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The homotopy type of toric arrangements

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The homotopy type of toric arrangements

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Abstract

A toric arrangement is a finite set of hypersurfaces in a complex torus, every hypersurface being the kernel of a character. In the present paper we build a CW-complex $\mathcal{S}$ homotopy equivalent to the arrangement complement $\mathcal{R}_X$, with a combinatorial description similar to that of the well-known Salvetti complex. If the toric arrangement is defined by a Weyl group, we also provide an algebraic description, very handy for cohomology computations. In the last part we give a description in terms of tableaux for a toric arrangement of type $\widetilde{A}_n$ appearing in robotics.

Keywords:
Arrangement of hyperplanes, toric arrangements, CW complexes, Salvetti complex, Weyl groups, integer cohomology, Young Tableaux

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52C35, 32S22, 20F36,17B10

Introduction

A toric arrangement is a finite set of hypersurfaces in a complex torus $T = (\mathbb{C}^*)^n$, in which every hypersurface is the kernel of a character $\chi \in X \subset Hom(T, \mathbb{C}^*)$ of T.

Let $\mathcal{R}_X$ be the complement of the arrangement: its geometry and topology have been studied by many authors, see for instance [8], [9], [4], [12]. In particular, in [10] and [3] the De Rham cohomology of $\mathcal{R}_X$ has been computed, and recently in [13] a wonderful model has been built.

In the present paper we build a topological model $\mathcal{S}$ for $\mathcal{R}_X$. This model is a regular CW-complex, similar to the one introduced by Salvetti ([14]) for the complement of hyperplane arrangements.

Moreover for a wide class of arrangements, which we call *thick*, its cells are given by couples $[C \prec F]$, where C is a *chamber* of the *real* toric arrangement and F is a *facet* adjacent to it (according to the definitions given in Section 2).

The model $\mathcal{S}$ is well suited for homology and homotopy computations, which we hope to develop in future papers. Furthermore, the jumping loci in the

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local system cohomology of a CW-complex are affine algebraic varieties. In the theory of hyperplane arrangements such objects, called *characteristic varieties*, proved to be of fundamental importance. It is then a remarkable fact that the characteristic varieties can be defined also in the toric case.

In Section 3 we focus on the toric arrangement associated to an affine Weyl group $\widetilde{W}$. In this case the chambers are in bijection with the elements of the corresponding finite Weyl group W, and the cells of $\mathcal{S}$ are given by the couples $E(w,\Gamma)$, where $w \in W$ and Γ is a proper subset of the set S of generators of $\widetilde{W}$. This generalizes a construction introduced in [15] and [6].

In the last Section we give a description of the facets of the real toric arrangement defined by the Weyl group $\widetilde{A}_n$ in the torus corresponding to the root lattice. This description in terms of Young tableaux turns out to be interesting since it coincides with the complex describing the space of all periodic legged gaits of a robot body (see [2]).

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1 The CW-complex

1.1 Main definitions

Let $T = (\mathbb{C}^*)^n$ be a complex torus and $X \subset Hom(T, \mathbb{C}^*)$ be a finite set of characters of T. The kernel of every $\chi \in X$ is a hypersurface of T:

$$H_\chi := \{t \in T \mid \chi(t) = 1\}.$$

Then X defines on T the *toric arrangement*:

$$\mathcal{T}_X := \{H_\chi, \chi \in X\}.$$

Let $\mathcal{R}_X$ be the *complement* of the arrangement:

$$\mathcal{R}_X := T \setminus \bigcup_{\chi \in X} H_\chi$$

Let $\pi : V \longrightarrow T$ be the universal covering of T. Then V is a complex vector space of rank n, and π is the quotient map $\pi : V \longrightarrow V/\Lambda$, where Λ is a lattice in V. Then the preimage $\pi^{-1}(H_\chi)$ of a hypersurface $H_\chi \in \mathcal{T}_X$ is an infinite family of parallel hyperplanes. Thus

$$\mathcal{A}_X := \{\pi^{-1}(H_\chi), \quad \chi \in X\}$$

is a periodic affine hyperplane arrangement in V. Let $\mathcal{M}_X$ be its complement:

$$\mathcal{M}_X := V \setminus \bigcup_{\chi \in X} \pi^{-1}(H_\chi).$$

By definition, π maps $\mathcal{M}_X$ on $\mathcal{R}_X$. Moreover the equations defining the hyperplanes in $\mathcal{A}_X$ can always be assumed to have integral (hence real) coefficients since they are given by elements of Λ. Thus by [14] there is an (infinite) CW-complex $\widetilde{\mathcal{S}} \subset \mathcal{M}_X$ and a map $\varphi : \mathcal{M}_X \longrightarrow \widetilde{\mathcal{S}}$ giving a homotopic equivalence. Furthermore, we can build $\widetilde{\mathcal{S}}$ in such a way that it is invariant under the action of translation in Λ, for instance by building the cells relative to a fundamental domain, and then defining the others by translation. Thus $\pi(\widetilde{\mathcal{S}})$ is a finite CW-complex, which will be denoted by $\mathcal{S}$, and the image of every cell of $\widetilde{\mathcal{S}}$ is a cell of $\mathcal{S}$. Moreover, since φ is $\Lambda-$equivariant, it is well defined the map

$$\varphi_\pi(t) := (\pi \circ \varphi)(\pi^{-1}(t))$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{M}_X & \stackrel{\varphi}{\rightarrow} & \widetilde{\mathcal{S}} \\ \pi \downarrow & & \pi \downarrow \\ \mathcal{R}_X & \stackrel{\varphi_\pi}{\longrightarrow} & \mathcal{S} \end{array} \tag{1}$$

Lemma 1.1 *The map φ_π is a homotopy equivalence between $\mathcal{R}_X$ and $\mathcal{S}$.*

Proof. The map φ is a homotopy equivalence hence, by definition, there is a continuous map $\psi : \widetilde{\mathcal{S}} \to \mathcal{M}_X$ such that $\psi\varphi$ is homotopic to the identity map $id_{\mathcal{M}_X}$ and $\varphi\psi$ is homotopic to $id_{\widetilde{\mathcal{S}}}$. Namely, since $\widetilde{\mathcal{S}}$ is a deformation retract, the homotopy inverse ψ is simply the inclusion map, which is clearly $\Lambda-$equivariant. Hence the map

$$\psi_\pi(t) := (\pi \circ \psi)(\pi^{-1}(t))$$

is well defined and makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{\mathcal{S}} & \stackrel{\psi}{\rightarrow} & \mathcal{M}_X \\ \pi \downarrow & & \pi \downarrow \\ \mathcal{S} & \stackrel{\psi_\pi}{\longrightarrow} & \mathcal{R}_X. \end{array} \tag{2}$$

