangle_t \right) \quad .$$

From this equality the statement immediately follows.

A.4 Proof of Proposition 5.1

It is straightforward to get (5.3) and the statement of the item *(i)*, as soon as the conditions $\pi_t = \pi_{t+1} = \pi^*$ and $r_t = r_{t+1} = r^*$ are imposed in the system. To prove the second item we use the general results refreshed in the Appendix B. The Jacobian matrix J of the system in a fixed point reads:

$$J = \begin{bmatrix} 0 & f' \\ -1/(\pi^*(1-\pi^*)) & (1+r^*)\, f'/(\pi^*(1-\pi^*)) \end{bmatrix} \quad .$$

Thus, the trace is $\mathrm{Tr}(J) = (1+r^*)\, f'/(\pi^*(1-\pi^*))$ and the determinant is $\mathrm{Det}(J) = f'/(\pi^*(1-\pi^*))$. Applying now the result of the Theorem B.1 we get the conditions (5.5).

A.5 Proof of Proposition 5.3

The items *(i)* and *(ii)* are obvious. The Jacobian matrix J of the system in a fixed point reads:

$$J = \begin{bmatrix} 0 & f' \\ -(1-\lambda)/(\pi^*(1-\pi^*)) & \lambda + (1-\lambda)(1+y^*)\, f'/(\pi^*(1-\pi^*)) \end{bmatrix} \quad . \tag{A.1}$$

Thus, the trace is $\mathrm{Tr}(J) = \lambda + (1-\lambda)(1+y^*)\, f'/(\pi^*(1-\pi^*))$ and the determinant is $\mathrm{Det}(J) = (1-\lambda)\, f'/(\pi^*(1-\pi^*))$. Applying now the result of Theorem B.1 we get the conditions (5.13).

A.6 Proof of Proposition 5.5

The items *(i)* and *(ii)* are obvious. Let $x = \pi^*(1-\pi^*)$. The Jacobian matrix J in a fixed point reads:

$$J = \begin{bmatrix} 0 & f'_y & f'_z \\ -(1-\lambda)/x & \lambda + (1-\lambda)(1+y^*)\, f'_y/x & (1-\lambda)(1+y^*)\, f'_z/x \\ 0 & 0 & \lambda \end{bmatrix} \quad ,$$

where f'_z is the partial derivative of the function f with respect to z. Thus, one of the eigenvalues is, obviously, $\lambda < 1$, while the others are the eigenvalues of the matrix (A.1). The only change is that f' becomes now f'_y in our notation. Then conditions (5.20) immediately follow.

A.7 Proof of Proposition 7.2

We start with writing the Jacobian J of the system (7.2). The dimension of this matrix is 7×7 and the order of variables will be the following: $\pi_1, \pi_2, y_1, y_2, z_1, z_2, \varphi_1$. We should compute the Jacobian in a fixed point characterized in the item *(i)* of this Proposition. The Jacobian reads:

$$
J = \left[\begin{array}{cccc} 0 & 0 & f'_{1\,y} & \dots \\ 0 & 0 & 0 & \dots \\ (1-\lambda_1)\, r'_{\pi_1} & (1-\lambda_1)\, r'_{\pi_2} & \lambda_1 + (1-\lambda_1)\, r'_{y_1} & \dots \\ (1-\lambda_2)\, r'_{\pi_1} & (1-\lambda_2)\, r'_{\pi_2} & (1-\lambda_2)\, r'_{y_1} & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \varphi'_{\pi_1} & \varphi'_{\pi_2} & \varphi'_{y_1} & \dots \end{array} \right. \tag{A.2}
$$

$$
\left. \begin{array}{ccccc} \dots & 0 & f'_{1\,z} & 0 & 0 \\ \dots & f'_{2\,y} & 0 & f'_{2\,z} & 0 \\ \dots & (1-\lambda_1)\, r'_{y_2} & (1-\lambda_1)\, r'_{z_1} & (1-\lambda_1)\, r'_{z_2} & (1-\lambda_1)\, r'_{\varphi} \\ \dots & \lambda_2 + (1-\lambda_2)\, r'_{y_2} & (1-\lambda_2)\, r'_{z_1} & (1-\lambda_2)\, r'_{z_2} & (1-\lambda_2)\, r'_{\varphi} \\ \dots & 0 & \lambda_1 & 0 & 0 \\ \dots & 0 & 0 & \lambda_2 & 0 \\ \dots & \varphi'_{y_2} & \varphi'_{z_1} & \varphi'_{z_2} & \varphi'_{\varphi} \end{array} \right] \quad ,
$$

