Laboratory of Economics and Management

Sant'Anna School of Advanced Studies
Piazza Martiri della Libertà, 33 - 56127 PISA (Italy)
Tel. +39-050-883-343 Fax +39-050-883-344
Email: lem@sssup.it Web Page: http://www.lem.sssup.it/

LEM

Working Paper Series

# Evolution and market behavior with endogeneous investment rules

Giulio Bottazzi [§]
Pietro Dindo [§]

[§] Institute of Economics, Scuola Superiore Sant'Anna, Pisa, Italy

2010/20 February 2014

ISSN (online) 2284-0400

# Evolution and market behavior with endogenous investment rules*

Giulio Bottazzi Pietro Dindo

January 28, 2014

Istituto di Economia, Scuola Superiore Sant'Anna
Piazza Martiri della Libertà 33, 56127 Pisa, Italy

**Abstract**

In a repeated market for short-lived assets, we investigate wealth-driven selection among boundedly rational traders whose investment rules depend on endogenous market variables, such as current and past prices. We study the random dynamical system describing prices and wealth dynamics and characterize local stability of the long-run equilibria in which one or a group of traders dominate. Multiplicity of stable and unstable equilibria, leading to endogenous fluctuations and assets mis-pricing, turns out to be a common phenomenon generated by two different mechanisms. Firstly, conditioning investment decisions on endogenous market variables implies that dominance of an investment rule on others, in terms of relative wealth, may be different for different prevailing prices, so that the market may fail to select a global winner. Secondly, the feedback existing between past asset prices and current investment decisions can lead to a form of deterministic overshooting.

*Keywords*: Market Selection; Evolutionary Finance; Price Feedbacks; Asset Pricing; Informational Efficiency; Bounded Rationality; Kelly Rule.

*JEL Classification*: D50, D80, G11, G12

*This work benefited from the interaction with Mikhail Anufriev, Larry Blume, David Easley, Cars Hommes, Jan Werner, two anonymous referees, and participants to a number of workshops and seminars. We thank all of them for their feedbacks. Pietro Dindo is supported by a Marie Curie International Outgoing Fellowship, proposal number 300637, within the 7th European Community Framework Programme. Pietro Dindo gratefully acknowledges the hospitality of the Economics Department at Cornell University. All usual disclaimers apply.

# 1 Introduction

The Efficient Market Hypotheses (EMH), advanced as a general interpretative and normative framework nearly forty years ago (Fama, 1970), has grown to become a widely accepted working tool for the economic profession. A reason for the widespread reliance on the EMH is to be found in the market selection argument that supports it: if poorly informed boundedly rational investors are persistently loosing wealth in favor of the better informed, the formers are ultimately driven out of the market by the latters so that, in the long run, prices reflect the best available information about fundamentals.[1]

Despite its pervasive influence in economics, a general formal proof of the selective capability of financial markets and, consequently, of the ultimate convergence of asset prices toward fundamental values is still lacking. Only fairly recently scholarshiply work has started to investigate this issue. Several behavioral models based on evidence collected from laboratory experiments and real markets (see Barberis and Thaler, 2003, and references therein) contend both the positive and normative aspects of the EMH. Rational and informed behavior does not appear a pervasive property of trading, nor does automatically guarantee, even if appropriately implemented, better performances and higher probability to "survive" the speculative struggle. The modeling effort of these studies has been, however, limited to partial equilibrium models with an exogenous price dynamics.

A general equilibrium model was firstly proposed in Blume and Easley (1992). In a repeated market for Arrow securities, they investigate wealth-driven selection, and the informational content of asset prices, among investment rules that optimize portfolio asset allocations and have fixed saving rates. They find that the best rule for long term survival is maximizing the expected log wealth. In fact, this rule can be seen as a generalization of the Kelly rule (Kelly, 1956) to equilibrium models. When the "best" informed trader employs the log-rule, she gains all the wealth in the long run and drives prices as close as possible to their correct values. However, when the "best" informed trader uses another rule, Blume and Easley are able to construct examples in which she vanishes, suggesting that the market selection argument supporting the EMH deserves further investigation. Two groups of contributions developed from their analysis.

A first group of works has investigated investment rules not necessarily coming from utility maximization but expressed as fractions of wealth to be invested in each asset (see Evstigneev et al., 2009, for a recent survey). Fractions are allowed to depend on histories of exogenous variables, in particular past dividends, but not on endogenous variables, such as equilibrium prices. Under the assumption that

[1] The same argument lies behind the evolutionary foundations of neoclassical economics (Alchian, 1950; Friedman, 1953).

saving rates are homogeneous across traders, a robust finding is that investing proportionally to asset expected dividends, a rule named Generalized Kelly by the authors, is the unique dominating rule. The result holds for both short- and long-lived assets with general dividend processes, as shown in Amir et al. (2005) and Evstigneev et al. (2008) respectively. These contributions are silent on the outcome of market selection when rules depend on endogenous market variables and/or when a full knowledge of the underlying dividend process is lacking.

A second group of works has instead focused on selection when investment decisions are derived from expected utility maximization and traders have perfect foresight on future prices but disagree about the dividend process. In these works portfolio rules and saving rules depend both on endogenous and exogenous market variables. Moreover, given maximization and perfect foresight, both rules are optimal. The main objective is to establish if, at least in this idealized market set-up, wealth reallocation selects for agents whose beliefs are "closest" to the underlying dividend process. Sandroni (2000) and Blume and Easley (2006) find that when markets are dynamically complete[2] the answer is positive. The selection argument behind informational efficiency is correct: in the long-run only the traders whose beliefs are "nearest" to the correct ones are selected for, provided that all discount future utility at the same rate.

There is however a general question that both groups of contributions leave un-answered: except for few specific examples, it is not known how wealth-driven selection works when investment rules depend on endogenous market variables but, at the same time, traders are boundedly rational and do not have perfect foresight on future prices, so that the employed rules may, ex-post, not be optimal. The aim of the present paper is to answer this question. In particular, we build upon the model in Blume and Easley (1992) and provide sufficient conditions for asymptotic survival and dominance of investment rules that depend on current and past equilibrium prices. Our final objective is to move closer to a formal general check of the selective capability of financial markets, even when traders are boundedly rational, and understand when there is support for the market selection argument behind the EMH and when not.

The dependence of investment rules on past prices implies a feedback from past to present market performances. This effect has already been investigated in several heterogeneous agents models. The main finding is that market instability and asset mis-pricing are in general possible (see Hommes, 2006; LeBaron, 2006, for a review). In these works, selection operates according to fitness measures different from relative wealth (Levy et al., 2000; Farmer, 2002; Chiarella and He, 2001,

[2]Notice that in an equilibrium model with sequential trading, as the one we consider here, market completeness cannot be achieved unless agents coordinate on having rational expectations on prices. On the issue of market completeness see also footnote 4.

are among the few exceptions). Moreover, results are often derived for specific investment behaviors and in a partial equilibrium framework. Some gaps have been filled by our previous works, see e.g. Anufriev and Bottazzi (2010), Anufriev et al. (2006), Anufriev and Dindo (2010), which study wealth-driven selection on the general class of price dependent investment rules in a deterministic and growing economy.

The present paper extends the analysis to stochastic economies. We study market selection by analyzing the random dynamical system that describes prices and wealth dynamics. In Section 3 we introduce our market model. In Section 4 we indentify the wealth distributions and prices that are invariant under the wealth-reallocation process and investigate their stochastic stability. We provide general sufficient conditions stating whether any given rule is locally dominating, or locally dominated by, other rules. Our findings apply to investment rules that are general (smooth enough) functions of current and past prices so that a wide spectrum of behaviors can be modeled, including those derived from the maximization of a Constant Relative Risk Aversion (CRRA) expected utility.

In Section 5 we introduce the S-rule, a price dependent generalization of the Kelly rule. When the S-rule trades in the market, it acts as the "local" champion: it destabilizes any long-run informationally inefficient market equilibrium and, at the same time, it determines a market equilibrium where risky assets are correctly priced proportionally to their expected dividends. This equilibrium is never unstable, no matter the number and type of other competing investment rules. However, when the S-rule is not used by any agent, the combination of endogenous investment rules and bounded rationality may lead to multiplicity of stable and unstable equilibria.

A first motivating example is in Section 2 while, in Section 6, different cases of market selection failure are reviewed. The discussion clarifies that the multiple equilibria and market instability are essentially related to two causes. Firstly, since average wealth growth rates depend on equilibrium prices, the relative average performance of rules can be different in different market states. For example, it may well happen that, given two rules, the first has the highest average wealth growth rate at the prices determined by the second while the second has the highest average wealth growth rate at the prices determined by the first, so that none can prevail. Alternatively either rule can prevail but only close to those market states where it possesses all the wealth and it determines asset prices. Secondly, instability might be related to price feedback. Even though a given rule has the highest average wealth growth rate for all possible market states, prices feedback may be strong enough to destabilize the market dynamics. In general, endogenous fluctuations may emerge so that asset prices may not converge as close as possible, given the competing rules, to their fundamental levels.

## 2 An instructive example

In this section we shall consider the simplest example where the implication of having endogenous rules used by boundedly rational agents can be fully appreciated. Time is discrete and uncertainty is modeled as a two states i.i.d. Bernoulli process $\{\omega_t\}$ with $\omega_t \in \{1, 2\}$ representing the state of the world at time $t$. For any $t$ the probability that $\omega_t = 1$ is $\pi$. In each period two Arrow securities can be traded. Security $i \in \{1, 2\}$ pays 1 if $\omega_t = i$ and 0 otherwise. Both securities are in unitary exogenous supply. Assets are traded sequentially by two agents in competitive markets. Each trader demand is expressed as a portfolio rule, that is, as a fraction of wealth to be invested in each asset. We do not specify how rules are chosen. They can be behavioral, rules of thumb, or come from utility maximization.[3] As we shall formalize in the next section, the constraint that we do impose on rules is that they are time independent (smooth enough) functions of current and past equilibrium prices. Typically the dependence on current prices is induced by the budget constraint while past prices can be used to learn the dividend process or to forecast future prices. Importantly, agents are not assumed to be able to coordinate their expectations on future prices. For this reason decisions taken to be optimal given price expectations may turn out not to be so. In fact, this is the only substantial difference between the present framework and the complete market case analyzed in Sandroni (2000) and Blume and Easley (2006, 2009), or in the recent work by Massari (2013).[4] In this example we further assume that traders do not consume so that portfolio and investment rules coincide. The question we address is what happens to agents' wealth and asset prices in the long run.

Name $w_t$ the wealth of agent 1. The total wealth in every period is equal to 1 so that the wealth of agent 2 is $1 - w_t$. Name $\alpha_t^i$ the fraction of wealth invested in asset 1 by agent $i$ in period $t$. Since traders do not consume, the budget constraint implies that a fraction $1 - \alpha_t^i$ is invested in asset 2. Asset prices are fixed by market clearing. In particular, for the first asset it holds

$$p_t = \alpha_t^1 w_t + \alpha_t^2 (1 - w_t) \,.$$

No consumption and unitary total wealth imply that the price of the second asset

[3]For example in Bottazzi and Dindo (2013) we consider the case of myopic CRRA expected utility maximizers who differ in their coefficients of risk aversion and/or in their beliefs about $\pi$.

[4] Because of the missing assumption of rational price expectations the number of possible prices is larger than the number of independent assets, so that the market is "endogenously" incomplete. However since for any given price vector all contingent wealth transfers can be traded, the market is "exogenously" complete. See also the difference between endogenous and exogenous uncertainty in Chichilnisky (1999) and Hahn (1999). Other authors refer to the repeated market for Arrow securities as sequentially complete but not necessarily dynamically complete, see e.g. Drèze and Herings (2008).

is $1-p_t$. Given prices and asset allocations in $t$, the wealth of agent 1 in period $t+1$ depends on the realization of the Bernoulli process $\omega_{t+1}$ and it is equal to the amount of Arrow securities of type $\omega_{t+1}$ purchased in $t$, that is

$$w_{t+1} = \frac{\alpha^1_{t,\omega_{t+1}}}{p_t}\, w_t\,.$$

We assume that portfolio decisions are (smooth) functions of current and past prices. For simplicity, in this example we restrict past prices to the last observed price, $p_{t-1}$. Thus we can define investment rules $\alpha^i(\cdot,\cdot)$ for $i=1,2$ such that

$$\alpha^i_t = \alpha^i(p_t, p_{t-1})\,, \quad i=1,2\,.$$

Given rules the market dynamics is summarized by

$$w_{t+1} = \begin{cases} \frac{\alpha^1(p_t,p_{t-1})}{p_t}\, w_t & \text{if} \quad \omega_{t+1}=1 \\ \\ \frac{1-\alpha^1(p_t,p_{t-1})}{1-p_t}\, w_t & \text{if} \quad \omega_{t+1}=2 \end{cases} \tag{2.1a}$$

where

$$p_t = \alpha^1(p_t,p_{t-1})w_t + \alpha^2(p_t,p_{t-1})(1-w_t)\,. \tag{2.1b}$$

Since rules depend also on contemporaneous prices, the existence and uniqueness of a positive market clearing price, i.e. a positive solution of (2.1b), cannot be given for granted. In order to proceed we simplify further our example and assume that rules depend either on current or past prices. We start from the latter dependence.