Let $I = [0, 1]$ be the unit interval and $F : \mathcal{M}_X \times I \to \mathcal{M}_X$ be the continuous map such that $F(x, 0) = \psi(\varphi(x))$ and $F(x, 1) = id_{M_X}(x)$. Again, since F is $\Lambda-$equivariant, we can define the map:

$$F_\pi(t) := (\pi \circ F)(\pi^{-1}(t))$$

In this way we get the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_X \times I & \stackrel{F}{\rightarrow} & \mathcal{M}_X \\ \pi \downarrow & & \pi \downarrow \\ \mathcal{R}_X \times I & \stackrel{F_\pi}{\longrightarrow} & \mathcal{R}_X. \end{array} \tag{3}$$

By construction map F_π is a continuous map such that $F_\pi(x, 1) = id_{\mathcal{R}_X}$ and

$$F_\pi(x, 0) = (\psi\varphi)_\pi(x) = \pi\psi\varphi\pi^{-1}(x) = \pi\psi\pi^{-1}\pi\varphi\pi^{-1}(x) = \psi_\pi \circ \varphi_\pi(x).$$

Hence F_π gives the required homotopy equivalence. $\square$

1.2 Salvetti complex for affine arrangements

In order to describe the structure of $\mathcal{S}$, we now have to focus on the real counterparts of the complex arrangements above.
Let $V_{\mathbb{R}}$ be the real part of V. In other words, let $V_{\mathbb{R}} \doteq \mathbb{R}^n$ be a real vector space, and let $V \doteq V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Then we identify $V_{\mathbb{R}}$ with a subspace of V via the map $v \mapsto v \otimes 1$.
Let $\mathcal{A}_{X,\mathbb{R}}$ be the corresponding hyperplane arrangement on $V_{\mathbb{R}}$ and $\mathcal{M}_{X,\mathbb{R}} = \mathcal{M}_X \cap V_{\mathbb{R}}$ its complement. Since the image of $\mathbb{R}$ under the map $\mathbb{C} \longrightarrow \mathbb{C}/\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^*$ is the circle

$$S^1 := \{z \in \mathbb{C} \mid \mid z \mid = 1\}$$

we have that the image of $V_{\mathbb{R}}$ under the map $\pi : V \to V/\Lambda \xrightarrow{\sim} T$ is a compact torus $T_{\mathbb{R}} \subset T$. A *real toric arrangement* $\mathcal{T}_{X,\mathbb{R}}$ is naturally defined on $T_{\mathbb{R}}$ with hypersurfaces $H_{\chi,\mathbb{R}} := H_\chi \cap T_{\mathbb{R}}$ and complement $\mathcal{R}_{X,\mathbb{R}} = \mathcal{R}_X \cap T_{\mathbb{R}}$. Furthermore π restricts to universal covering map $\pi : V_{\mathbb{R}} \longrightarrow T_{\mathbb{R}}$ and $\pi(\mathcal{M}_{X,\mathbb{R}}) = \mathcal{R}_{X,\mathbb{R}}$.

We recall the following definitions:

1. a *chamber* of $\mathcal{A}_{X,\mathbb{R}}$ is a connected component of $\mathcal{M}_{X,\mathbb{R}}$;
2. a *space* of $\mathcal{A}_{X,\mathbb{R}}$ is an intersection of elements in $\mathcal{A}_{X,\mathbb{R}}$;
3. a *facet* of $\mathcal{A}_{X,\mathbb{R}}$ is the intersection of a space and the closure of a chamber.

Let $\mathbf{S} := \{\widetilde{F}^k\}$ be the stratification of $V_{\mathbb{R}}$ into facets $\widetilde{F}^k$ induced by the arrangement $\mathcal{A}_{X,\mathbb{R}}$, where superscript k stands for codimension.

Then the k-cells of $\widetilde{\mathcal{S}}$ bijectively correspond to pairs

$$[\widetilde{C} \prec \widetilde{F}^k]$$

where $\widetilde{C} = \widetilde{F}^0$ is a chamber of $\mathbf{S}$ and $\widetilde{F}^i \prec \widetilde{F}^j \Leftrightarrow clos(\widetilde{F}^i) \supset \widetilde{F}^j$ is the standard partial ordering in $\mathbf{S}$.

Let $|\widetilde{F}|$ be the affine subspace spanned by $\widetilde{F}$, and let us consider the subarrangement

$$\mathcal{A}_{\widetilde{F}} = \{H \in \mathcal{A}_{X,\mathbb{R}} \ : \ \widetilde{F} \subset H\}.$$

A cell $[\widetilde{C} \prec \widetilde{F}^k]$ is in the boundary of $[\widetilde{D} \prec \widetilde{G}^j]$ $(k < j)$ if and only if

$$\begin{aligned} &\text{i) } \widetilde{F}^k \prec \widetilde{G}^j \\ &\text{ii) the chambers } \widetilde{C} \text{ and } \widetilde{D} \text{ are contained in the same chamber of } \mathcal{A}_{\widetilde{F}^k}. \end{aligned} \tag{4}$$

Previous conditions are equivalent to say that $\widetilde{C}$ is the chamber of $\mathcal{A}_{X,\mathbb{R}}$ which is the "closest" to $\widetilde{D}$ among those which contain $\widetilde{F}^k$ in their closure. The standard notation $[\widetilde{C} \prec \widetilde{F}^k] \in \partial_{\widetilde{\mathcal{S}}}[\widetilde{D} \prec \widetilde{G}^j]$ will be used.

1.3 Salvetti Complex for toric arrangements

In order to give a similar description for $\mathcal{S}$, we introduce the following definitions:

1. a *chamber* of $\mathcal{T}_{X,\mathbb{R}}$ is a connected component of $R_{X,\mathbb{R}}$;

2. a *layer* of $\mathcal{T}_{X,\mathbb{R}}$ is a connected component of an intersection of elements of $\mathcal{T}_{X,\mathbb{R}}$;

3. a *facet* of $\mathcal{T}_{X,\mathbb{R}}$ is an intersection of a layer and the closure of a chamber.