where for simplicity we use the variable φ instead of φ_1. In above expression the notation like $f'_{i\,y}$ and $f'_{i\,z}$ stand for the partial derivatives of the function $f_i\,(i = 1, 2)$ with respect to $y_{t,i}$ and $z_{t,i}$, correspondingly. Also we denote as r'_{π_1} (φ'_{π_1}) the partial derivative of r_{t+1} (φ_{t+1}) with respect to $\pi_{t,1}$. Analogously other notation is defined. The expression for r_{t+1} is given in (7.3), whereas the expression for φ_{t+1} is in the last row of the system (7.2). Notice also that a lot of "zeros" in the 5th and 6th rows are due to the squared term in (7.2) in the expressions for $z_{t+1,1}$ and $z_{t+1,2}$, correspondingly. The derivative of that term with respect to any variable will contain the factor $r_{t+1} - y_{t,i}$ which is equal to 0 in a fixed point.

In order to proceed further we have to evaluate some of the partial derivatives remaining in J in the fixed point. First of all, notice that

$$
r'_{\varphi} = \frac{\partial r_{t+1}}{\partial \varphi_{t,1}} = \frac{\bar{e}\, \pi^*_1\, \pi^*_2\, (\pi^*_1 - \pi^*_2)}{\pi^*_2\,(1-\pi^*_2) + \varphi^*_1\,(\pi^*_1\,(1-\pi^*_1) - \pi^*_2\,(1-\pi^*_2))^2} \quad ,
$$

which is equal to 0 in the equilibria with $\varphi_1 \in (0,1)$. On the other hand, if $\varphi_1 = 1$, then it is easy to see that

$$
\varphi'_{\pi_1} = \frac{\partial \varphi_{t+1}}{\partial \pi_{t,1}} = 0 \qquad \text{and} \qquad \varphi'_{\pi_2} = \frac{\partial \varphi_{t+1}}{\partial \pi_{t,2}} = 0 \quad .
$$

Moreover, using the chain rule and the fact that

$$
\frac{\partial \varphi_{t+1}}{\partial r_{t+1}} = 0 \quad , \tag{A.3}
$$

one can easily check that also

$$
\varphi'_{y_1} = \frac{\partial \varphi_{t+1}}{\partial y_{t,1}} = 0\,, \quad \varphi'_{y_2} = \frac{\partial \varphi_{t+1}}{\partial y_{t,2}} = 0\,, \quad \varphi'_{z_1} = \frac{\partial \varphi_{t+1}}{\partial z_{t,1}} = 0 \quad \text{and} \quad \varphi'_{z_2} = \frac{\partial \varphi_{t+1}}{\partial z_{t,2}} = 0 \quad .
$$

Thus, when $\varphi_1 \in (0,1)$ then the Jacobian has all zeros in the last column except φ'_{φ}. When $\varphi_1 = 1$ then the Jacobian has all zeros in the last raw except φ'_{φ}. In both cases, therefore the Jacobian has an eigenvalue

$$
\varphi'_{\varphi} = \frac{\partial \varphi_{t+1}}{\partial \varphi_{t,1}} = \frac{1 + \pi^*_2\,(r^* + \bar{e})}{1 + \pi^*_1\,(r^* + \bar{e})} \quad . \tag{A.4}
$$

This eigenvalue is inside of the unit circle only if $\pi_1^* > \pi_2^*$ which proves the relation (7.8). Moreover, if $\pi_1^* = \pi_2^*$ the eigenvalue is equal to 1 and the item *(ii)* is proven. Also it is clear that there are two eigenvalues $\mu_5 = \lambda_1$ and $\mu_6 = \lambda_2$ inside the unit circle. Now, in the fixed point where $\varphi_1^* = 1$ we have

$$r'_{\pi_2} = \frac{\partial r_{t+1}}{\partial \pi_{t,2}} = 0 \qquad \text{and} \qquad r'_{y_2} = \frac{\partial r_{t+1}}{\partial y_{t,2}} = 0 \quad ,$$

so that the Jacobian has eigenvalues $\mu_4 = \lambda_2$ and $\mu_2 = 0$, and the remaining are the eigenvalues of the matrix

$$J = \begin{bmatrix} 0 & f'_{1\,y} \\ (1-\lambda_1)\,r'_{\pi_1} & \lambda_1 + (1-\lambda_1)\,r'_{y_1} \end{bmatrix} \quad .$$