## 2.1 Dependence on past prices

First notice that with Arrow securities rules must be strictly positive, otherwise a trader would, almost surely, have zero or negative wealth in finite time. Moreover, given the budget constraint, invested wealth shares must also be in $(0,1)$. It follows that when rules depend only on past prices, (2.1b) defines a unique price in the interval $(0,1)$. The random dynamical system (2.1) is well defined and has two variables, the current wealth of agent 1 and the previous price of asset 1. We name $x_t=(w_t,p_{t-1})$ the market state vector and define the maps $f_1$ and $f_2$ so that (2.1) can be written as

$$x_{t+1} = f_{\omega_{t+1}}(x_t)\,.$$

To investigate the long-run behavior of wealth and prices we shall first focus on specific market configurations with constant prices and constant wealth. We shall

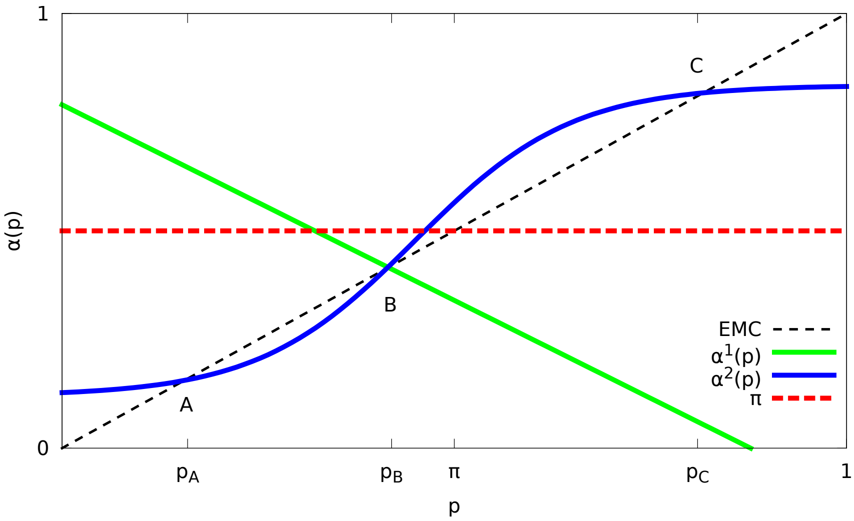


Figure 1: Investment rules $\alpha^1(p)$ and $\alpha^2(p)$ are plotted against the price of asset 1. Equilibrium prices are given by the coordinate of their intersections with the EMC, that is, points $p_A$, $p_B$ and $p_C$. The local stability of market selection equilibria $A, B, C$ can be appraised graphically with the aid of the line $\pi$.

name these states the Market Selection Equilibria (MSE, c.f. Section 3.3). A MSE is a deterministic fixed points, that is, a state $x^*$ such that

$$x^* = f_1(x^*) \quad \text{and} \quad x^* = f_2(x^*)\,.$$

Straightforward computations show that in general there exist only two types of MSE. Either one agent has all wealth, which occurs at $x_1^* = (w^* = 1, p_1^* = \alpha^1(p_1^*))$ and $x_2^* = (w^* = 0, p_2^* = \alpha^2(p_2^*))$, or both agents have some wealth, which occurs at $x_{1,2}^* = (w^*, p_{1,2}^* = \alpha^1(p_{1,2}^*) = \alpha^2(p_{1,2}^*))$. In both cases prices are fixed points of any investment rule $i$ with positive wealth at the MSE, $p^* = \alpha^i(p^*)$.

It is useful to visualize the location of a MSE in a plot. In Fig. 1 we plot two generic portfolio rules as a function of the (lagged) price $p$. Each intersection of a rule (demand) with the diagonal (supply) —points $A$, $B$ and $C$ in the plot— represent the price of a possible MSE. In $A$ and $C$ the associated MSE has $w^* = 0$: all the wealth is owned by agent 2. These are *single survivor* equilibria and we say that in $A$ and $C$ agent 2 dominates and agent 1 vanishes (c.f. Definition 3.5 for a formal statement). Conversely, at a *multiple survivor* equilibria of the $x_{1,2}^*$ type, like point $B$, both agents survive and none vanishes. Notice that a *multiple survivor* equilibrium requires that both rules intersect the supply curve at the same price, requirement that makes it non-generic.

No matter the shape of the investment rules, MSE lie on the function $f(p) = p$, the value of the first asset supply. In our previous work, (Anufriev et al., 2006; Anufriev and Bottazzi, 2010; Anufriev and Dindo, 2010) we have named this set the "Equilibrium Market Curve" (EMC) since it is the locus of all long-run equilibria. We keep this terminology here. In this example $p_B$ is closer than $p_A$ or $p_C$ to the fundamental price of asset 1, i.e. its (discounted) expected dividend $\pi$. Thus, we can say that market selection implies informationally efficiency if long-run prices converge almost surely and for almost all initial conditions to $p_B$. A necessary condition is that $B$ must be asymptotically stable, and $A$ and $C$ unstable. It turns out that in this example with two traders and one lagged price, there are two conditions that, if contemporaneously verified, are sufficient for the local stability of a MSE.

The first condition is related to average wealth growth rates at MSE prices $p^*$. Consider for example $x_2^* = (0, p_2^* = \alpha^2(p_2^*))$, where trader 2 has all the wealth. If close to $x_2^*$ the average wealth growth rate of trader 1 is positive then, on average, when $p$ is near to $p_2^*$ the wealth of trader 1 increases while the wealth of trader 2 shrinks (the total wealth is constant). The wealth $w_t$ does not converge to $w^* = 0$ and $x_2^*$ is unstable. Thus for $x_2^*$ to be asymptotically stable the average wealth growth rate of agent 1 must be negative in a neighborhood of $x_2^*$. The stability of $x_1^*$ can be appraised along the same argument.

By the Law of Large Numbers the average (gross) growth rate at the MSE $x^* = (w^*, p^*)$ can be computed from (2.1a) as

$$\mu(w^*, p^*) = \left( \left( \frac{\alpha^2(p^*)}{\alpha^1(p^*)} \right)^{\pi} \left( \frac{1-\alpha^2(p^*)}{1-\alpha^1(p^*)} \right)^{1-\pi} \right)^{2w^*-1} . \tag{2.2}$$

The exponent $(2w^* - 1)$ ensures that at any $x^*$ where a single agent survives we are computing the average (gross) growth rate of the vanishing agent at the prices set by the surviving one. It follows that if $\mu(w^*, p^*) < 1$ the MSE $x^*$ is asymptotically stable: close enough to $x^*$ the wealth of the vanishing agent decreases and the market is driven back to $x^*$. When instead $\mu(w^*, p^*) > 1$, $x^*$ in unstable. Notice that if both agents survive, as in $B$ in Fig. 1, then the average growth rate $\mu(w^*, p^*)$ is equal to one and it is not informative about the local stability.

Interestingly, these conditions can be given in terms of the relative entropy of the investment rule of agent $i$ at price $p$, $(\alpha^i(p), 1 - \alpha^i(p))$, with respect to the (invariant) distribution $(\pi, 1 - \pi)$:

$$I_\pi(\alpha^i(p)) := \pi \log \frac{\pi}{\alpha^i(p)} + (1-\pi) \log \frac{1-\pi}{1-\alpha^i(p)} . \tag{2.3}$$

Since $\log(\mu(w^*, p^*))$ is equal to the relative entropy of the surviving rule minus the relative entropy of the vanishing rule, both computed at equilibrium prices, the

MSE $x^* = (w^*, p^*)$ can be asymptotically stable only if the surviving agent is the one whose investment rule has, at equilibrium prices, the lowest relative entropy. Stating stability in terms of relative entropy has the advantage that survival can be directly appreciated in the EMC plot. In Fig. 1 the line $\pi$ represents the probability of occurrence of state 1. The distance between this line and $\alpha^i(p)$ at a given price $p$ is an increasing function[5] of $I_\pi(\alpha^i(p))$. Consider the point $A$ where agent 2 dominates and 1 vanishes, that is, $w^* = 0$ and $p^* = p_A = \alpha^2(p_A)$. Since at $p_A$ the distance from $\pi$ is larger for $\alpha^2(p_A)$ than for $\alpha^1(p_A,)$, it is $I_\pi(\alpha^1(p_A)) < I_\pi(\alpha^2(p_A))$ and the equilibrium $(w^*, p^*) = (0, p_A)$ is unstable. Conversely, at $p_C$ the curve nearest to $\pi$ is $\alpha^2$, so that, at least according to this criterion, the equilibrium $(0, p_C)$ is asymptotically stable.

Relative entropy, or relative average wealth growth rate, is not the only determinant of local stability. When past realized prices influence current investment decisions it could happen that the strength of price feedbacks is too strong for the dynamics to settle down. In a specific MSE $x^*$, this strength can be directly appraised by looking at

$$\lambda(w^*, p^*) = w^* \left.\frac{\partial \alpha^1(p)}{\partial p}\right|_{p=p^*} + (1 - w^*) \left.\frac{\partial \alpha^2(p)}{\partial p}\right|_{p=p^*} . \tag{2.4}$$

If $|\lambda(w^*, p^*)| < 1$ then $x^*$ is stable with respect to price feedbacks. Otherwise, when $|\lambda(w^*, p^*)| > 1$, $x^*$ is unstable.

Also this second source of instability can be appraised graphically. In the example of Fig. 1, given the slope of $\alpha^2(p)$ at $p_A$ and $p_C$, we infer that both $A$ and $C$ are stable under past price feedbacks. Since $C$ is also stable in terms of relative entropy, it represents an asymptotically stable single survivor equilibrium and a possible outcome of the long-run market dynamics. $C$ is the unique single survivor stable equilibrium, but may not the unique stable long-run equilibrium. We have still to evaluate the stability with respect to past price feedbacks of $B$, where both agents survive. Locally, the market dynamics is equivalent to the dynamics generated by a single agent whose investment rule is the wealth weighed average of both surviving rules. Since $|\partial \alpha^1(p_B)| < 1$ and $|\partial \alpha^2(p_B)| > 1$, if $w$ is large enough then, by continuity, $|\lambda(w^*, p^*)| < 1$ and the point is stable. Conversely, for smaller values of $w$, the over-reaction to price movements of $\alpha^2$ destabilizes the equilibrium. Notice at last that since in $B$ investment decisions of both agents are equal, the distance of their rules in terms of relative entropy is zero and $\mu(w^*, p_B) = 1$. For this reason if $|\lambda(w^*, p^*)| < 1$ the fixed point is stable but not asymptotically stable. A perturbation can indeed generate a permanent change in the distribution

[5] Moreover when $\pi = 0.5$, as we often use, the relative entropy is symmetric around the line $\pi$.

of wealth. Prices converge back to their equilibrium level $p_B$ but the system ends up in a fixed point with a different value of $w^*$.

We have established that whereas $C$ is stable, $A$ is unstable, and $B$ may be stable. Depending on initial conditions and realizations of $\omega$, long-run prices may converge to $p_C$ or to $p_B$. Only in the second case they converge as close as possible, given the two investment rules, to assets' fundamental value. Moreover long-run prices may also fail to converge at all and instead exhibit persistent fluctuations. This is always the case when all selection equilibria are unstable. We shall give more examples in Section 6.

## 2.2 Dependence on current prices

Consider the case in which investment rules depend on *current* but not on past prices. The random dynamical system (2.1) becomes

$$w_{t+1} = \begin{cases} \frac{\alpha^1(p_t)}{p_t} w_t & \omega_{t+1} = 1 \\ \\ \frac{1-\alpha^1(p_t)}{1-p_t} w_t & \omega_{t+1} = 2 \end{cases} , \tag{2.5a}$$

where $p_t$ is a solution of

$$p_t = \alpha^1(p_t) w_t + \alpha^2(p_t)(1 - w_t)\,. \tag{2.5b}$$

Contrary to (2.1b), the pricing equation (2.5b) can possess multiple or no positive solutions at all, so that the global dynamics may be ill-defined. However, MSE can be defined as in the previous case. Moreover, as long as the intersections of each investment rule with the EMC define positive and isolated prices, the dynamics implied by (2.5a-2.5b) is well defined in a neighborhood of the MSE. Due to the differentiability assumption of the $\alpha$s, it is sufficient to require that $w^*\partial_p\alpha^1(p^*) + (1 - w^*)\partial_p\alpha^2(p^*) \neq 1$.

Here stability of $x^*$ does not depend on $\lambda(w^*, p^*)$: since the investment rules do not depend on past prices there is no room for the destabilizing role of price feedbacks. If we interpret the rules in Fig. 1 as functions of current prices, the slopes at the MSE $A$, $B$, and $C$, are no more related to feedback stability but to the fact that the local dynamics is well defined. Graphical analysis implies that in $A$ and $C$ the local dynamics is well defined. Instead in $B$ there exists a value of $w^*$ such that the weighted investment rule has slope 1 and the local dynamics is degenerate. Whenever the local dynamics around $x^*$ is well defined, its local stability depends only on $\mu(w^*, p^*)$, the average growth rate of the dominated trader, or, graphically, by the relative distance of each investment rule from $\pi$.

## 2.3 Relative Entropy and Price Feedback

How do the two sources of instability discussed above, relative entropy differences and price feedback, compare with known results from the related literature?