Lemma 1.2

1. If $\widetilde{C}$ is a chamber of $\mathcal{A}_{X,\mathbb{R}}$, $\pi(\widetilde{C})$ is a chamber of $\mathcal{T}_{X,\mathbb{R}}$;

2. If $\widetilde{L}$ is a space of $\mathcal{A}_{X,\mathbb{R}}$, $\pi(\widetilde{L})$ is a layer of $\mathcal{T}_{X,\mathbb{R}}$;

3. If $\widetilde{F}$ is a facet of $\mathcal{A}_{X,\mathbb{R}}$, $\pi(\widetilde{F})$ is a facet of $\mathcal{T}_{X,\mathbb{R}}$;

Proof. The first statement is clear, as well as the second one since $\pi(\widetilde{L})$ must be connected. The third claim is a direct consequence of the previous two. □

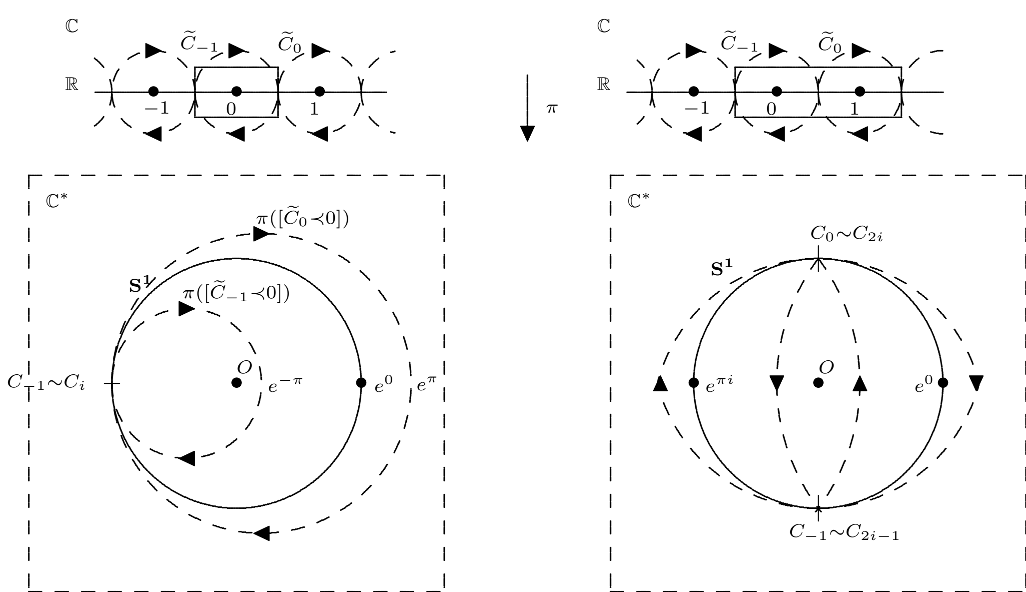
Now, let us consider the set $\overline{\mathcal{S}}$ of pairs

$$[C \prec F^k]$$

where $C = F^0$ is a chamber of $\mathcal{T}_{X,\mathbb{R}}$, F^k a k-codimensional facet of $\mathcal{T}_{X,\mathbb{R}}$ and $F^i \prec F^j \Leftrightarrow clos(F^i) \supset F^j$.

By Lemma 1.2 the quotient map $\pi(\widetilde{F})$ of a facet is still a facet in the real torus and, by π surjective, we get that any facet F in $\mathcal{T}_{X,\mathbb{R}}$ is the image $F = \pi(\widetilde{F})$ of an affine one.

In general the cells of the complex $\mathcal{S}$ cannot be described using the above notation, i.e. $\mathcal{S} \neq \overline{\mathcal{S}}$ as a set. Let us consider the very simple example defined by $\mathcal{A} = \{x \in \mathbb{R} \mid x \in \mathbb{Z}\}$. The chambers $\widetilde{C}_i$ for $i \in \mathbb{Z}$ are the open intervals $(i, i+1)$ and the 1-codimensional facets are the points. The toric arrangement depends on the chosen lattice. For example we can quotient in two different way as in the following figure. Namely, the picture on the left corresponds to the choice $\Lambda = \mathbb{Z}$, i.e. $\pi : x \mapsto e^{2\pi i x}$, whereas the picture on the right is given by $\Lambda = 2\mathbb{Z}$ and $\pi : x \mapsto e^{\pi i x}$.



As shown in the pictures the complex in the former example cannot be described by pairs $[C_{-1} \prec C_{-1}]$, $[C_{-1} \prec e_0]$ while the one in the latter can.

In the first example, the vertices $\widetilde{F}_i = i$ and $\widetilde{F}_{i+1} = i+1$ in the closure of the chamber $\widetilde{C}_i$ have the same image $e_0 = \pi(\widetilde{F}_i) = \pi(\widetilde{F}_{i+1})$ and the boundary of the 1-cell $[C_{-1} \prec e_0]$ is the only vertex $[C_{-1} \prec C_{-1}]$. Thus, if we define $[C_i \prec F_j] = [\pi(\widetilde{C}_i) \prec \pi(\widetilde{F}_j)] := \pi([\widetilde{C}_i \prec \widetilde{F}_j])$ for $j = i, i+1$, we get that $[C_{-1} \prec e_0] = \pi([\widetilde{C}_{-1} \prec 0])$ and $[C_{-1} \prec e_0] = \pi([\widetilde{C}_0 \prec 0])$ which is clearly a bed definition as $\pi([\widetilde{C}_{-1} \prec 0]) \neq \pi([\widetilde{C}_0 \prec 0])$.

We notice that

$$\pi([\widetilde{C} \prec \widetilde{F}]) = \pi([\widetilde{D} \prec \widetilde{G}]) \Longrightarrow [\pi(\widetilde{C}) \prec \pi(\widetilde{F})] = [\pi(\widetilde{D}) \prec \pi(\widetilde{G})].$$

Indeed if $\pi([\widetilde{C} \prec \widetilde{F}]) = \pi([\widetilde{D} \prec \widetilde{G}])$ there is a translation $t \in \Lambda$ which sends $[\widetilde{C} \prec \widetilde{F}]$ in $[\widetilde{D} \prec \widetilde{G}]$. As a simple consequence $\widetilde{D} = t.\widetilde{C}$ and $\widetilde{F} = t.\widetilde{G}$, i.e. $\pi(\widetilde{C}) = \pi(\widetilde{D})$ and $\pi(\widetilde{F}) = \pi(\widetilde{D})$.

The converse is not necessarily true as seen before.