This matrix coincides with the matrix (A.1), since

$$r'_{\pi_1} = \frac{\partial r_{t+1}}{\partial \pi_{t,1}} = \frac{\pi_1^*\,\bar{e} - 1 - r^*\,(1-\pi_1^*)}{\pi_1^*\,(1-\pi_1^*)} = -\frac{1}{\pi_1^*\,(1-\pi_1^*)} \quad ,$$

$$r'_{y_1} = \frac{\partial r_{t+1}}{\partial y_{t,1}} = \frac{f'_{1\,y}}{\pi_1^*\,(1-\pi_1^*)}\left(1 + \bar{e}\,\frac{\pi_1^*}{1-\pi_1^*}\right) = \frac{f'_{1\,y}\,(1+y_1^*)}{\pi_1^*\,(1-\pi_1^*)} \quad .$$

Thus the conditions (7.9) are also proven.

B Stability Conditions and Bifurcation Types for 2×2 Matrices

Here we present some classical results from the theory of difference equation. For the extensive theory of the stability and bifurcation analysis of the autonomous dynamical systems see e.g Guckenheimer and Holmes (1983) or Kuznetsov (1995). Consider a general 2-dimensional non-linear dynamical system

$$\begin{cases} x_{1,t+1} = f_1(x_{1,t}, x_{2,t}) \\ x_{2,t+1} = f_2(x_{1,t}, x_{2,t}) \end{cases} \quad , \tag{B.1}$$

and suppose that $(x_1^\star, x_2^\star)$ is a fixed point of the system. Moreover, let $J(x_1^\star, x_2^\star)$ denote the Jacobian matrix of the system (B.1) computed in this fixed point:

$$J(x_1^\star, x_2^\star) = \begin{Vmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{Vmatrix} \quad . \tag{B.2}$$

Let $t = J_{1,1} + J_{2,2}$ and $d = J_{1,1}\,J_{2,2} - J_{1,2}\,J_{2,1}$ be, respectively, the trace and the determinant of matrix (B.2). Then the following result takes place:

Theorem B.1. ***Sufficient conditions for the local stability.***
The fixed point $(x_1^\star, x_2^\star)$ *of the system* (B.1) *is locally asymptotically stable if the following conditions are satisfied*

$$\begin{aligned} &-1-d < t < 1+d \quad , \\ &d < 1 \quad . \end{aligned} \tag{B.3}$$

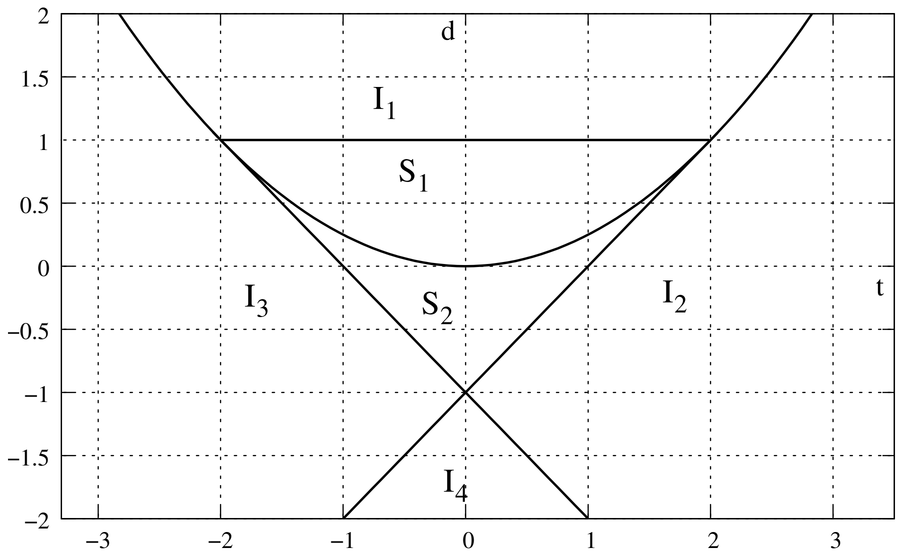


Figure 10: Stability region for a fixed point of a 2d dynamical system in Trace-Determinant coordinates.