Relative entropy determines average relative wealth growth. This is the same force driving results in Blume and Easley (1992) or the works surveyed by Evstigneev et al. (2009). Our toy-market example shows that relative entropy differences, and thus the relative performance of endogenous rules, is a function of market states. As a consequence, it can well happen, as in the example of Fig. 1, that multiple selection equilibria exist and are stable, or that they are all unstable. As we shall show in Section 6, in both cases the trader whose selection equilibrium prices are closest to fundamentals is not granted to dominate. Conversely, in markets for Arrow securities with exogenous, i.e. not price dependent, rules local stability always implies global stability and the trader who sets prices closest to fundamentals attracts all the wealth in the long run with probability one (c.f. Proposition 3.1 in Blume and Easley, 1992). Relative wealth growth rates determine stability and instability of long-run equilibria also in the deterministic, and growing, economies studied in Anufriev et al. (2006); Anufriev and Bottazzi (2010); Anufriev and Dindo (2010). The main difference of the present example is that due to its stochastic nature, what matters for stability are expected growth rates, or relative entropy differences.

The second source of instability, price feedback, is typical in the literature on financial market models with heterogeneous agents (see e.g. the survey by Hommes, 2006). The results we find are again similar to those in Anufriev et al. (2006); Anufriev and Bottazzi (2010); Anufriev and Dindo (2010): a rule might fail to dominate the market because of its overreaction to price movements. This source of instability is similar to that of the deterministic overshooting observed in price adjustment processes.

In the next section we present our full model and, in Section 4, our formal results. Apart a few subtle details we have to cope with, the general conclusions remain the same.

# 3 The model

Given the set of states of the world $\Sigma = \{1, \ldots, s, \ldots, S\}$, we define the set of sequences $\Omega := \prod_{-\infty}^{+\infty} \Sigma$ with elements $\omega = (\ldots, \omega_0, \ldots, \omega_t, \ldots)$. For all $t \in \mathbb{Z}$ we denote with $\{\omega\}_t = \omega_t$ and with $\omega^t$ all the elements of $\Omega$ with the same history till time $t$. Let $\mathcal{R}$ be the cylinder $\sigma$-algebra and $\rho$ a probability measure on $\mathcal{R}$ so that $(\Omega, \mathcal{R}, \rho)$ is a well-defined probability space. Each $\mathcal{R}^t$ is defined as the $\sigma$-algebra generated by partial histories $\omega^t$. Name $\theta$ the shift operator on $\Omega$, so that

$\{\theta\omega\}_t = \{\omega\}_{t+1}$, and $\theta^t$ the $t$-times composition of $\theta$, so that $\{\theta^t\omega\}_{t'} = \{\omega\}_{t'+t}$. We assume that the measure $\rho$ is such that if $f$ is an integrable real random variable on $\Omega$ it holds

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f(\theta^t\omega) = \int_\Omega d\rho(\omega)\, f(\omega) \quad \text{almost surely (a.s.) on } \Omega \,. \tag{3.1}$$

When $\theta$ in a measure-preserving and ergodic transformation of $\Omega$ the above is a consequence of the Ergodic Theorem.

We study a market where, in each period $t$, $I$ agents are trading $K$ short-lived assets in constant exogenous unitary supply. We denote agent $i$ wealth in period $t$ as $W_t^i$, and asset $k$ price in period $t$ as $P_{k,t}$, using the vector notations $W_t = (W_t^1, \ldots, W_t^I)$ and $\mathbf{P}_t = (P_{1,t}, \ldots, P_{K,t})$ when appropriate.

Asset dividend payoffs are paid in terms of an homogeneous good, the numéraire of the economy. They depend on the past histories $\omega^t$ but not explicitly on time. Formally, we consider $K$ real random variables on $\Omega$, $D_k$ with $k = 1, \ldots, K$, and define the dividend payed at time $t+1$ by asset $k$ traded in $t$ as

$$D_{k,t}(\omega) = D_k(\theta^t\omega) \,.$$

We assume that $D(\omega) = (D_1(\omega), \ldots, D_K(\omega))$ is measurable with respect to $\mathfrak{R}^1$, so that the dividend process $D_t(\omega) = (D_{1,t}(\omega), \ldots, D_{K,t}(\omega))$ is adapted to the filtration $\{\mathfrak{R}^t\,,\, t \in \mathbb{Z}\}$. From (3.1) it is[6]

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} D_k(\theta^t\omega) = \int_\Omega d\rho(\omega)\, D_k(\omega) \quad \text{a.s. on } \Omega \,.$$

Without loss of generality, we assume that that there are no redundant assets and the process $D$ is non-trivial, that is, $\sum_{k=1}^{K} D_k(\omega) > 0$ almost surely and, for each $k$, $D_k(\omega) > 0$ with positive probability. The first assumption assigns probability zero to the event that the total wealth in a given period is zero. The second assumption implies that each asset will sooner or later pay a positive dividend.

Although we shall give results for the general dividend process, in most examples we assume a fixed dividend matrix, that is, the existence of a $K \times S$ matrix $D$ such that

$$D_k(\omega) = D_{k,\omega_1}\,, \quad k = 1, \ldots, K \,.$$

Non-triviality of the dividend process is equivalent to assume that $D$ is positive and has no zero rows nor zero columns. With a fixed matrix, if we define the probability measure $\pi = (\pi_1, \ldots, \pi_S)$ on $\Sigma$ such that

$$\rho(\{\omega \in \Omega | \omega_t = s\}) = \pi_s\,, \quad s = 1, \ldots, S \quad \text{for every } t \in \mathbb{Z}\,,$$

[6]This will be a key property in the proofs of Section 4.

we can easily compute

$$\int_\Omega d\rho(\omega)\, D_k(\omega) = \sum_{s=1}^{S} \pi_s D_{k,s}\,.$$

Notice at last that dividend matrices, both when they are fixed and when they are history dependent as in the general case, could have rank lower than $S$, making short-term trading incomplete.

Agents use assets to transfer wealth inter-temporally. We denote the fractions of wealth invested in asset $k$ by agent $i$ at time $t$ with $\alpha^i_{k,t}$. The wealth in period $t+1$ depends on these fractions and on the dividends payed at time $t+1$

$$W^i_{t+1} = \sum_{k=1}^{K} \frac{\alpha^i_{k,t} W^i_t}{P_{k,t}} D_{k,t}(\omega)\,, \quad i = 1, \ldots, I\,, \tag{3.2a}$$

where, having normalized asset supplies to one, market clearing prices are set by

$$P_{k,t} = \sum_{i=1}^{I} \alpha^i_{k,t} W^i_t\,, \quad k = 1, \ldots, K\,. \tag{3.2b}$$

Since assets are in unitary supply and their dividend is the only source of future wealth, the total wealth in each period is given by the sum of asset dividends paid in the state of the world just realized. Thus the total wealth can be used to define normalized dividends, individual wealth, and prices

$$d_{k,t}(\omega) = \frac{D_{k,t}(\omega)}{\sum_{h=1}^{K} D_{h,t}(\omega)}\,, \quad w^i_t = \frac{W^i_t}{\sum_{h=1}^{K} D_{h,t}(\omega)}\,, \quad p_{k,t} = \frac{P_{k,t}}{\sum_{h=1}^{K} D_{h,t}(\omega)}\,,$$

so that

$$\sum_{i=1}^{I} w^i_t = \sum_{k=1}^{K} d_{k,t}(\omega) = 1\,, \quad t \in \mathbb{Z}\,. \tag{3.3}$$

In what follows we shall use $d$ to denote the normalized dividend process, or normalized dividend matrix. Notice that equations (3.2a-3.2b) are unchanged when written in terms of normalized variables.

## 3.1 Endogenous investment rules

The dynamics of wealth distribution and its effect on asset prices depends on the types of investment behaviors considered. A first, natural, assumption to be imposed on asset demands is that they should satisfy a budget constraint or

$$\sum_{k=1}^{K} \alpha^i_{k,t} = 1 - \alpha^i_{0,t}\,, \quad i = 1, \ldots, I, \quad t \in \mathbb{Z}\,,$$

where $\alpha^i_{0,t}$ denotes the fraction of wealth consumed by agent $i$ in $t$. We assume that consumption cannot be negative nor exhaust all the wealth, so that

$$\sum_{k=1}^{K} \alpha^i_{k,t} \in (0,1]\,, \quad i = 1,\dots,I, \quad t \in \mathbb{Z}\,. \tag{3.4}$$

Together with the market clearing equation (3.2b) the latter implies a bound on normalized prices

$$\sum_{k=1}^{K} p_{k,t} = \sum_{i=1}^{I} (1-\alpha^i_{0,t}) w^i_t \le 1, \quad t \in \mathbb{Z}. \tag{3.5}$$

A second assumption is that demands in $t$ are such that the probability that an agent is left without wealth at time $t+1$ is zero, i.e.

$$w^i_{t+1} = \sum_{k=1}^{K} \frac{\alpha^i_{k,t} d_{k,t}(\omega)}{p_{k,t}} w^i_t > 0 \quad \text{a.s. on } \Omega. \tag{3.6}$$

Assumptions (3.4) and (3.6) are natural because consuming all the wealth or not being diversified enough imply an (almost) sure exit and thus no impact on the long-run market dynamics.

Finally we assume that investment fractions depend on market variables, in particular current and past $L$ prevailing prices, but not on the histories of states of the world nor on past dividends. Within the expected utility framework, this is consistent with the maximization of CRRA expected utilities with any coefficient of relative risk aversion, a fixed dividend matrix, and fixed i.i.d. beliefs about the probabilities of the states of the world. However, since rules do not depend on wealth, other types of maximizing behaviors, such as CARA utilities, are excluded. Rules not coming from the maximization of expected utility can be also considered, as long as they are expressed as functions of prices and do not depend on histories $\omega^t$ nor on past dividends. Notice however that investment rules may depend also on past prices. This dependence could exist for a number of reasons: past prices could be employed to predict future asset prices, which is needed when an agent investment horizon is longer than one period; they could be used to infer about the dividend process, acknowledging that an asset prices may reflect the properties of its dividend process; or they could be used to evaluate if assets are mis-priced.[7]

All the assumptions on rules are formalized in the following

[7]We are aware that the most general investment rules should depend, other than on prices, also on the realizations of states of the world and/or of the stochastic dividend process, as well as on agent wealth. Mathematical tractability constraints us to the present setting. We leave the inclusion of a feedback from the dividend process and other dependencies to future work.

**Definition 3.1.** An *endogenous investment rule* is a vector function $\alpha : \mathbb{R}^{(L+1)K} \to \mathbb{R}^K$ such that

**A1** the fraction of wealth invested in period $t$ on asset $k$ is

$$\alpha_{k,t} = \alpha_k(p_t) \quad k = 1, \dots, K\,, \tag{3.7}$$

where $p_t = (\mathbf{p}_t, \mathbf{p}_t^1, \dots, \mathbf{p}_t^L)$ with[8] $\mathbf{p}_t^l = \mathbf{p}_{t-l}$;

**A2** the total wealth invested in assets is always positive and never greater than one, $0 < \sum_{k=1}^K \alpha_k(p) \leq 1$, and what is not invested is consumed, $\alpha_{0,t} = 1 - \sum_{k=1}^K \alpha_k(p_t)$;

**A3** assets portfolios are sufficiently diversified, that is $\sum_{k=1}^K \alpha_k(p) d_k(\omega)/p_k > 0$ almost surely.

Throughout the paper we shall name $\mathcal{A}$ the set of endogenous investment rules and assume that $\alpha^i \in \mathcal{A}$ for each agent $i$. **A1** specifies that given a rule, investment decisions are time-invariant functions of normalized prices.[9] **A2** comes from (3.4) and the consumption bounds. **A3** comes from (3.6) and guarantees that agents' portfolios are sufficiently diversified. Since we have assumed that the dividend process is non-trivial, the set of positive rules always belong to $\mathcal{A}$. Moreover, if the dividend process $d$ is diagonal (as in the case of Arrow securities) **A3** implies that all rules are positive, $\alpha_k^i(p) > 0$, a no short-selling constraint. Notice, however, that for the general dividend process short-selling is allowed. As a consequence, when prices are such that arbitrage opportunities exist our rules can exploited them.[10] Some examples of rules in the class $\mathcal{A}$ are provided in the applications of Sections 5 and 6 as well as in Bottazzi and Dindo (2013).

## 3.2 Market Dynamics

Given asset dividends, a set of investment rules, a history of prices, and an initial wealth distribution, wealth update (3.2a) and market clearing (3.2b) determine the dynamics of asset prices and agents' wealth for all subsequent periods.

[8] The compact notation for lagged prices allows $l$ to be equal to 0, in which case trivially $\mathbf{p}_t^0 = \mathbf{p}_t$ and $p_{k,t}^0 = p_{k,t}$.

[9] In our framework rules depend on normalized prices $\mathbf{p_t}$ rather than on $\mathbf{P_t}$. If demands come from utility maximization, rules are homogeneous of degree zero, so that the two formulations are equivalent.

[10] Since dividends are assumed to be positive an arbitrage must involves a short position in some assets. Given prices $\mathbf{p}$ and the dividend process $d$, the portfolio $\mathbf{y} \in \mathbb{R}^K$ is an arbitrage if $\mathbf{p}\,\mathbf{y} \leq 0$ and $\mathbf{y}\,d(\omega) \geq 0$ almost surely, with at least one of the inequalities being strict.

If investment rules do not depend on current prices, (3.2b) explicitly determines the unique vector of market clearing prices. Conversely, when some of the rules depend on contemporaneous prices, prices are implicitly fixed through a system of $K$ equations. Given the generality of $\mathcal{A}$, existence and uniqueness of a set of positive market clearing prices is in general not guaranteed.