We now want to focus on the case $\mathcal{S} = \overline{\mathcal{S}}$ in which the description of the complex $\mathcal{S}$ is particularly striking.

As $\mathcal{S} = \pi(\widetilde{\mathcal{S}})$ is a complex homotopic to the complement R_X then $\mathcal{S}$ is described by couple $[C \prec F]$ if and only if the definition

$$[C \prec F] = [\pi(\widetilde{C}) \prec \pi(\widetilde{F})] := \pi([\widetilde{C} \prec \widetilde{F}]) \text{ for all } [\widetilde{C} \prec \widetilde{F}] \in \widetilde{\mathcal{S}} \text{ and } \widetilde{C}, \widetilde{F} \in \mathbf{S} \tag{5}$$

is a good one. Under this condition, we get $\mathcal{S} = \pi(\widetilde{\mathcal{S}}) = \overline{\mathcal{S}}$ as set.

Moreover, if the definition (5) holds then we can define the boundary in $\overline{\mathcal{S}}$. We need first to introduce new notations.

Notations. Let $P_0 \subset V$ be a fundamental parallelogram for $\pi : V \to T$ containing the origin of V. Let $\mathcal{A}_{0,X}$ be the subarrangement of $\mathcal{A}_X$ made by all the hyperplanes that intersect P_0 (see, for istance, figure (8) in the next Section).

We will say that a maximal dimensional cell $[\widetilde{C} \prec \widetilde{F}^n]$ is in $\mathcal{A}_{0,X}$ if its support $\mid \widetilde{F}^n \mid$ is the intersection of hyperplanes in $\mathcal{A}_{0,X}$. While a k-cell $[\widetilde{C} \prec \widetilde{F}^k]$ is in $\mathcal{A}_{0,X}$ if it is in the boundary of a n-cell in $\mathcal{A}_{0,X}$. Let $\widetilde{\mathcal{S}}_0$ be the set of all such cells.

With previous notations if (5) holds we define the boundary in $\overline{\mathcal{S}}$ as follow:

$[C \prec F^k]$ is in the boundary of $[D \prec G^j]$ $(k < j)$ if and only if there are cells $[\widetilde{C} \prec \widetilde{F}^k] \in \pi^{-1}([C \prec F^k]) \cap \widetilde{\mathcal{S}}_0$ and $[\widetilde{D} \prec \widetilde{G}^j] \in \pi^{-1}([D \prec G^j]) \cap \widetilde{\mathcal{S}}_0$ such that $[\widetilde{C} \prec \widetilde{F}^k] \in \partial_{\widetilde{\mathcal{S}}}[\widetilde{D} \prec \widetilde{G}^j]$.

Obviously this boundary map commutes with the one in $\widetilde{\mathcal{S}}$ and we get $\mathcal{S} = \pi(\widetilde{\mathcal{S}}) = \overline{\mathcal{S}}$ as CW-complexes.

Toric arrangement for which $\mathcal{S} = \overline{\mathcal{S}}$ are easily characterized as follows.

Definition 1.3 *A toric arrangement* $\mathcal{T}_X$ *is* thick *if the quotient map*

$$\pi : V \longrightarrow T$$

is injective on the closure $clos(\widetilde{C})$ *of every chamber* $\widetilde{C}$ *of the associated affine arrangement* $\mathcal{A}_{X,\mathbb{R}}$.

We notice that every toric arrangement is covered by a thick one and the fiber of the covering map is finite; hence our assumption is not very restrictive.

We have the following

Lemma 1.4 *A toric arrangement* $\mathcal{T}_X$ *is thick if and only if*

$$[\pi(\widetilde{C}) \prec \pi(\widetilde{F})] = [\pi(\widetilde{D}) \prec \pi(\widetilde{G})] \Longleftrightarrow \pi([\widetilde{C} \prec \widetilde{F}]) = \pi([\widetilde{D} \prec \widetilde{G}])$$

for any two cells $[\widetilde{C} \prec \widetilde{F}], [\widetilde{D} \prec \widetilde{G}] \in \widetilde{\mathcal{S}}$

Proof. By previous considerations, it is enough to prove that the thick condition is equivalent to

$$[\pi(\widetilde{C}) \prec \pi(\widetilde{F})] = [\pi(\widetilde{D}) \prec \pi(\widetilde{G})] \Longrightarrow \pi([\widetilde{C} \prec \widetilde{F}]) = \pi([\widetilde{D} \prec \widetilde{G}])$$

$\Rightarrow$: Let $\mathcal{T}_X$ be thick and $[\pi(\widetilde{C}) \prec \pi(\widetilde{F})] = [\pi(\widetilde{D}) \prec \pi(\widetilde{G})]$ for two given k-cells in $\widetilde{\mathcal{S}}$. This implies that $\pi(\widetilde{C}) = \pi(\widetilde{D})$ and $\pi(\widetilde{F}) = \pi(\widetilde{G})$, i.e. there are translations $t, t' \in \Lambda$ such that $\widetilde{D} = t.\widetilde{C}$ and $\widetilde{G} = t'.\widetilde{F}$.

By construction $t.\widetilde{F}$ is a facet in the closure $clos(D)$. We get two facets $t.\widetilde{F}$ and $\widetilde{G}$ both in $clos(D)$ and with the same image $\pi(t.\widetilde{F}) = \pi(\widetilde{F}) = \pi(\widetilde{G})$. By hypothesis π is injective on $clos(D)$ then $t.\widetilde{F} = G$, i.e. $t = t'$ which implies that $\pi([\widetilde{C} \prec \widetilde{F}]) = \pi([\widetilde{D} \prec \widetilde{G}])$.

$\Leftarrow$ Let $\widetilde{F}$ and $\widetilde{G}$ two facets in $clos(\widetilde{C})$ such that $\pi(\widetilde{F}) = \pi(\widetilde{G})$ then

$$\pi([\widetilde{C} \prec \widetilde{F}]) = [\pi(\widetilde{C}) \prec \pi(\widetilde{F})] = [\pi(\widetilde{C}) \prec \pi(\widetilde{G})] = \pi([\widetilde{C} \prec \widetilde{G}]).$$

As a consequence if $t \in \Lambda$ is the translation such that $\widetilde{F} = t.\widetilde{G}$ then $t.\widetilde{C} = \widetilde{C}$. It follows that t is the identity and we get $\widetilde{F} = \widetilde{G}$, i.e. π is injective on $clos(\widetilde{C})$ $\square$

By previous considerations together with Lemma 1.4 we get the following theorem

Theorem 1 *Let* $\mathcal{T}_X$ *be a thick toric arrangement. Then its complement* $\mathcal{R}_X$ *has the same homotopy type of the CW-complex* $\overline{\mathcal{S}}$.