Proof. The characteristic polynomial computed in a fixed point in our notations reads

$$\lambda^2 - t\,\lambda + d = 0$$

Thus, the stability analysis reduces to the analysis of the region in the (t, d) parameters space, where the absolute values of both roots

$$\lambda_{\pm} = \frac{t \pm \sqrt{t^2 - 4\,d}}{2} \tag{B.4}$$

are less than 1. There are two cases.

If $t^2 \geq 4\,d$ there are two real roots $\lambda_- \leq \lambda_+$ (coinciding when $t^2 = 4\,d$) and for the local stability we need that both $\lambda_+ < 1$ and also $\lambda_- > -1$. It is straightforward to see that it leads to the following conditions on trace and determinant

$$t < 2 \qquad \text{and} \qquad 1 - t + d > 0 \tag{B.5}$$

$$t > -2 \qquad \text{and} \qquad 1 + t + d > 0 \tag{B.6}$$

The area S_1 on Fig. B contains those points where these conditions are satisfied and both roots are real.

Instead, if $t^2 < 4\,d$ there are two complex conjugate roots and the stability condition reads

$$|\lambda| = \sqrt{\lambda_+ \lambda_-} = \sqrt{d} < 1 \tag{B.7}$$

Corresponding region is labeled S_2 on Fig. B. Now, all points corresponding to stable fixed point lie in the triangle shaped by the union of S_1 and S_2. This triangle is described by the conditions (B.3). □

Notice that if the fixed point is stable there are two possible qualitative behavior of the system in a neighborhood of it. If eigenvalues of Jacobian are not real, then the local behavior of the system is characterized by the superposition of the rotation and exponential convergence towards the fixed point, so that the resulting time series of both state variables have cyclical component. Such situation is referred as a *focus*. If the eigenvalues of Jacobian are real, then we have a *node*.

According to the Theorem B.1, if all 3 inequalities in (B.3) hold, the fixed point is stable. The fixed point is locally unstable if at least one of that inequalities has an opposite sign. The situation in which with the change of one or more parameters the fixed point loses its stability is called a *bifurcation*. The following theorem summarize the information about types of bifurcations.

Theorem B.2. *The fixed point of the system* (B.1) *looses its stability when one of the inequalities in conditions* (B.3) *is changing its sign. Moreover, the system has*

(i) fold bifurcation, if $t = 1 + d$;

(ii) flip bifurcation, if $t = -1 - d$;

(iii) Neimark-Sacker (secondary Hopf) bifurcation, if $d = 1$.

Proof. From the proof of the previous theorem it is clear that when the parameters cross the line separating region S_1 from I_1 at Fig. B (when $d = 1$), the modulus of the complex eigenvalues become larger than 1. Thus the system undertakes a Neimark bifurcation. When the line between S_2 and I_2 is crossed (and so $t = 1 + d$) the largest real eigenvalue becomes greater than 1 a fold bifurcation is observed. Finally, if the line between S_2 and I_3 is crossed (i.e. $t = -1 - d$) the smallest real eigenvalue becomes less than -1, and the flip bifurcation is observed. □

References

Arthur, W.B., J.H. Holland, B. LeBaron, R. Palmer, P. Tayler, (1997). "Asset Pricing under Endogenous Expectations in an Artificial Stock Market", in Arthur W.B., S.N. Durlauf and D.A. Lane (eds.) *The Economy as an Evolving Complex System II*, Addison-Wesley.

Beja, A. and M.B. Goldman, (1979). "Market prices vs. equilibrium prices: return variance, serial correlation, and the role of the specialist", *Journal of Finance*, 34, 595-607.

Beja, A. and M.B. Goldman, (1980). "On the Dynamic Behaviour of Prices in Disequilibrium", *Journal of Finance*, 35, 235-248.

Bottazzi, G. (2002). "A Simple Micro-Model of Market Dynamics. Part I: the "Large Market" Deterministic Limit" *L.E.M. Working Paper*, 2002-10, Sant'Anna School for Advanced Studies, Pisa.

Bottazzi, G., G. Dosi and I. Rebesco, (2004). "Institutional Achitectures and Behavioural Ecologies in the Dynamics of Financial Markets: a Preliminary Investigation", *Journal of Mathematical Economics*, forthcoming.