The derivation of sufficient conditions on the set of rules that guarantee local uniqueness is postponed to the formal analysis of Section 4. At the present stage we shall just assume, for the sake of exposition, that for almost all sequences $\omega$ and all states $x_t = (w_t, p_t)$ there exist $K$ positive maps $f_k$ such that

$$p_{k,t+1} = f_k(x_t; \theta^t \omega)\,, \quad k = 1, \dots, K\,, \tag{3.8}$$

and the market dynamics implied by (3.2) and endogenous rules is equivalent to the following system of $I + K(L+1)$ equations

$$\mathcal{F}(\theta^t\omega)x_t := \begin{cases} w^1_{t+1} &= \sum_{k=1}^K \frac{\alpha^i_k(p_t)\, w^i_t}{p_{k,t}}\, d_k(\theta^t\omega)\,, & i = 1, \dots, I\,, \\ p_{k,t+1} &= f_k(x_t; \theta^t\omega)\,, & k = 1, \dots, K\,, \\ p^l_{k,t+1} &= p^{l-1}_{k,t}\,, & k = 1, \dots, K\ , l = 0, \dots, L\,. \end{cases}$$

Let $\Delta^{K-1}$ denote the simplex in $\mathbb{R}^K$ with the canonical base as vertexes and $\Delta^K_c$ denote the simplex in $\mathbb{R}^K$ with the canonical base plus the origin as vertexes, or

$$\Delta^K_c = \left\{ \mathbf{x} \in \mathbb{R}^K \,\middle|\, \sum_{k=1}^K x_k \le 1 \quad \text{and} \quad x_k \ge 0,\ k = 1, \dots, K \right\}.$$

Name $\Delta^{K-1}_+$ and $\Delta^K_{c+}$ their respective subsets with all positive components. Due to normalizations in (3.3) and (3.5), and to the assumed positivity of (3.8), each $\mathcal{F}(\cdot)$ maps the set $\mathcal{X} = \Delta^{I-1} \times (\Delta^K_{c+})^{L+1}$ in itself. The first $I$ components of $\mathcal{F}$ characterize the dynamics of agents' wealth fractions, whereas the other components fix prices using market clearing and keeps track of their past values. The Random Dynamical System $\varphi : \mathbb{Z} \times \Omega \times \mathcal{X} \to \mathcal{X}$ representing the market dynamics is defined iterating $\mathcal{F}(\cdot)$ and $\theta$. For any given period $t$, initial state $x_0$, sequence $\omega$, the market state in $t$ is

$$\varphi(t, \omega, x_0) = \mathcal{F}(\theta^{t-1}\omega) \circ \ldots \circ \mathcal{F}(\theta\omega) \circ \mathcal{F}(\omega)x_0. \tag{3.9}$$

A set of initial conditions $x_0 \in \mathcal{X}$ and a realization of the stochastic process $\omega \in \Omega$ define a trajectory of the system. The map $\mathcal{F}$ may not preserve the properties of the measure $\rho$. As a result, even if we assume an ergodic dividend process, the price and wealth process may not be ergodic. For this reason, and given the arbitrariness of the dividend process, population size $I$, memory span $L$, and the generality of rules in $\mathcal{A}$, the following analysis concentrates on the local dynamics. We postpone the analysis of the global dynamics generated by (3.9) to future work.

### 3.3 Market selection equilibria

Our approach to market selection consists in studying the existence and stability of long-run market states where one or a group of traders gain all the wealth and (normalized) asset prices are positive and constant.[11] As already anticipated in Section 2, we name these long-run outcomes Market Selection Equilibria (MSE). Technically MSE are given by the deterministic fixed points of (3.9) or

**Definition 3.2.** The state $x^* = (w^*, p^*) \in \mathcal{X}$ is a *Market Selection Equilibrium* if it is a deterministic fixed point of the Random Dynamical System $\varphi$ generated by the map $\mathcal{F}(\cdot)$ and by the shift operator $\theta$, i.e.

$$\mathcal{F}(\theta^t\omega)x^* = x^* \tag{3.10}$$

for almost all $\omega \in \Omega$ and for every $t \in \mathbb{Z}$.

In a Market Selection Equilibrium the allocation of wealth implied by the asset exchanged has come to a rest point. Those trader who retain a positive amount of wealth are thus the winners of the competitive struggle and, given the pricing equation, they set equilibrium prices.

Not all MSE represent interesting asymptotic states, however. Indeed, in order for the market dynamics to actually converge to a MSE, at least when starting from a neighborhood of it, the equilibrium must be asymptotically stable.

**Definition 3.3.** A Market Selection Equilibrium $x^*$ is *asymptotically stable* if, for almost all $\omega \in \Omega$ and for all $x$ in a neighborhood $U(\omega)$ of $x^*$, $\lim_{t\to\infty} ||\varphi(t, \omega, x) - x^*|| = 0$.

For some equilibria we use the weaker notion of stability, which is sufficient to guarantee that orbits do not diverge from a deterministic fixed point when initial conditions are sufficiently close to it.

**Definition 3.4.** A Market Selection Equilibrium $x^*$ is *stable* if, for any neighborhood $V$ of $x^*$ and for almost all $\omega \in \Omega$, there exists a neighborhood $U(\omega) \subseteq V$ of $x^*$ such that $\lim_{t\to\infty} \varphi(t, \omega, x) \in V$ for all $x$ in $U(\omega)$.

In the previous definitions the neighborhood $U$ might depend on the realization of the process $\omega$. If a MSE is neither asymptotically stable nor stable we shall say that it is *unstable*.

When characterizing MSE and their local stability the following terminology is useful.

[11] Since we have assumed that investment rules do not depend on histories of states of the world nor on past dividends, prices can be constant when wealth fractions are constant.

**Definition 3.5.** An agent $i$ is said to *survive on a given trajectory* generated by the dynamics (3.9) if $\limsup_{t\to\infty} w_t^i > 0$ on this trajectory. Otherwise, an agent $n$ is said to *vanish on that trajectory.* A surviving agent $i$ is said to *dominate on a given trajectory* if she is the unique survivor on that trajectory, that is, $\liminf_{t\to\infty} w_t^i = 1$

Notice that survival and dominance are defined only with respect to a given trajectory. A trader may survive on a given trajectory but vanish on others. A similar definition is given in Blume and Easley (1992) for a stochastic setting like ours[12] and in Anufriev and Bottazzi (2010) and Anufriev and Dindo (2010) for a deterministic setting.

Applying the previous definition to a MSE $x^*$, we shall say that agent $i$ dominates at $x^*$ if $w^{*i} = 1$, she survives if her wealth share is strictly positive, $w^{*i} > 0$, while she vanishes if $w^{*i} = 0$. Such taxonomy can be applied both to a stable or unstable MSE, but the implications are very different in the two cases. When the equilibrium is stable, all trajectories starting in a neighborhood of the equilibrium stay close to it, so that a survivor in the MSE also survives on all these trajectories. If, moreover, the agent is the unique survivor and the equilibrium is also asymptotically stable, then that agent dominates on all trajectories starting inside a proper neighborhood. Conversely, when the equilibrium is unstable, the trader who dominates or has positive wealth in the MSE might not dominate or might even vanish with positive probability when initial conditions are close to the MSE.

In the rest of the paper we shall show that the ergodicity imposed on the dividend process, the market clearing equations, and the wealth evolution are sufficient to uniquely characterize all Market Selection Equilibria and to derive their local stability conditions.

# 4 Local Stability of Market Selection Equilibria

In this section we derive results about the existence and local stability of Market Selection Equilibria for the market dynamics described by (3.9). In presenting our findings it is convenient to treat the case of single survivor equilibria first and move to the multiple survivors case at a later stage.

## 4.1 Single survivor equilibria

As previously noted one cannot assume that positive market clearing prices exist and are unique for any set of rules in $\mathcal{A}$ and any initial conditions $x_0$. Since all our results about long-run prices and wealth are local, it is sufficient to concentrate

[12] Notice however that in Blume and Easley (1992) *dominance* is defined as $\liminf_{t\to\infty} w_t^i > 0$ so that, as for survival, more than one rule may dominate on a given trajectory.

on the dynamics close to specific market states. In what follows, our first step is to characterize single survivor MSE with positive prices. Then, we will provide sufficient conditions for local uniqueness of market clearing prices around a MSE. This guarantees that the *local* dynamics is well defined. Finally we characterize the asymptotic stability of single survivor MSE.

At a MSE where a single trader has all the wealth and prices are constant, the market clearing conditions (3.2b) imply that asset prices are fixed points of the investment rule that dominates. Using the terminology and graphical examples of Section 2, this is equivalent to find the intersection of every component of the rule assumed to dominate with the corresponding component of the EMC by setting $\alpha(\mathbf{p}, \mathbf{p}^1, \ldots, \mathbf{p}^L) = \mathbf{p}$. If no other rules share the same fixed point prices, i.e. if no other rule intersects the EMC at the same prices, we have a *single survivor equilibrium.*

**Theorem 4.1.** *Consider a market for $K$ short-lived assets with non-trivial dividend process $d$, where $I$ agents invest according to rules in $\mathcal{A}$ using $L$ price lags. Assume agents' wealth and asset prices evolve according to $\varphi$ in* (3.9)*. The state $x^* = (w^*, p^*)$ with $w^{i*} = 1$ and $w^{j*} = 0$ for $j \neq i$ is a single survivor Market Selection Equilibrium where trader $i$ dominates if and only if $p^* = (\mathbf{p}^*, \ldots, \mathbf{p}^*) \in (\Delta^K_{c+})^{L+1}$ solves $\alpha^i(p^*) = \mathbf{p}^*$, but $\alpha^j(p^*) \neq \mathbf{p}^*$ for $j \neq i$.*

At a single survivor MSE $x^*$, a unique trader has all the wealth and fixes positive asset prices. Moreover, a transfer of wealth to any other trader would imply a price change. Given $x^*$, a well-defined local dynamics exists provided that the excess demand function has an isolated zero in $p^*$. Sufficient conditions can be obtained using the implicit function theorem:

**Theorem 4.2.** *Under the hypothesis of Theorem 4.1, let $x^*$ be a single survivor MSE where, without loss of generality, the $I$-th trader dominates. Assume further that all investment rules $i \in \{1, \ldots, I\}$ are continuously differentiable in a neighborhood of $p^*$, $\alpha^i \in \mathcal{C}^1(p^*)$. If the matrix*

$$H := \begin{pmatrix} (\alpha^I_1)^{1,0} - 1 & (\alpha^I_1)^{2,0} & (\alpha^I_1)^{3,0} & \ldots & (\alpha^I_1)^{K,0} \\ (\alpha^I_2)^{1,0} & (\alpha^I_2)^{2,0} - 1 & (\alpha^I_2)^{3,0} & \ldots & (\alpha^I_2)^{K,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\alpha^I_K)^{1,0} & (\alpha^I_K)^{2,0} & (\alpha^I_K)^{3,0} & \ldots & (\alpha^I_K)^{K,0} - 1 \end{pmatrix} , \tag{4.1}$$

*where*

$$(\alpha^i_k)^{h,l} := \left. \frac{\partial \alpha^i_k\,(p)}{\partial p^l_h} \right|_{x^*} , \quad i = 1, \ldots, I \,, \quad l = 0, 1, \ldots, L \,, \quad k, h = 1, \ldots, K \,, \tag{4.2}$$

*is non-singular, then the dynamics is locally well-defined, that is, there exists a neighborhood $X$ of $x^*$ where prices are positive and* (3.9) *is well defined.*

Importantly the set $X$ does not depend on $\omega$ so that the Random Dynamical System $\varphi$ is locally well defined. Notice that when the investment rule of agent $I$ does not depend on current prices, then $H = -\mathbb{I}$ so that the non-singularity condition is trivially fulfilled.

The crucial issue is now to assess whether an agent dominating or vanishing at a MSE is also dominating or vanishing on all trajectories starting close enough to it. The next theorem provides sufficient conditions for the asymptotic stability or instability of a single survivor MSE.

**Theorem 4.3.** *Under the hypothesis of Theorem 4.2, consider the fixed point $x^* = (w^*, p^*)$ of Theorem 4.1 where $w^{I*} = 1$ and $\alpha^I(p^*) = \mathbf{p}^*$ and assume that the matrix $H$ defined in* (4.1) *is non-singular. Define*

$$\mu_i := \exp \int_\Omega d\rho(\omega) \log \left( \sum_{k=1}^{K} \frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} d_k(\omega) \right) , \quad i = 1, \ldots, I-1 , \tag{4.3}$$

*and*

$$P(\lambda) := \sum_{l_1=1}^{L} \cdots \sum_{l_K=1}^{L} \lambda^{LK - \sum_j l_j} \sum_{\sigma} sgn(\sigma) \prod_{k=1}^{K} \left( (\bar{\alpha}_k^I)^{\sigma_k, l_{\sigma_k}} - \lambda\, \delta_{k,\sigma_k}\, \delta_{l_{\sigma_k},1} \right) \tag{4.4}$$

*where*

$$(\bar{\alpha}_k^I)^{h,l} := -\sum_{k'=1}^{K} \{H^{-1}\}_{k,k'} (\alpha_{k'}^I)^{h,l} \quad l = 0, 1, \ldots, L , \quad k, h = 1, \ldots, K ,$$

*$(\alpha_{k'}^I)^{h,l}$ are defined in* (4.2)*, $\delta$ is Kronecker's delta, and $\sigma$ are the permutations of the set $\{1, \ldots, K\}$. If $\mu_i < 1$ for all $i = 1 \ldots, I-1$ and all the roots of $P(\lambda)$ have absolute value smaller than one, then the fixed point $x^*$ is asymptotically stable. If, for some $i$, $\mu_i > 1$ or if a root of $P(\lambda)$ has absolute value greater than one, then the fixed point $x^*$ is unstable.*

The quantity $\mu_i$ defined in (4.3) is trader $i$ average wealth growth rate when prices are determined by trader $I$. If $\mu_i$ is lower than one, then at the prices set by trader $I$ the wealth of trader $i$ decreases, on average. The exogenous transfer of a small amount of wealth from agent $I$ to agent $i$ would be reverted back by market forces. Conversely, if $\mu_i$ is greater than one, the dominance of trader $I$ can be effectively challenged by trader $i$.[13] Thus, a sufficient condition for a MSE in which agent $I$ dominates to be unstable is that there exists a $i \in \{1, \ldots, I-1\}$ such that $\mu_i > 1$.