Then in this case the complex $\mathcal{S}$ has a nice combinatorial description, totally analogue to that of the classical Salvetti complex [14].

Moreover if a toric arrangement is thick then the maximal dimensional cells $[\widetilde{C} \prec \widetilde{F}^n]$ in $\mathcal{A}_{0,X}$ are in one to one correspondence with the n-dimensional facets of $\overline{\mathcal{S}}$. Then the boundary in a thick toric arrangement $\mathcal{T}_X$ can be completely described knowing the boundary in the associated finite complex $\mathcal{A}_{0,X}$.

This allows to better understand the fundamental group of the complement and to perform computations on integer cohomology.

Furthermore, in this case $\mathcal{S}$ is a *regular* CW-complex.

Remark 1.5 *The number of chambers of $\mathcal{T}_{X,\mathbb{R}}$ can be computed by formulae given in [7] and [12]. However the combinatorics of the layers in $\mathcal{T}_{X,\mathbb{R}}$ is more complicated than the one of spaces of $\mathcal{A}_{X,\mathbb{R}}$; hence an enumeration of the faces is not easy to provide in the general case. Thus from now on we focus on the arrangements defined by roots systems. In this case the chambers are parametrized by the elements of the Weyl group, and the poset of layers has been described in [11].*

2 The case of Weyl groups

In this section we give a simpler and nicer description of the above complex for the particular case of toric arrangements associated to affine Weyl groups when the lattice Λ is spanned by coroots. Indeed in this case the toric arrangement is thick. Using this description, we give an example of computation for the integer cohomology of these arrangements.

2.1 Notations and Recalls.

Toric arrangement for Weyl group Let Φ be a root system, $\langle\Phi^\vee\rangle$ be the lattice spanned by the coroots, and Λ be its dual lattice (which is called the *cocharacters* lattice). Then we define a torus $T = T_\Lambda$ having Λ as group of characters. In other words, if $\mathfrak{g}$ is the semisimple complex Lie algebra associated to Φ and $\mathfrak{h}$ is a Cartan subalgebra, T is defined as the quotient $T \doteq \mathfrak{h}/\langle\Phi^\vee\rangle$.

Each root α takes integer values on $\langle\Phi^\vee\rangle$, so it induces a character

$$e^\alpha : T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*.$$

Let X be the set of this characters; more precisely, since α and $-\alpha$ define the same hypersurface, we set

$$X \doteq \left\{e^\alpha,\ \alpha \in \Phi^+\right\}.$$

In this way to every root system Φ is associated a toric arrangement that we will denote by $T_{\widetilde{W}}$, where $\widetilde{W}$ is the affine Weyl group associated to Φ.

Remark 2.1 1. *Let G be the semisimple, simply connected linear algebraic group associated to $\mathfrak{g}$. Then T is the maximal torus of G corresponding to $\mathfrak{h}$, and $\mathcal{R}_X$ is known as the set of* regular points *of T.*

2. *One may take as Λ the lattice spanned by the roots. But then one obtains as T a maximal torus of the semisimple* adjoint *group G^a, which is the quotient of G by its center.*

Let $(\widetilde{W}, S)$ be the Coxeter system associated to $\widetilde{W}$ and

$$\mathcal{A}_{\widetilde{W}} = \{H_{\widetilde{w}s_i\widetilde{w}^{-1}} \mid \widetilde{w} \in \widetilde{W} \text{ and } s_i \in S\}$$

the arrangement in $\mathbb{C}^n$ obtained by complexifying the reflection hyperplanes of $\widetilde{W}$, where, in a standard way, the hyperplane $H_{\widetilde{w}s_i\widetilde{w}^{-1}}$ is simply the hyperplane fixed by the reflection $\widetilde{w}s_i\widetilde{w}^{-1}$.
We can view Λ as a subgroup of $\widetilde{W}$, acting by translations. Then it is well

known that $\widetilde{W}/\Lambda \simeq W$, where W is the finite reflection group associated to $\widetilde{W}$. As a consequence, the toric arrangement can be described as:

$$T_{\widetilde{W}} = \{H_{[w]s_i[w^{-1}]} \mid w \in W \text{ and } s_i \in S\}$$

where two hypersurfaces $H_{[w]s_i[w^{-1}]}$ and $H_{[\overline{w}]s_i[\overline{w}^{-1}]}$ are equal if and only if there is a translation $t \in \Lambda$ such that $tws_i(tw)^{-1} = \overline{w}s_i\overline{w}^{-1}$, i.e. $\overline{w} = tw$. Moreover, by [11], these hypersurfaces intersect in

$$\frac{\mid W \mid}{\mid W_{S\setminus\{s_i\}} \mid}$$

local copies of the finite hyperplane arrangement $A_{W_{S\setminus\{s_i\}}}$ associated to the group generated by $S \setminus \{s_i\}$, $s_i \in S$.

For example in the affine Weyl group $\widetilde{A}_n$ generated by $\{s_0, \ldots, s_n\}$ for any generator s_i the finite reflection group associated to $S \setminus \{s_i\}$ is simply a copy of the finite Coxeter group A_n.

The above condition is equivalent to say that $T_{\widetilde{W}}$ is thick. Then we can construct the Salvetti complex for these arrangements in a very similar way to the affine one.

Salvetti Complex for affine Artin groups It is well known (see, for instance, [6], [15]) that the cells of Salvetti complex $\mathcal{S}_W$ for arrangements $\mathcal{A}_{\widetilde{W}}$ are of the form $E(\widetilde{w}, \Gamma)$ with $\Gamma \subset S$ and $\widetilde{w} \in \widetilde{W}$. Indeed if $\widetilde{\alpha} \in \{\widetilde{w}s\widetilde{w}^{-1} | s \in S, \widetilde{w} \in \widetilde{W}\}$ is a reflection, the chambers are in one to one correspondence with the elements of the group $\widetilde{W}$ as follows:

fixed a base chamber C_0, it will correspond to $1 \in \widetilde{W}$ and if C corresponds to $\widetilde{w}$, then the chamber D separated from C by the reflection hyperplane $H_{\widetilde{\alpha}}$ will correspond to the element $\widetilde{\alpha}\widetilde{w} \in \widetilde{W}$. The notation $D \simeq \widetilde{\alpha}\widetilde{w}$ will be used.