Brock, W.A. and C.H. Hommes, (1998). "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model", *Journal of Economic Dynamics and Control* 22, 1235-1274

Brock, W.A., C.H. Hommes and F.O.O. Wagener, (2004). "Evolutionary Dynamics in Markets with Many Trader Types", *Journal of Mathematical Economics*, forthcoming.

Chiarella, C. (1992). "The Dynamics of Speculative Behaviour", *Annals of Operational Research*, 37, 101-123.

Chiarella, C., R. Dieci and L. Gardini, (2002). "Speculative behaviour and complex asset price dynamics: a global analysis", *Journal of Economic Dynamics and Control* 49, 173-197.

Chiarella, C. and X. He, (2001). "Asset Price and Wealth Dynamics under Heterogeneous Expectations", *Quantitative Finance*, 1, 509-526.

Chiarella, C. and T. He, (2002a). "Heterogeneous Beliefs, Risk and Learning in a Simple Asset Pricing Model", *Computational Economics*, 19 (1), 95-132.

Chiarella, C. and T. He, (2002b). "An Adaptive Model on Asset Pricing and Wealth Dynamics with Heterogeneous Trading Strategies", Quantitative Finance Research Centre, University of Technology, Sydney, Working Paper Number: 84.

Chiarella, C. and T. He, (2003). "Heterogeneous Beliefs, Risk and Learning in a Simple Asset-Pricing Model with a Market Maker", *Macroeconomic Dynamics*, 7(4), pp. 503 - 536.

Chiarella, C. and G. Iori, (2002). "A simulation analysis of the microstructure of double auction markets", *Quantitative Finance*, 2, 346-353.

Farmer, J.D. (2002). "Market force, ecology, and evolution", *Industrial and Corporate Change*, 11, 895-953.

Farmer, J.D. and S. Joshi (2002). "The price dynamics of common trading strategies", *Journal of Economic Behaviour and Organization*, 49, 149-171.

Gaunersdorfer, A. (2000). "Endogenous Fluctuations in a Simple Asset Pricing Model with Heterogeneous Agents", *Journal of Economic Dynamics and Control*, 24, 799-831

Guckenheimer, J. and P. Holmes (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer, New York.

Kirman, A. (1993). "Ants, rationality and recruitment", *Quarterly Journal of Economics*, 108, 137-156.

Kuznetsov, Y.A., (1995). *Elements of Applied Bifurcation Theory.* Springer, New York.

LeBaron B., W.B. Arthur and R.Palmer (1999). "Time series properties of an artificial stock market", *Journal of Economic Dynamics and Control*, 23, 1487-1516.

Levy M., H. Levy and S. Solomon (1994). "A Microscopic Model of the Stock Market: Cycles, Booms, and Crashes", *Economics Letters* 45, 1, 103-111.

Levy M., H. Levy and S. Solomon (1995). "Microscopic simulation of the stock market: the effect of microscopic diversity", *Journal of Physica I (France)* 5, 1087-1107.

Levy M. and H. Levy (1996). "The danger of assuming homogeneous expectations" *Financial Analysts Journal*, May/June, 65-70.

Levy M., H. Levy and S. Solomon, (2000). *Microscopic Simulation of Financial Markets*, Academic Press, London.

Lux T., (1995). "Herd Behaviour, Bubbles and Crashes", *The Economic Journal*, 105, no. 431, 881-896.

Lux T., (1997). "Time variation of second moments from a noise trader/infection model", *Journal of Economic Dynamics and Control*, 22, 1-38.

Lux T., (1998). "The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distribution", *Journal of Economic Behaviour and Organization*, 33, 143-165

Lux, T. and M. Marchesi, (1999). "Scaling and criticality in a stochastic multi-agent model of financial market", *Nature*, vol. 397, 498-500.

Lux, T. and M. Marchesi, (2000). "Volatility clustering in financial markets: a microsimulation of interacting agents", *International Journal of Theoretical and Applied Finance*, 3, 675-702.

Mossin J., (1968). "Optimal Multiperiod Portfolio Policies", *Journal of Business*, v.41, n.2, 215-229.

Zschischang, E. and T. Lux, "Some new results on the Levy, Levy and Solomon microscopic stock market model", *Physica A*, v.291, 563-573.