[13] When arbitrage opportunities exist the value of $\mu_i$ may even indicate that the wealth of agent $i$ grows or shrinks deterministically in each period.

Notice that the $\mu$s take also into account different consumption patterns. At a MSE in which trader $I$ dominates, if she invests proportionally to an agent $i$ but consumes more, then there exists a constant $c \in (0,1)$ such that $\alpha^I = c\,\alpha^i$. As a result $\mu^i = 1/c > 1$ so that the MSE in which agent $I$ dominates is unstable. With equal portfolio rules, the agent who consumes the most can never dominate.

The second set of stability conditions concerns the roots of a polynomial which depends on the derivatives of the surviving investment rule. They account for potential market instability related to the strength of price feedbacks.

The polynomial (4.4) can be heavily simplified if the rules are such that the $k$-th component of the $I$-th investment rule, $\alpha^I_k$, depends only on the prices (present and past) of asset $k$. In the following we shall name this property of investment rules the *no-cross-dependence condition.* It holds

**Corollary 4.1.** *Under the assumptions of Theorem 4.3, if trader $I$ investment rule satisfies the no-cross-dependence condition then $P(\lambda)$ simplifies to*

$$P(\lambda) = \prod_{k=1}^{K} \left( \lambda^L - \sum_{l=1}^{L} \lambda^{L-l} (\bar{\alpha}^I_k)^{(k,l)} \right) , \tag{4.5}$$

*and each $(\bar{\alpha}^I_k)^{h,l}$ reduces to*

$$(\bar{\alpha}^I_k)^{h,l} = \frac{(\alpha^I_k)^{h,l}}{1 - (\alpha^I_k)^{h,0}} \quad l = 0, 1, \ldots, L\,, \quad k, h = 1, \ldots, K\,.$$

Whether or not the no-cross dependence condition is satisfied for rule $I$, if the latter is a sufficiently flat function of past prices, so that its partial derivatives are close to zero, the corresponding MSE is stabilized. Indeed, as a straightforward application of Theorem 4.3 one has the following

**Corollary 4.2.** *Under the hypothesis of Theorem 4.3, if trader $I$ investment rule depends only on current prices then the asymptotic stability of $x^*$ depends only on the value of $\mu$s as defined in* (4.3).

## 4.2 Multiple survivors equilibria

We move now to investigate MSE where more rules have positive wealth, or *multiple survivors equilibria.* These equilibria are associated with prices that are fixed simultaneously by all the survivors. Thus it must be that all survivors take, at equilibrium, the same portfolio decisions.

**Theorem 4.4.** *Consider a market for $K$ short-lived assets with non-trivial dividend process $d$, where $I$ agents invest according to a rule in $\mathcal{A}$ using $L$ price lags.*

*Assume agents' wealth and asset prices evolve according to $\varphi$ in* (3.9)*. The state $x^* = (w^*, p^*)$ with shares $w^{M*}, \ldots, w^{I*}$ such that $\sum_{m=I-M+1}^{I} w^{m*} = 1$ is a manifold of multiple survivors Market Selection Equilibria where the last $M$ traders dominate if and only if $p^* = (\mathbf{p}^*, \ldots, \mathbf{p}^*) \in (\Delta^K_{c+})^{L+1}$ solves $\alpha^i(p^*) = \mathbf{p}^*$ for $i = I-M+1, \ldots, I$ but $\alpha^j(p^*) \neq \mathbf{p}^*$ for $j = 1, \ldots, I-M$.*

By fixing the same prices, the rules of the surviving traders have a common "intersection" with the EMC. Their common intersection defines a manifold of MSE because each reallocation of wealth among surviving agents does not change the equilibrium prices and it is still a MSE. As a result some potentially surviving agents can even possess a zero wealth share. The manifold of multiple survivor equilibria has dimension equal to the number of potential survivors minus one and is isomorphic to $\Delta^{M-1}$. We turn now to the specification of the sufficient conditions for the stability or instability of $x^* = (w^*, p^*)$. The following theorem generalizes both Theorem 4.2 and 4.3 to the present case.

**Theorem 4.5.** *Consider the manifold of MSE $x^* = (w^*, p^*)$ of Theorem 4.4 and assume that all investment rules $i \in \{1, \ldots, I\}$ are continuously differentiable in a neighborhood of $p^*$, $\alpha^i \in \mathcal{C}^1(p^*)$. Sufficient conditions for the existence of a well-defined local dynamics in a neighborhood of each $x^*$ and for the stability or instability of each $x^*$ are the same as those specified, respectively, in Theorem 4.2 and 4.3 provided that*

*(i) condition* (4.3) *is checked only for the last $I - M$ rules,*

*(ii) in the definition of $(\bar{\alpha}_k)^{h,l}$, $H$, and thus $P(\lambda)$, the expression $(\alpha^I_k)^{h,l}$ is replaced by*

$$\langle \alpha_k \rangle^{h,l} := \sum_{m=I-M+1}^{M} (\alpha^m_k)^{h,l} w^{m*} \quad l = 0, 1, \ldots, L\,, \quad k, h = 1, \ldots, K\,. \tag{4.6}$$

Intuitively, results for multiple survivors MSE mimic those for a single survivor MSE with the dominating rule equal to the weighted average of all surviving rules, the weights being equal to their equilibrium wealth shares. If at a certain MSE all $I$ agents take the same investment decision, all quantities $\mu$ will be equal to one, so that the only necessary conditions for local stability that are binding are those related to the roots of the polynomial $P(\lambda)$, representing the strength of the "average" price feedback. This is exactly what occurs at $p = p_B$ in the two-trader example of Section 2. Notice also that while the statement in Theorem 4.3 concerns asymptotic stability, the conditions of Theorem 4.5 only assure stability. This is the obvious consequence of the fact that multiple survivor equilibria are non-isolated equilibria.

# 5 Never vanishing rules

Knowing about the determinants of local survival and dominance, the next issue we address is whether there exist rules that never vanish. In Section 5.1 we characterize such a rule, which we name S-rule or $\alpha^\star$. In Section 5.2 we compare it with the Kelly rule and its generalizations. In the Section 5.3 we show that in the specific class of investment rules which depend on some given statistics of past prices it is possible to find rules that successfully adapt to $\alpha^\star$.

Before we start it is convenient to introduce a survival relation on rules.

**Definition 5.1.** Given two rules $\alpha^1, \alpha^2 \in \mathcal{A}$ it is $\alpha^1 \succeq \alpha^2$, if, for almost all initial conditions $x_0 \in \mathcal{X}$ and almost all $\omega \in \Omega$, rule $\alpha^1$ does not vanish when it trades only with rule $\alpha^2$.

Local stability and instability of MSE are informative on the survival relation. If there are no stable MSE in which $\alpha^1$ dominates then $\alpha^2 \succeq \alpha^1$, and *vice versa*. If instead we can identify a stable MSE in which $\alpha^1$ dominates then $\alpha^2 \not\succeq \alpha^1$, but not necessarily $\alpha^1 \succeq \alpha^2$, since there could be also a stable MSE in which $\alpha^2$ dominates. It follows that the relation $\succeq$ needs not to be complete. Notice, however, that we can asses the stability of a MSE using the results of the previous Section only when the rules considered satisfy the assumptions of Theorem 4.2 in a neighborhood of their MSE.

## 5.1 A Star rule

Let $\mathcal{A}^0$ denotes the subset of $\mathcal{A}$ made of constant (exogenous) rules. Given a measure $\rho$ and a non-trivial dividend process $d$, on the set $\mathcal{A}^0 \times \Delta^K_{c+}$ we define

$$I_{\rho,d}(\alpha, \mathbf{p}) = -\int_\Omega d\rho(\omega) \log\left(\sum_{k=1}^K \frac{\alpha_k}{p_k} d_k(\omega)\right) . \tag{5.1}$$

$I_{\rho,d}(\alpha, \mathbf{p})$ is a generalization of the relative entropy (2.3) to the general dividend process $d$, measure $\rho$, and to vectors $\alpha$ and $\mathbf{p}$ possibly non adding-up to one.[14] Its exponential is the inverse of the average growth rate of $\alpha$ when prevailing prices are equal to $\mathbf{p}$. The rule which maximizes this growth rate for each $\mathbf{p}$, or minimizes (5.1), never vanishes in a pairwise comparison.

For fixed $\rho$ and $d$, we define the S-rule $\alpha^\star(\mathbf{p})$ as the rule that minimizes $I_{\rho,d}(\alpha, \mathbf{p})$ for each given price vector $\mathbf{p}$. Since the minimizing vector in $\mathcal{A}^0$ might be different for different prices, $\alpha^\star(\mathbf{p})$ generically depends on prices and belongs to $\mathcal{A}$. If

[14] More precisely when the dividend process $d$ is defined by the identity matrix we recover $I_{\rho,d}(\alpha^i(p), \pi) = I_\pi(\alpha^i(p))$.

prices $\mathbf{p}$ are such that there are no arbitrage opportunities, compactness and strict convexity of the minimization problem ensures that then S-rule is well defined and continuous.

**Theorem 5.1.** *On the set of* $\mathbf{p} \in \Delta^K_{c+}$ *for which there are no arbitrages the rule*

$$\alpha^\star(\mathbf{p}) = \operatorname*{argmin}_{\alpha \in \mathcal{A}^0} \{I_{\rho,d}(\alpha, \mathbf{p})\} \tag{5.2}$$

*is a well defined function of* $\mathbf{p}$*. Moreover* $\alpha^\star(\mathbf{p})$ *is of class* $\mathcal{C}^1$*, satisfies* $\sum_{k=1}^K \alpha^\star_k(\mathbf{p}) = 1$*, and* $\alpha^\star(\mathbf{p}) = \mathbf{p}$ *if and only if* $p_k = \int_\Omega d\rho(\omega) d_k(\omega)$ *for every* $k = 1, \dots, K$*.*

If instead $\mathbf{p}$ is such that there are arbitrage opportunities, then there exist unbounded portfolios in $\mathcal{A}^0$ that give infinite wealth with positive probability. In this case the S-rule would exploit the arbitrage opportunities taking unbounded positions. An immediate consequence of the unboundedness of the S-rule under arbitrage is that when the S-rule is trading market clearing prices do not exhibit arbitrage.

Consider a finite set of rules $\mathcal{E} \subset \mathcal{A}$ such that the assumptions of Theorem 4.2 are satisfied for all possible MSE. For instance $\mathcal{E}$ can be made of rules belonging to $\mathcal{A}^0$, as those considered in the works surveyed by Evstigneev et al. (2009), or of rules derived by the maximization of CRRA utility functions of wealth, as the ones considered in Bottazzi and Dindo (2013), or any mixture of them.

**Theorem 5.2.** *Given the set* $\mathcal{E}$*, if* $\alpha^\star \in \mathcal{E}$ *then* $\alpha^\star \succeq \alpha$ *for every* $\alpha \in \mathcal{E}$*.*

The previous theorem exploits the fact that rules are sufficiently well behaved to allow the inference of global properties of the dynamics from the local analysis of Section 4: for almost all $\omega \in \Omega$ the S-rule never vanishes. The reason is that otherwise the market would converge to a MSE in which the other rule dominates, but, since $\alpha^\star$ has the maximal average growing rate for all possible market states, the MSE where the other rule dominates must be unstable, leading to a contradiction. Notice also that, since in absence of arbitrage Theorem 5.1 guarantees that $\alpha^\star$ is of class $\mathcal{C}^1$, and in particular at its intersection with the EMC, the sets $\mathcal{E}$ where Theorem 5.2 can be applied is not trivial.

When many rules compete in the same market against the S-rule, one can prove that the the S-rule survives almost surely and assets are priced correctly.

**Theorem 5.3.** *Consider a set* $\mathcal{E}$ *with* $\alpha^\star \in \mathcal{E}$*. All deterministic fixed points* $x^* = (w^*, p^*)$ *where* $\alpha^\star$ *vanishes are unstable. Moreover, there exists at least one stable deterministic fixed point in which* $\alpha^\star$ *survives and long-run asset prices are equal to* $p^*_k = \int_\Omega d\rho(\omega) d_k(\omega)$ *for all* $k = 1, \dots, K$*.*

Notice at last that one can define an S-rule associated with any consumption level $\alpha_0$ by conditioning the minimization in (5.2) to the set of vectors obeying to $\sum_{k=1}^{K} \alpha_k \leq 1 - \alpha_0$. The resulting rule will still satisfy Theorems 5.2 and 5.3 if the set $\mathcal{E}$ is restricted to the rules having the same (or higher) consumption rates.