If $\widetilde{F}^k$ is a k-codimensional facet then the k-cell $[\widetilde{C} \prec \widetilde{F}^k]$ corresponds to the couple $E(\widetilde{w}, \Gamma)$ where $\widetilde{w} \simeq \widetilde{C}$ and $\Gamma = \{s_{i_1}, \ldots, s_{i_k}\}$ is the unique subset of cardinality k in S such that

$$\mid F^k \mid = \bigcap_{j=1}^{k} H_{\widetilde{w}s_{i_j}\widetilde{w}^{-1}}.$$

If $\widetilde{W}_\Gamma$ is the finite subgroup generated by $s \in \Gamma$, by [6] the integer boundary map can be expressed as follows:

$$\partial_k(E(\widetilde{w}, \Gamma)) = \sum_{s_j \in \Gamma} \sum_{\beta \in \widetilde{W}_\Gamma^{\Gamma\setminus\{s_j\}}} (-1)^{l(\beta)+\mu(\Gamma, s_j)} E(\widetilde{w}\beta, \Gamma \setminus \{s_j\}). \qquad (6)$$

where $\widetilde{W}_\Gamma^{\Gamma\setminus\{\sigma\}} = \{w \in \widetilde{W}_\Gamma : l(ws) > l(w) \forall s \in \Gamma \setminus \{\sigma\}\}$ and $\mu(\Gamma, s_j) = \sharp\{s_i \in \Gamma | i \leq j\}$.

Remark 2.2 *Instead of the co-boundary operator we prefer to describe its dual, i.e. we define the boundary of a k-cell $E(\widetilde{w},\Gamma)$ as a linear combination of the $(k-1)$-cells which have $E(\widetilde{w},\Gamma)$ in their co-boundary, with the same coefficient of the co-boundary operator. We make this choice since the boundary operator has a nicer description than co-boundary operator in terms of the elements of $\widetilde{W}$.*

2.2 Description of the complex

Let $\mathcal{S}_W$ be the CW-complex associated to $T_{\widetilde{W}}$. By the previous considerations, $\mathcal{S}_W$ admits a description similar to that of $\widetilde{\mathcal{S}}_W$. Indeed each chamber C is in one to one correspondence with an equivalence class $[w] \in \widetilde{W}/\Lambda$ and then with an element $w \in W \simeq \widetilde{W}/\Lambda$ of the finite reflection group W. We will write $C \simeq [w]$.

In the same way, the couple $[C \prec F^k] \in \mathcal{S}_W$ corresponds to the cell $E([w],\Gamma)$ where $C \simeq [w]$ and $\Gamma = \{s_{i_1},\ldots,s_{i_k}\}$ is the unique subset of cardinality k in S such that

$$\mid F^k \mid = \bigcap_{j=1}^{k} H_{[w]s_{i_j}[w^{-1}]}.$$

We now want to describe the boundary of each cell: this is done in a standard way by characterizing the cells that are in the boundary of a given cell, and by assigning an orientation to all cells.

By construction the toric CW complex is locally isomorphic to the affine one and it can inherit the affine orientation. Then the integer boundary operator for Coxeter toric arrangements can be written as the affine one:

$$\partial_k(E([w],\Gamma)) = \sum_{\sigma\in\Gamma} \sum_{\beta \in W_\Gamma^{\Gamma\setminus\{\sigma\}}} (-1)^{l(\beta)+\mu(\Gamma,\sigma)} E([w\beta],\Gamma\setminus\{\sigma\}) \tag{7}$$

where, instead of elements of the affine group $\widetilde{W}$, we have equivalence classes with representatives in the finite group W.

Using this operator it is possible to compute the integer cohomology for Weyl toric arrangements.

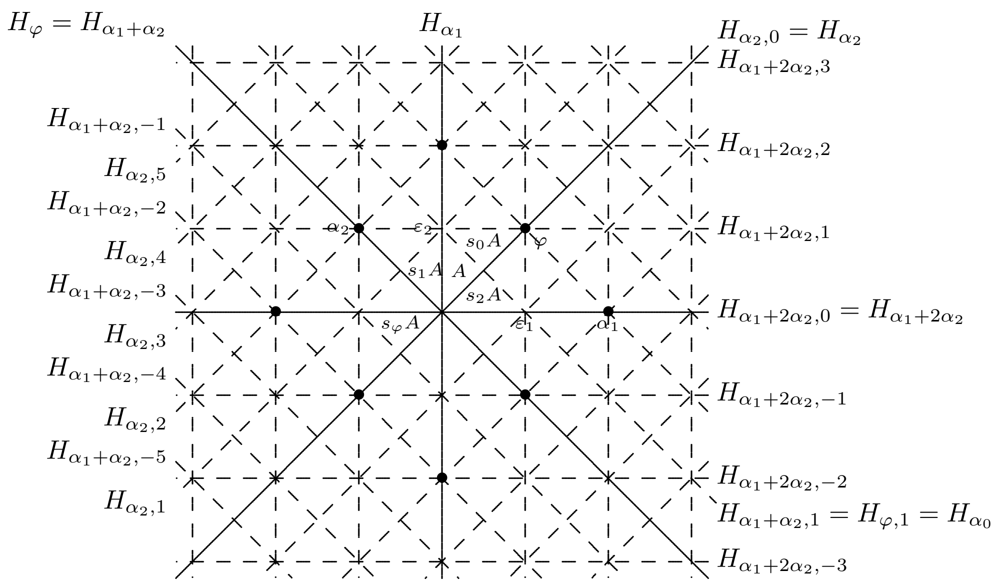
Example. Let us consider the affine Weyl group $\widetilde{B}_2$ (see [1]) with Coxeter graph:

$$\begin{array}{ccccc} \circ & \overset{4}{-} & \circ & \overset{4}{-} & \circ \\ s_0 & & s_1 & & s_2 \end{array}$$