## 5.2 Generalized Kelly rules

Consider a dividend process defined in terms of a constant and non-trivial diagonal matrix $d$ as in the example of Section 2. In this case the S-rule coincides with the so called Kelly rule, that is, $\alpha_k^\star(\mathbf{p}) = \pi_k$ (Kelly, 1956). When $d$ is not diagonal, $\alpha^\star$ is price dependent and can be seen as a generalization of the Kelly rule. As an example consider $d = \left(\begin{smallmatrix} 1/2\,, & 0 \\ 1/2\,, & 1 \end{smallmatrix}\right)$ and an invariant measure $\pi = (2/3, 1/3)$. The S-rule can be easily found to be

$$\alpha_1^\star(p_1, p_2) = \frac{2p_2 - 3p_1}{3(p_2 - p_1)}\,, \quad \alpha_2^\star = 1 - \alpha_1^\star\,,$$

when prices do not allow arbitrage, $p_2 > p_1$, and any unbounded arbitrage otherwise. When prices are bound to add up to one, we can name $p_1 = p$, as in the example of Section 2. In absence of arbitrage, $p < \frac{1}{2}$, the fraction $\alpha^\star(p)$ to be invested in the first asset becomes

$$\alpha^\star(p) = \frac{2 - 5p}{3(1 - 2p)}\,. \tag{5.3}$$

The S-rule plays a central role in the graphical analysis introduced in Section 2. In fact it is the relative wealth growth rate of a rule with respect to the S-rule that determines its survival possibility. Since for any given price vector, and thus also at any given MSE, the S-rule is maximal with respect to the average wealth growth rate, to have the highest wealth growth rate is equivalent to be "nearest" to that rule. We shall exploit this fact in Section 6.2 where we use the EMC plot to analyze local survival and dominance with a non-diagonal matrix $d$.

Another generalization of the Kelly rule in repeated markets for short-lived assets has been proposed by Amir et al. (2005) (see also Evstigneev et al., 2009). When the dividend process is formed using a dividend matrix $d$ this generalized Kelly rule, which we name $\alpha^{AEHS}$ from the initials of its proponents, is $\alpha_k^{AEHS} = \sum_{s=1}^{S} \pi_s d_{k,s}$ for $k = 1, \ldots, K$. The rule states to invest in proportion to assets expected dividends. If $d$ is diagonal $\alpha^\star$ and $\alpha^{AEHS}$ coincide. If $d$ is not diagonal, they differ. It is still true, however, that the intersections of the two rules with the EMC coincide (c.f. Th. 5.1) so that the two rules set the same prices when trading together. In Amir et al. (2005) it is shown that the $\alpha^{AEHS}$ dominates globally on the class of rules that do not depend on endogenous market variables, such as prices, but do depend on partial histories of the dividend process.[15]

[15]Since $\alpha^{AEHS}$ does belong to the set of rules $\mathcal{A}^0 \subset \mathcal{A}$, and given that it does not vanish when

## 5.3 Learning from prices

In Section 5.1 we have established that the S-rule, not being dominated by any other rule, never vanishes. Is the S-rule the unique rule having this property? The answer is negative as we show in this Section by considering a class of "price learning" rules that, using only market information given by past prices, adapt to any other rule and are never dominated. In particular they are not dominated by the S-rule.

Consider a rule $\alpha^L$ of class $\mathcal{C}^1$ that satisfies the no-cross-dependence condition, has zero consumption, and is consistent, that is, $\alpha^L_k(p) = p_k$, $k = 1, \ldots, K$ for any constant price vector $p = (\mathbf{p}, \ldots, \mathbf{p})$. We assume the following: $\alpha^L$ does not depend on current prices but only on some statistics, like the mean and/or or variance, of $L$ past realized prices[16]; the statistics assigns equal weights to all past prices; for every $k = 1, \ldots, K$, all partial derivatives of $\alpha^L$ are equal, or

$$(\alpha^L_k)^{k,l} = (\alpha^L_k)^{k,l'}\,, \quad \text{for every} \quad l, l' = 1, \ldots, L\,, \quad k = 1, \ldots, K\,. \tag{5.4}$$

The previous assumptions imply a substantial simplification in the sufficient condition for local stability of the market selection equilibrium where $\alpha^L$ dominates.

**Theorem 5.4.** *Consider a MSE $x^*$ in which only the investment rule $\alpha^L$ survives. Define $(\alpha^L_k)_{x^*}$ the common value of the partial derivative of the $k$-th component of the investment rule at the fixed point $x^*$. All the roots of polynomial $P(\lambda)$ defined in Theorem 4.3 are inside the unit circle provided that*

$$(\alpha^L_k)_{x^*} \in \left(-1, \frac{1}{L}\right)\,, \quad \textit{for every} \quad k = 1, \ldots, K\,. \tag{5.5}$$

The extension of the previous result to the case of multiple survivors case is straightforward: conditions are not on partial derivatives $(\alpha^L_k)^{k,l}$ but on convex combinations of partial derivatives of the type $\langle \alpha_k \rangle^{k,l}$. In this case the equilibrium could be stable for some mixtures of strategies and unstable for others. When this is the case, the stability condition can be re-written in terms of which wealth distributions among survivors guarantee stability.

We can now apply the previous result considering a market populated by a price learner using $\alpha^L$ and a trader using the S-rule $\alpha^\star$. Consider a fixed point $x^* = (w^*, p^*)$, with $\mathbf{p}^* = \alpha^\star(\mathbf{p}^*) = \alpha^L(\mathbf{p}^*)$ and where $w^* \in (0, 1)$ is the wealth

trading with $\alpha^\star$, the reader might wonder whether it also does not vanish when trading with any other price dependent rule. We do not have an answer to this question at this stage of our research. Notice that surviving against $\alpha^\star$ is necessary but not sufficient for surviving against any other rule. The right panel of Fig. 5 illustrates this last point with an example.

[16] Several so called "technical" rule of chartist inspiration, like trend detection, ceiling and floor crossing, fall in this category.

share of $\alpha^\star$. Then, if the assumptions of Theorem 4.2 are satisfied in $x^*$, using the notation of Theorem 5.4, one has

**Corollary 5.1.** *If for every* $k = 1, \dots, K$

$$w^* > 1 - \frac{1}{(\alpha_k^L)_{x^*} L} \quad \textit{when} \quad (\alpha_k^L)_{x^*} > 0\,,$$

*and*

$$w^* > 1 - \frac{1}{|(\alpha_k^L)_{x^*}|} \quad \textit{when} \quad (\alpha_k^L)_{x^*} < 0\,.$$

*then the MSE* $x^*$ *is stable.*

The intuition behind this result is simple. Given a value of $(\alpha_k^L)_{x^*}$ there always exists an appropriate bound on the fraction of wealth of the S-rule that assures that the portfolio of a price learner asymptotically approaches fast enough the market portfolio. The market portfolio, $\alpha(\mathbf{p}) = \mathbf{p}$, has constant wealth and thus never vanishes. As a result, a price learner never vanishes when trading with an agent using the S-rule, in that there always exists a finite wealth fraction of the former that stabilizes the MSE where both survive. Since it is never the case that the S-rule dominates a price learner, we have established[17] that $\alpha^L \succeq \alpha^\star$. The assumption of zero-consumption is essential to the proof. Indeed any rule with a positive consumption rate would vanish against the S-rule.

# 6 Assets mis-pricing and endogenous fluctuations

As discussed in Section 2.3, when rules are exogenous and assets are Arrow securities market selection achieves informational efficient outcomes. With more general rules this is not the case. In Section 6.1 we show that, despite the existence of never vanishing rules, the survival relation on the set $\mathcal{A}$ of endogenous rules is not transitive and thus rules cannot be ordered according to their ability to survive (or dominate). This result is responsible for the possible inefficiency of very simple market settings. We illustrate this point discussing two examples of market selection failure taken from previous contributions. First, in Section 6.2 we consider the example presented in Evstigneev et al. (2009) in which market incompleteness is responsible of persistent mis-pricing. Using the graphical analysis based on the EMC plot we re-obtain the original result and we show that when the rules are endogenous the same phenomenon is present also in markets for Arrow securities. Second, in Section 6.3 we review an example presented in Blume and Easley

[17]In fact, along the same lines, it is straightforward to show that $\alpha^L \succeq \alpha$ for every $\alpha$ of class $\mathcal{C}^1$ at its equilibria.

(1992). We show that persistent mis-pricing is not only the result of market selecting the least informed trader, as in the original example, but can also derive from persistent heterogeneity of beliefs that the repeated trading is not able to resolve.

## 6.1 Dominance and ordering

The relation introduced in Definition 5.1 does not induce an order relation on any finite set of rules $\mathcal{E} \subset \mathcal{A}$. The crucial idea behind this result is that although rules relative "distance" with respect to the S-rule does imply an ordering, this ordering is only local and not global, as it depends on prevailing prices. To see it we propose the following example. In the context of the toy market of Section 2, consider the following three zero-consumption investment rules. The fraction of wealth to be invested in the first asset, whose price is denoted as $p$, is respectively

$$\alpha^1(p) = 0.3\,, \quad \alpha^2(p) = \begin{cases} 0.9 & p \leq 0.2 \\ 1.5 - 3p & 0.2 < p \leq 0.3 \\ 0.6 & p > 0.3 \end{cases} \;, \quad \alpha^3(p) = \begin{cases} 0.2 & p \leq 0.3 \\ p - 0.1 & p > 0.3 \end{cases} \;.$$

These rules are depicted, together with the S-rule $\alpha^\star = \pi$, in Fig. 2. All three rules form a set $\mathcal{E}$ where the local stability conditions of Section 4 can be used. Moreover, when two of them are trading, the market dynamics is well defined for all possible wealth distributions. For example when only rules 1 and 2 compete on the market, naming $w$ the wealth fraction of trader 1 and solving (2.5b) for market prices gives

$$p_t = 0.6 - 0.3 w_t\,.$$

The price of asset 1 is always between 0.3 (when $w = 1$) and 0.6 (when $w = 0$). Plugging this price equation in (2.5a) one obtains the 1-dimensional dynamical system describing the evolution of the market. The same analysis can be repeated for the other two possible pairings.

From the local stability conditions, whose results can be inferred by graphical inspection of Fig. 2, we conclude that at prices where trader 1 or trader 2 dominates $\alpha^2$ is closer to the S-rule than $\alpha^1$. The MSE were rule 1 dominates is thus unstable while the MSE where rule 2 dominates is stable, so that $\alpha^2$ never vanishes and $\alpha^2 \succeq \alpha^1$. The inverse statement, $\alpha^1 \succeq \alpha^2$, is not true since the MSE where rule 2 dominates is stable. The same reasoning can be applied to the other pairings concluding that $\alpha^1$ never vanishes when trading with $\alpha^3$, but not the inverse, and $\alpha^3$ never vanishes when trading with $\alpha^2$, but not the inverse. Since $\alpha^2 \succeq \alpha^1$, $\alpha^1 \succeq \alpha^3$ but $\alpha^2 \not\succeq \alpha^3$, we can state that the relation is not transitive.

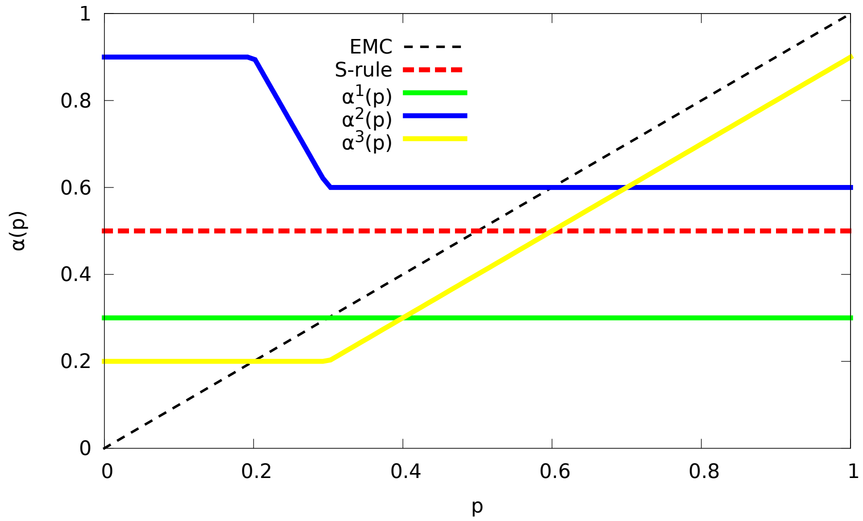


Figure 2: The relation $\succeq$ is not transitive as $\alpha^2 \succeq \alpha^1$ , $\alpha^1 \succeq \alpha^3$, but $\alpha^3 \succeq \alpha^2$.

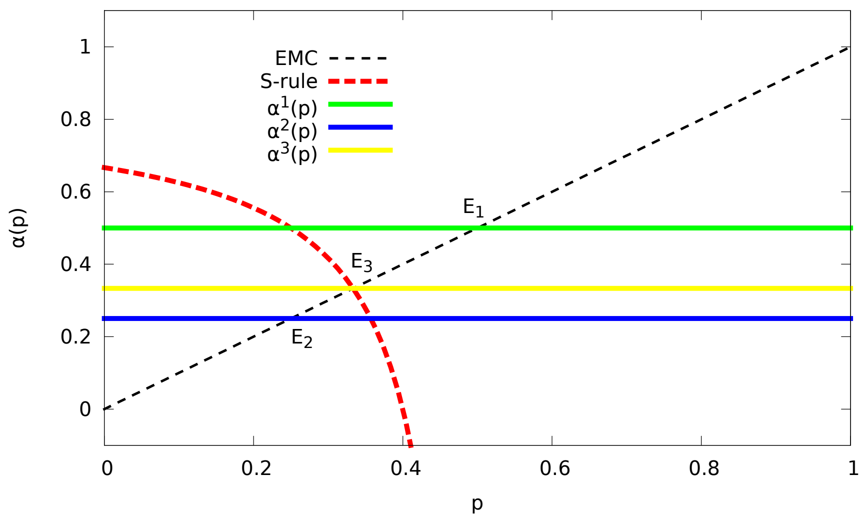


Figure 3: Coexistence of unstable MSE and long-run heterogeneity in the example of Sec. 3.3 from Evstigneev et al. (2009).

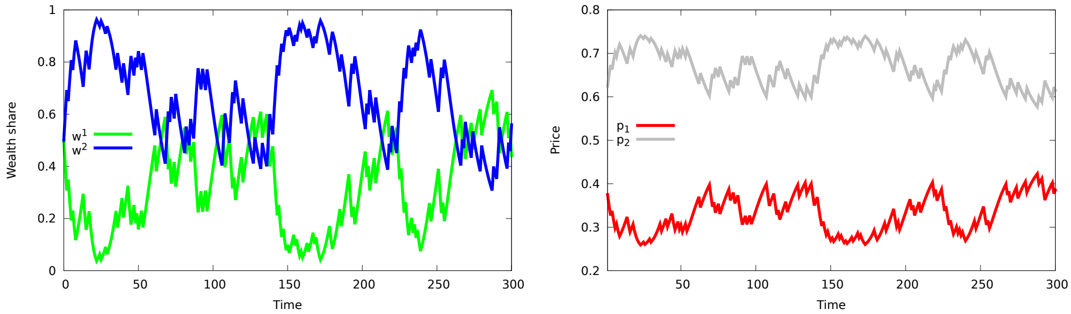


Figure 4: Wealth shares and prices dynamics for the example of Sec. 3.3 from Evstigneev et al. (2009) when rules $\alpha^1$ and $\alpha^2$ are trading.

## 6.2 Endogenous fluctuations and path dependency

In Section 3.3 of Evstigneev et al. (2009) an incomplete market with 3 states of the world but only 2 assets is considered. Using our notation $S = 3$, $K = 2$, and the normalized dividend matrix is $d = \begin{pmatrix} 1/2, & 1/2, & 0 \\ 1/2, & 1/2, & 1 \end{pmatrix}$. The process driving the states of world is i.i.d. with $\pi = (1/3, 1/3, 1/3)$. The authors consider three rules: $\alpha^1 = (1/2, 1/2)$, $\alpha^2 = (1/4, 3/4)$, and $\alpha^3 = (1/3, 2/3)$. Fig. 3 illustrates the three rules in a EMC plot. Since all rules do not consume, prices add up to one and we set $p_1 = p$, $p_2 = 1 - p$. The plot also reports the S-rule found by solving the maximization problem (5.2).The solution coincides with (5.3).

When only rules $\alpha^1$ and $\alpha^2$ compete there are two possible MSE labeled $E_1$ and $E_2$ in Fig. 3. In $E_1$ trader 1 dominates and $p = 1/2$. In $E_2$ trader 2 dominates and $p = 1/4$. Both equilibria are unstable: when $p = 1/4$ trader 1 is closer to the S-rule than trader 2 while when $p = 1/2$ is trader 2 to be closer.[18] Since neither trader prevails in the long run, heterogeneity is persistent and prices do not converge to constant values. Simulations reported in Fig. 4 show how wealth shares are time varying and the price of the first asset keeps fluctuating in the interval $(1/4, 1/2)$. The fact that prices in $E_2$ are closer to discounted dividends than prices in $E_1$ is not enough to guarantee that trader 2 ultimately dominates. We can interpret this as a form of asset mis-pricing induced by persistent heterogeneity. It is also shown that when $\alpha^3$ is added to the trading rules prices stabilize and converges to the correct values.[19] In fact $E_3$, the MSE in which trader 3 dominates, is the unique stable MSE.[20]

[18] When $p \geq 1/2$ there exist arbitrage opportunities. Despite rule $\alpha^2$ is not an arbitrage, on average it still invests "better" than rule $\alpha^1$.

[19] The proof is based on global results previously presented in Amir et al. (2005)

[20] Notice that $\alpha^3 = \alpha^{AEHS}$ and even if the S-rule and $\alpha^{AEHS}$ are rather different, they coincide at $p = \sum_{s=1}^{3} \pi_s d_{1,s} = 1/3$.

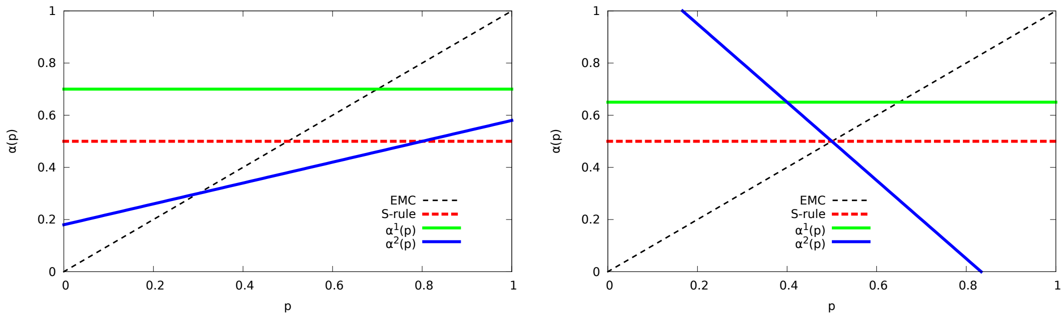


Figure 5: Coexistence of unstable MSE (left panel) and coexistence of stable MSE (right panel) in the toy market of Section 2.

Moving from exogenous to endogenous rules our contribution shows that incomplete markets or non-diagonal assets are no longer required to observe market instability. Consider the complete market described in Section 2, two Arrow securities and two equally likely states, populated in turn by the couples of rules depicted in the left and right panel of Fig. 5. In the left pane, linear rule 2 invests a larger share of wealth in asset 1 as its price increases and can thus be thought of as a market follower rule. In the right panel, rule 2 does the opposite and can thus be thought of as a contrarian rule. In the left panel, both MSE are unstable, despite the fact that assets are better priced in the MSE where rule 2 dominates. This is the same kind of market failure discussed in the previous example. Conversely, in the right panel, both MSE are locally stable. Thus rule 1 can dominate with finite probability despite the fact that it is rule 2 that correctly prices the assets. In this second case heterogeneity is not persistent but asset mis-pricing, caused by dependency on initial conditions and on the realization of $\omega$, may be quite severe.

In Bottazzi and Dindo (2013) we present more examples where coexistence of stable or of unstable equilibria occurs generically when traders traders are CRRA myopic utility maximizers, have different beliefs over the invariant distribution, and different coefficients of relative risk aversion.

### 6.3 Vanishing of the informed trader

In Theorem 5.4 of Blume and Easley (1992) it is shown that in an i.i.d. economy with aggregate risk, that is, where $\sum_k D_{k,s} \neq \sum_k D_{k,s'}$ if $s \neq s'$, having correct beliefs is not sufficient to avoid vanishing. In particular, among myopic expected utility maximizers with constant coefficient of relative risk aversion $\gamma$, given a rule with correct beliefs and $\gamma \neq 1$, there always exists a rule with worst beliefs but $\gamma$

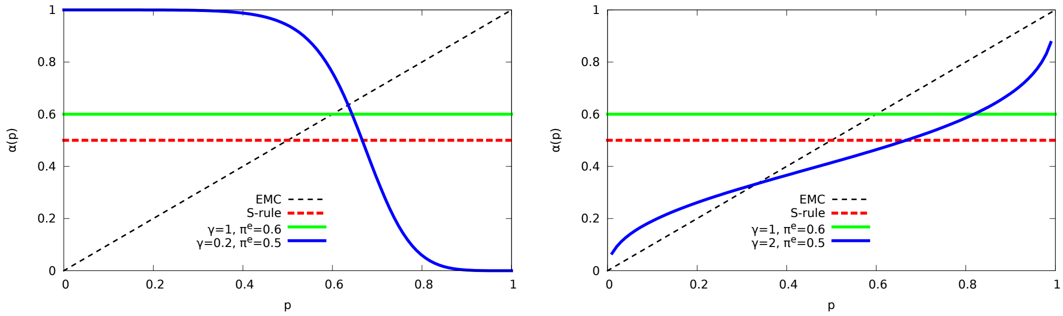


Figure 6: Left panel: Dominance of the uninformed trader as in Blume and Easley (1992). Right panel: long-run coexistence of uninformed and informed traders. In both examples demands are as in (6.1), $D_1 = 2$, $D_1 = 1$, and $\pi = (0.5, 0.5)$.

closer to one, such that the former vanishes when trading against the latter.

To see an example, consider a market for two securities with dividends $D = \begin{pmatrix} D_1, & 0 \\ 0, & D_2 \end{pmatrix}$, $D_1 \neq D_2$. Name trader $i$ beliefs $(\pi^i, 1-\pi^i)$ and her coefficient of relative risk aversion $\gamma^i$. By solving the related maximization problem, the fraction to be invested in asset 1 by trader $i$ as a function of its normalized price $p$ can be found to be

$$\alpha^i(p) = \frac{\left(\pi^i \left(\frac{D_1}{p}\right)^{1-\gamma^i}\right)^{\frac{1}{\gamma^i}}}{\left(\pi^i \left(\frac{D_1}{p}\right)^{1-\gamma^i}\right)^{\frac{1}{\gamma^i}} + \left((1-\pi^i)\left(\frac{D_2}{1-p}\right)^{1-\gamma^i}\right)^{\frac{1}{\gamma^i}}} . \tag{6.1}$$

In the left panel of Fig. 6 trader 2 has correct beliefs but has $\gamma^2 < 1$. Trader 1 has incorrect beliefs but has $\gamma^1 = 1$. Parameters are chosen so that for all possible equilibrium prices, and thus also at the two MSE, trader 1 is closer to the S-rule than trader 2. It follows that the MSE where 1 dominates is stable while the MSE where 2 dominates is unstable. The correct beliefs of trader 2 will never be reflected in prevailing prices. However the dominance of the worst informed is not the unique possible failure of wealth-driven selection. Another possibility is represented by the right panel of Fig. 6. Here we change the preferences of the second trader to make both MSE unstable and obtain long-run heterogeneity of beliefs and persistent price fluctuations.

# 7 Conclusion

We investigate wealth-driven selection in a repeated market for short-lived assets where traders are boundedly rational and investment rules depend on current and past equilibrium prices. We derive local stability conditions of long-run Market Selection Equilibria where a rule, or a group of rules, dominates and determines asset prices. A strength point of our results is that they can be applied to any repeated market for short-lived assets as long as their dividend process is ergodic. In particular we do not have restrictions on the number of traders, nor on the type of rules they use, as long as they are smooth enough, diversified, and satisfy the usual budget constraint. Our analysis shows that coexistence of stable and unstable equilibria are generic and might lead to persistent heterogeneity, endogenous fluctuations, path-dependency, and mis-pricing. We identify two different sources of market selection failures. The first is related to the fact that rules' long-run average growth rates can be ordered only for a given set of prevailing prices. A rule might outperform other rules at certain prices, having there the highest average growth rate, and fail to do so at other prices. The second source of selection failure is that an investment rule may have a too strong past prices feedback to let the price dynamics converge.

Our results cast doubts on the validity of the selection argument underlying the informational efficiency of financial markets, at least when traders are boundedly rationals so that markets are not dynamically complete. The market does not generically selects a unique log-run winner and asset prices do not need to be as close as possible, given the competing rules, to their fundamental values. Also a trader with perfect knowledge about the underlying dividend process is not in general guaranteed to survive. Only if a trader has perfect knowledge regarding the underlying dividend process and exploits it at best using the S-rule, then the selection equilibria where she survives are the unique stable equilibria. In this case prices correctly reflect, in the long-run, assets' fundamental values.

Our future work shall be oriented at establishing whether our local results are also informative of the global dynamics; whether they carry over also when trader rules depend on partial histories of the dividend process; and, most importantly, if the market selection failures we identify are able to make sense of the financial anomalies found by the empirical literature.

# References


Alchian, A. (1950). Uncertainty, evolution, and economic theory. *Journal of Political Economy 58*, 211–221.

Amir, R., I. Evstigneev, T. Hens, and K. Schenk-Hoppé (2005). Market selection and survival of investment strategies. *Journal of Mathematical Economics 41*, 105–122.

Anufriev, M. and G. Bottazzi (2010). Market equilibria under procedural rationality. *Journal of Mathematical Economics 46*, 1140–1172.

Anufriev, M., G. Bottazzi, and F. Pancotto (2006). Equilibria, stability and asymptotic dominance in a speculative market with heterogeneous agents. *Journal of Economic Dynamics and Control 30*, 1787 – 1835.

Anufriev, M. and P. Dindo (2010). Wealth-driven selection in a financial market with heterogeneous agents. *Journal of Economic Behavior and Organization 73*, 327–358.

Barberis, N. and R. Thaler (2003). A survey of behavioral finance. In G. Constantinides, M. Harris, and R. Stultz (Eds.), *Handbook of the Economics of Finance.* North-Holland (Handbooks in Economics Series), Amsterdam.

Blume, L. and D. Easley (1992). Evolution and market behavior. *Journal of Economic Theory 58*, 9–40.

Blume, L. and D. Easley (2006). If you are so smart why aren't you rich? Belief selection in complete and incomplete markets. *Econometrica 74*, 929–966.

Blume, L. and D. Easley (2009). The market organism: Long-run survival in markets with heterogenous traders. *Journal of Economic Dynamics and Control 33*, 1023–1035.