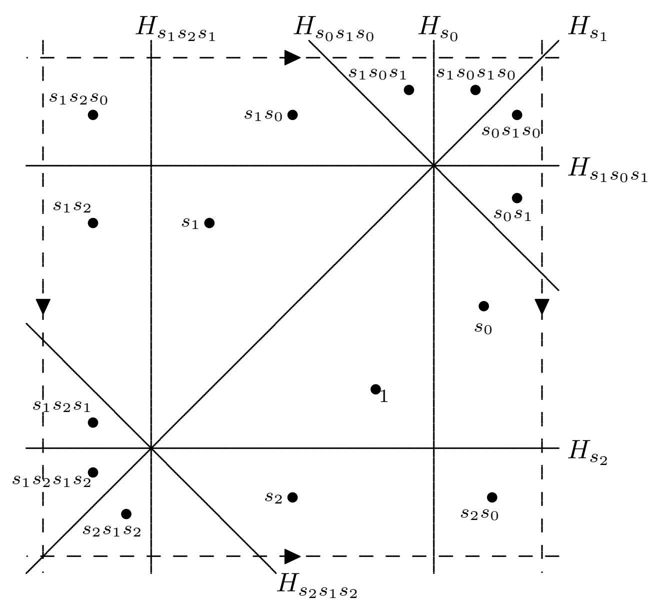
and associated finite group B_2

$$\begin{array}{ccc} \circ & \overset{4}{-} & \circ \\ s_1 & & s_2 \end{array}$$

In this case we get translations $t_1 = s_0s_1s_2s_1$ and $t_2 = s_2s_1s_0s_1$ and the affine arrangement is represented as:



If $\mathcal{A}_0$ is the finite subarrangement defined in Section 2.3, then the real toric arrangement is obtained quotienting it as shown in the following figure, where arrows indicate identified edges:



(8)

Here, for brevity, the vertices $E(w, \emptyset)$ are labelled simply by the element $w \in \widetilde{W}$.

We get, for example, that the cell $E([1], \emptyset)$ is the vertex in the chamber containg $1 \in \widetilde{W}$, while the vertices $E([s_0], \emptyset)$ and $E([s_1s_2s_1], \emptyset)$ correspond to the same chamber in the toric arrangement; indeed $s_0 = t_1 s_1 s_2 s_1$, then $[s_0] = [s_1 s_2 s_1]$.

Notice that the number of chambers in the real torus is 8 in one to one correspondence with the finite Weyl group B_2 with cardinality 8. Then we get exactly:

8	0-cells of the form $E([w], \emptyset)$ for $w \in B_2$,
24	1-cells of the form $E([w], \{s_i\})$ for $w \in B_2$ and $i = 0, 1, 2$,
24	2-cells of the form $E([w], \{s_i, s_j\})$ for $w \in B_2$ and $0 \leq i < j \leq 2$.

These cells locally correspond to four finite Coxeter arrangements, two of type B_2 and two of type $A_1 \times A_1$ appearing in the figure above. In particular the 2-cells can be written as:

$E([w], \{s_i, s_{i+1}\})$ with a representative w chosen in the Coxeter group B_2 generated by $\{s_i, s_{i+1}\}$), i=0,1;

$E([w], \{s_0, s_2\})$ and $E([s_1 w], \{s_0, s_2\})$ with a representative w chosen in the group $\{1, s_0, s_2, s_0 s_2\}$ generated by $\{s_0, s_2\}$.

The representatives can be chosen in the more suitable way for computations. The boundary map (7) for the 1-cells is:

$$\partial_1 E([w], \{s_i\}) = E([w], \emptyset) - E([ws_i], \emptyset)$$

and it gives rise to a matrix of 24 columns and 8 rows with entries 0, 1 and -1. On the other hand, the second boundary map is given by

$$\begin{aligned} \partial_2 E([w], \{s_i, s_{i+1}\}) = E([w], \{s_i\}) - E([ws_{i+1}], \{s_i\}) + E([ws_i s_{i+1}], \{s_i\}) - \\ - E([w], \{s_{i+1}\}) + E([ws_i], \{s_{i+1}\}) - E([ws_{i+1} s_i], \{s_{i+1}\}) \end{aligned}$$

$$\partial_2 E([w], \{s_0, s_2\}) = E([w], \{s_0\}) - E([ws_2], \{s_0\}) - E([w], \{s_2\}) + E([ws_0], \{s_2\}).$$

In this way we get that the homology, and hence the cohomology, is torsion free and $H_0(R_{B_2}, \mathbb{Z}) = \mathbb{Z}$, $H_1(R_{B_2}, \mathbb{Z}) = \mathbb{Z}^8$ and $H_2(R_{B_2}, \mathbb{Z}) = \mathbb{Z}^{15}$, which agrees with the Betti numbers computed in [11, Ex. 5.14].

In general we have the following

Conjecture 2.3 *Let $\widetilde{W}$ be an affine Weyl group and $\mathcal{T}_{\widetilde{W}}$ be the corresponding toric arrangement. Then the integer cohomology of the complement is torsion free (and hence it coincides with the De Rham cohomology computed in [3]).*

3 An interesting example of non-thick toric arrangement.

In this section we give an example of non-thick arrangement: the one coming from the affine Weyl arrangement $\mathcal{A}_{\widetilde{A}_n}$ when the lattice $\Lambda_{\widetilde{A}_n}$ is spanned by the roots of the Weyl group $\widetilde{A}_n$ (see the second part of Remark 3.1).

Indeed in this case the underlying real toric arrangement has a very nice description in terms of Young tableaux. More precisely the facets of $\mathcal{T}_{\widetilde{A}_n, \mathbb{R}}$ are in one to one correspondence with a family of Young tableaux which turn out to be the same tableaux describing the space of all periodic legged gaits of a robot body (see [2]).

It is clear that, in this case, the finite arrangement $\mathcal{A}_{0,\widetilde{A}_n}$ is exactly the braid arrangement $\mathcal{A}_{A_n}$.

3.1 Tableaux description for the complex $\widetilde{\mathcal{S}}_{A_n}$

We indicate simply by A_n the symmetric group on $n+1$ elements, acting by permutations of the coordinates. Then $\mathcal{A} = \mathcal{A}_{A_n}$ is the braid arrangement and $\widetilde{\mathcal{S}}_{A_n}$ is the associated CW-complex (even if the arrangement is finite we continue to use the same notation used above for the affine case to distinguish it from the toric one).

Given a system of coordinates in $\mathbb{R}^{n+1}$, we describe $\widetilde{\mathcal{S}}_{A_n}$ through certain tableaux as follow.

Every k-cell $[\widetilde{C} \prec \widetilde{F}]$ is represented by a tableau with $n+1$ boxes and $n+1-k$ rows (aligned on the left), filled with all the integers in $\{1, ..., n+1\}$. There is no monotony condition on the lengths of the rows. One has:

- $(x_1, \ldots, x_{n+1})$ is a point in $\widetilde{F}$ if and only if:

1. i and j belong to the same row if and only if $x_i = x_j$,
2. i belongs to a row less than the one containing j if and only if $x_i < x_j$;

- the chamber $\widetilde{C}$ belongs to the half-space $x_i < x_j$ if and only if:

1. either the row which contains i is less than the one containing j or
2. i and j belong to the same row and the column which contains i is less than the one containing j.

Notice that the facets of the real stratification are represented by standard Young tableaux, since the order of the entries in each row does not matter, and hence we can assume it to be strictly increasing.
Notice also that the geometrical action of A_n on the stratification induces a natural action on the complex $\widetilde{\mathcal{S}}_{A_n}$ which, in terms of tableaux, is given by a left action of A_n: $\sigma.\,T$ is the tableau with the same shape as T, and with entries permuted through σ.

3.2 Tableaux description for the facets of $\mathcal{T}_{\widetilde{A}_n,\mathbb{R}}$

Let $\mathcal{A}_{0,\widetilde{A}_n} \subset \mathcal{A}_{\widetilde{A}_n}$ be the braid arrangement passing through the origin and $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}/\Lambda_{\widetilde{A}_n} = T_{\mathbb{R}}$ the projection map.

If $\mathbf{S}_{\widetilde{A}_n}$ is the stratification of $\mathbb{R}^{n+1}$ into facets induced by the arrangement $\mathcal{A}_{\widetilde{A}_n}$, we define the set:

$$\mathbf{S}_{0,\widetilde{A}_n} = \{\widetilde{F}^k \in \mathbf{S}_{\widetilde{A}_n} \mid clos(F^k) \supset \bigcap_{H \in \mathcal{A}_{0,\widetilde{A}_n}} H\}.$$

Obviously $\mathbf{S}_{0,\widetilde{A}_n}$ is in one to one correspondence with the stratification $\mathbf{S}_{A_n}$ induced by the braid arrangement $\mathcal{A}_{A_n}$ and the restriction $\pi_{\mathbf{S}_{0,\widetilde{A}_n}}$ is surjective on $T_{\mathbb{R}}$.

It follows that in order to understand how $\Lambda_{\widetilde{A}_n}$ acts on $\mathbf{S}_{\widetilde{A}_n}$ it is enough to study how it acts on $\mathbf{S}_{0,\widetilde{A}_n}$. Moreover it is enough to consider facets in the closure of the base chamber $\widetilde{C}_0$ corresponding to $1 \in \widetilde{A}_n$; the action on the others will be obtained by symmetry.

Let us remark that a facet $\widetilde{F}^k$ is in $\mathbf{S}_{0,\widetilde{A}_n}$ if and only if it intersects any ball B_0 around the origin. Let B_0 be a ball of sufficiently small radius and

$$x = (x_1, \dots, x_{n+1}) \in clos(\widetilde{C}_0) \cap B_0$$

be a given point in a facet $\widetilde{F}^k \in \mathbf{S}_{0,\widetilde{A}_n}$. Then $x_1 \leq x_2 \leq \dots \leq x_{n+1}$ and the standard Young tableaux $Tb_{\widetilde{F}^k}$ associated to $\widetilde{F}^k$ will have entries increasing along both, rows and columns.

Let $t_1, \dots, t_n \in \Lambda_{\widetilde{A}_n}$ be a base such that t_i translates the reflection hyperplane $H_{i,i+1} = Ker(x_i - x_{i+1})$ fixing all hyperplanes $H_{j,j+1} = Ker(x_j - x_{j+1})$ for $j \neq i$ (i.e. each point in $H_{j,j+1}$ is sent in a point still in $H_{j,j+1}$).
Then we can assume that translation t_i acts on the entry x_i as $t_i.x_i = x_i + t$ with $x_i + t > x_{i+1}$ and, as $H_{j,j+1}$, for $j \neq i$, are invariant under the action of t_i, it follows that $t_i.x_{i-1} = x_{i-1} + t$ and, by induction, $t_i.x_j = x_j + t$ for all $j < i$, while $t_i.x_j = x_j$ for all $j > i$.

Recall that, by construction, given a standard Young tableaux, a point $(x_1, \dots, x_{n+1})$ is a point in $\widetilde{F}$ if and only if:

1. i and j belong to the same row if and only if $x_i = x_j$,
2. i belongs to a row less than the one containing j if and only if $x_i < x_j$;

It follows that if Tb is a tableau such that $i \in r_k$ and $i+1 \in r_{k+1}$ are in two different rows, then t_i acts on Tb sending it in a tableau Tb' with rows $r'_1 = r_{k+1}, \dots, r'_{h-k} = r_h, r'_{h-k+1} = r_1, \dots, r'_h = r_k$. While if $i, i+1 \in r_k$ are in the same row, then t_i acts sending the corresponding facet in a facet which is not anymore in $\mathcal{A}_{0,\widetilde{A}_n}$.

Then $\Lambda_{\widetilde{A}_n}$ acts on the h rows of a tableau $Tb_{\widetilde{F}}$ as a power of the cyclic permutation $(1, \dots, h)$.

Equivalently let $Y(n+1, k+1)$ be the set of standard Young tableaux with $k+1$ rows and $n+1$ entries and $Tb \in Y(n+1, k+1)$ be a tableau of rows $(r_1, \dots, r_{k+1})$. Then each facet F^k of the toric arrangement $\mathcal{T}_{\widetilde{A}_n,\mathbb{R}}$ is in one to one correspondence with the set

$$Y(n+1, k+1)/\sim$$

where a tableau $Tb' \sim Tb$ if and only if the rows of Tb' are $(r_{\sigma^s(1)}, \dots, r_{\sigma^s(k+1)})$ for a power σ^s of the cyclic permutation $\sigma = (1, \dots, k+1)$. So far we get exactly the tableaux described in [2].

Finally let us recall that a facet $\widetilde{F}^k \prec \widetilde{F}^{k+1}$ if and only if the tableau $Tb_{\widetilde{F}^{k+1}}$ corresponding to $\widetilde{F}^{k+1}$ is obtained by attaching two consecutive rows of $Tb_{\widetilde{F}^k}$. As a consequence if F^k and F^{k+1} are facets in the toric arrangement $\mathcal{T}_{\widetilde{A}_n,\mathbb{R}}$, $F^k \prec F^{k+1}$ if and only if the tableau $Tb_{F^{k+1}}$ corresponding to F^{k+1} is obtained by attaching two consecutive rows of Tb_{F^k} or attaching the first one to the last one.

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