Bottazzi, G. and P. Dindo (2013). Selection in asset markets: the good, the bad, and the unknown. *Journal of Evolutionary Economics 23*, 641–661.

Chiarella, C. and X. He (2001). Asset price and wealth dynamics under heterogeneous expectations. *Quantitative Finance 1*, 509–526.

Chichilnisky, G. (1999). Existence and optimality of a general equilibrium with endogenous uncertainty. In G. Chichilnisky (Ed.), *Markets, Information, and Uncertainty.* Cambridge University Press, Cambridge.

Coayla-Teran, E. and P. Ruffino (2004). Stochastic versions of Hartman-Grobman theorems. *Stochastics and Dynamics 4*, 571–593.

Drèze, J. and P. Herings (2008). Sequentially complete markets remain incomplete. *Economic Letters 100*, 445–447.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2008). Globally evolutionary stable portfolio rules. *Journal of Economic Theory 140*, 197–228.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2009). Evolutionary finance. In T. Hens and K. Schenk-Hoppé (Eds.), *Handbook of Financial Markets: Dynamics and Evolution*. North-Holland (Handbooks in Economics Series).

Fama, E. (1970). Efficient capital markets: a review of theory and empirical work. *Journal of Finance 25*, 383–422.

Farmer, J. (2002). Market force, ecology, and evolution. *Industrial and Corporate Change 11*, 895–953.

Friedman, M. (1953). *Essays in Positive Economics*. Univ. Chicago Press.

Ginsburgh, V. and M. Keyzer (1997). *The Structure of Applied General Equilibrium Models*. The MIT Press, Cambridge, Massachusetts.

Hahn, F. (1999). A remark on incomplete market equilibrium. In G. Chichilnisky (Ed.), *Markets, Information, and Uncertainty*. Cambridge University Press, Cambridge.

Hommes, C. (2006). Heterogeneous agent models in economics and finance. In K. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics*. North-Holland (Handbooks in Economics Series), Amsterdam.

Kelly, J. (1956). A new interpretation of information rates. *Bell System Technical Journal 35*, 917–926.

Lang, S. (1993). *Complex Analysis*. Springer, New York.

LeBaron, B. (2006). Agent-based computational finance. In K. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics*. North-Holland (Handbooks in Economics Series).

Levy, M., H. Levy, and S. Solomon (2000). *Microscopic Simulation of Financial Markets*. Academic Press, London.

Massari, F. (2013). *Essays on Price Dynamics and Market Selection*. Elctronic Theses and Dissertation. Paper 1080.

Sandroni, A. (2000). Do markets favor agents able to make accurate predictions. *Econometrica 68*(6), 1303–1341.

# A Appendix: Proofs

## A.1 Section 4

For the sake of the proofs, it is useful to write the system of $I+K(L+1)$ equations characterizing the market dynamics in terms of the two functions $\mathcal{W}$ and $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_K)$ so that

$$
\mathcal{F}(\theta^t\omega)x_t := \left[\begin{array}{lll}
\mathcal{W}(x_t;\theta^t\omega) & := & \left[\begin{array}{lll} w^1_{t+1} & = & \mathcal{W}^1(x_t;\theta^t\omega) \\ \vdots & \vdots & \vdots \\ w^I_{t+1} & = & \mathcal{W}^I(x_t;\theta^t\omega) \end{array}\right. \\
\mathcal{P}(x_t;\theta^t\omega) & := & \left[\begin{array}{lll}
\mathcal{P}_1(x_t;\theta^t\omega) & := & \left[\begin{array}{lll} p_{1,t+1} & = & f_1(x_t;\theta^t\omega) \\ p^1_{1,t+1} & = & p_{1,t} \\ \vdots & \vdots & \vdots \\ p^L_{1,t+1} & = & p^{L-1}_{1,t} \end{array}\right. \\
\vdots & & \vdots \\
\mathcal{P}_K(x_t;\theta^t\omega) & := & \left[\begin{array}{lll} p_{K,t+1} & = & f_K(x_t;\theta^t\omega) \\ p^1_{K,t+1} & = & p_{K,t} \\ \vdots & \vdots & \vdots \\ p^L_{K,t+1} & = & p^{L-1}_{K,t} \end{array}\right.
\end{array}\right.
\end{array}\right. . \tag{A.1}
$$

**Proof of Theorem 4.1** The result follows from substitution of $x^*$ in (A.1). $\mathcal{W}(x^*;\omega) = w^*$ holds because, for every $\omega$, if $i \neq I$ $w^{i*} = 0$ is a fixed point of $\mathcal{W}^j(\cdot,\omega)$; if $i = I$, since by assumption $p^*_k = \alpha^I_k(p^*)$ for $k = 1,\dots,K$, $w^{I*} = 1$ is a fixed point of $\mathcal{W}^I(\cdot,\omega)$. $\mathcal{P}(x^*;\omega) = p^*$ holds because current prices are fixed by $I$ (all other agents have zero wealth) so that $p^*_k = \alpha^I_k(p^*)$ holds by assumption for $k = 1,\dots,K$, lagged prices are instead all equal by constructions given that $p^* = (\mathbf{p}^*,\dots,\mathbf{p}^*)$. $\square$

**Proof of Theorem 4.2** After noting that prices are implicitly defined by the set of $K$ equations in (3.2b) or

$$
p_{k,t+1} = \sum_{i=1}^{I} w^i_{t+1}\alpha^i_k(p_{t+1})\,, \quad k = 1,\dots,K
$$

with $w^I_{t+1} = 1$ and $w^i_{t+1} = 0$ for $i \neq I$, the result follows by applying the implicit function theorem. More specifically we apply the theorem for each of the $S$ maps $\mathcal{F}$, one for each state of the world. If we take $X$ as the intersection of all $S$

neighborhoods $X(s)$ where the explicit map is well defined, the resulting set $X$ does not depend on $s$ and thus also not on $\omega$. $\square$

**Proof of Theorem 4.3** Consider the reduced system in $[0,1]^{I-1} \times (0,1)^{K(L+1)}$ of dimension $I-1+K(L+1)$ obtained by substituting $w_t^I = 1 - \sum_{i=1}^{I-1} w_t^i$. With an abuse of notation we will keep using the same names for the map $f$, and thus also $\mathcal{F}$, even though its definition has actually changed. In particular the definition of $f$ given in (3.8) becomes

$$f_k(x_t;\omega) = \sum_{i=1}^{I-1} \mathcal{W}^i(x_t;\omega)(\alpha_{k,t+1}^i - \alpha_{k,t+1}^I) + \alpha_{k,t+1}^I\,, \quad k = 1,\dots,K\,. \tag{A.2}$$

$\mathcal{F}$ defined in (A.1) and $x^*$ vary accordingly, in particular $x^* = (0,\dots,0,p^*)$.

Given $x^*$ and $X$ as in Theorem 4.2, the map $\mathcal{F}$ and the shift map $\theta$ define a Random Dynamical System $\varphi : \mathcal{N} \times \Omega \times X \to X$ such that

$$\varphi(t,\omega,x) = \mathcal{F}(\theta^{t-1}\omega) \circ \ldots \circ \mathcal{F}(\theta\omega) \circ \mathcal{F}(\omega)x.\,,$$

see e.g. Def. 2.1 in Coayla-Teran and Ruffino (2004). We shall show that sufficient conditions for the stability and instability of $x^*$ can be given in terms of the Lyapunov spectrum of its Jacobian.

The Jacobian $J(\omega,x)$ of $\mathcal{F}$ can be written as

$$J(\omega,x) = \begin{pmatrix} \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{P}} \\ \frac{\partial\mathcal{P}}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}}{\partial\mathcal{P}} \end{pmatrix}\,, \tag{A.3}$$

or, subdividing the part relative to prices, with obvious notation,

$$J(\omega,x) = \begin{pmatrix} \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{W}}{\partial\mathcal{P}_K} \\ \frac{\partial\mathcal{P}_1}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}_1}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{P}_1}{\partial\mathcal{P}_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\mathcal{P}_K}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}_K}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{P}_K}{\partial\mathcal{P}_K} \end{pmatrix}\,. \tag{A.4}$$

The element $i,j$ of each block matrix is the partial derivative of the $i$-th component of the numerator with respect to the $j$-th component of the denominator.

In each sub-block $\partial\mathcal{W}/\partial\mathcal{P}_k$ the first column reads

$$\left\{\frac{\delta\mathcal{W}}{\delta\mathcal{P}_k}\right\}_{i,1} = \left(\sum_{k'} \frac{(\alpha_{k'}^i)^{k,1}}{p_{k',t}} d_{k',t}(\omega) - \frac{\alpha_{k,t}^i}{(p_{k,t})^2 d_{k,t}(\omega)}\right) w_t^i\,, \quad i = 1,\dots,I-1\,,$$

while for $l > 1$ it is

$$\left\{ \frac{\delta \mathcal{W}}{\delta \mathcal{P}_k} \right\}_{i,l>1} = \left( \sum_{k'} \frac{(\alpha^i_{k'})^{k,l-1}}{p_{k't}} d_{k',t}(\omega) \right) w^i_t \,, \quad i = 1, \ldots, I-1 \,, \quad L = 2, \ldots, L+1 \,.$$

Since $w^{*j} = 0$ if $j \neq I$, the previous expressions at $x^*$ read

$$\left\{ \left. \frac{\partial \mathcal{W}}{\partial \mathcal{P}} \right|_{x^*} \right\}_{i,j} = 0 \quad \text{for all} \quad i, j \,.$$

As a result, the Jacobian matrix evaluated at $x^*$, $J^*(\omega) = J(\omega, x^*)$, is lower block triangular and the eigenvalues of $J^*(\omega)$ are those of the left-upper wealth/wealth block and right-lower price/price block. These blocks turn out to have a peculiar structure at $x^*$.

Let us start from the left-upper block $\partial \mathcal{W}/\partial \mathcal{W}$,. Taking the partial derivatives of wealth fractions gives

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{W}} \right\}_{i,j} = \frac{\partial \mathcal{W}^i(x_t; \omega)}{\partial w^j_t} = \delta_{i,j} \sum_{k=1}^{K} \frac{\alpha^i_{k,t}}{p_{k,t}} d_{k,t}(\omega) \quad i, j = 1, \ldots, I-1$$

so that the block computed in $x^* = (0, \ldots, 0, p^*)$ becomes diagonal and reads

$$\left. \frac{\partial \mathcal{W}}{\partial \mathcal{W}} \right|_{x^*} = \begin{pmatrix} \mu_1(\omega_{t+1}) & 0 & \cdots & 0 \\ 0 & \mu_2(\omega_{t+1}) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{I-1}(\omega_{t+1}) \end{pmatrix}, \tag{A.5}$$

where, using the fact that prices are fixed by trader $I$,

$$\mu_i(\omega_{t+1}) = \sum_{k=1}^{K} \frac{\alpha^i_k(p^*)}{\alpha^I_k(p^*)} d_{k,t}(\omega) \,. \tag{A.6}$$

Concerning the right-lower block $\partial \mathcal{P}/\partial \mathcal{P}$, in a neighborhood of the fixed point $x^*$ it holds that

$$\begin{aligned} \left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h} \right\}_{1,l} &= \left. \frac{\partial f_k(x_t; \omega)}{\partial p^{l-1}_{h,t}} \right|_{x^*} \qquad l = 1, \ldots, L+1 \,, \\ &= -\sum_{k'=1}^{K} H^{-1}_{k,k'}(x_t) M_{k',h,l} \end{aligned} \tag{A.7}$$

where $H^{-1}$ is the inverse of the matrix $H_{k,k'}(x_t) = \sum_{i=1}^{I} w_{t+1}^i (\alpha_k^i(p_{t+1}))^{k',l}$, defined in Theorem 4.2 and non-singular buy assumption, and

$$M_{k',h,l} = \sum_{i=1}^{I-1} \left( \left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_h} \right\}_{1,l} + w_{t+1}^i \left( \left(\alpha_{k'}^i\right)^{h,l} - \left(\alpha_{k'}^I\right)^{h,l} \right) \right) + \left(\alpha_{k'}^I\right)^{h,l} . \qquad \text{(A.8)}$$

Substituting in (A.7) the expression of (A.8) computed at $x^*$ and using the matrix defined in (4.1) leads to

$$\left\{ \left. \frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h} \right|_{x^*} \right\}_{1,l} = - \sum_{k'=1}^{K} H_{k,k'}^{-1} (\alpha_{k'}^I)^{h,l} = (\bar{\alpha}_k^I)^{h,l} , \quad l = 1, \dots, L \qquad \text{(A.9)}$$

and $\{\partial \mathcal{P}_k / \partial \mathcal{P}_h |_{x^*}\}_{1,L+1} = 0$. The other rows are all zero but for the diagonal blocks which have a "Jordan" form, that is,

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h} \right\}_{i>1,l} = \frac{\partial p_{k,t+1}^l (x_t; \omega)}{\partial p_{k,t}^{l-1}} = \delta_{k,h} \delta_{i+1,i} , \quad i = 2, \dots, L+1 \quad l = 1, \dots, L+1 .$$

As a result

$$\left. \frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h} \right|_{x^*} = \begin{pmatrix} (\bar{\alpha}_k^I)^{h,1} & (\bar{\alpha}_k^I)^{h,2} & \dots & (\bar{\alpha}_k^I)^{h,L} & 0 \\ \delta_{k,h} & 0 & \dots & 0 & 0 \\ 0 & \delta_{k,h} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_{k,h} & 0 \end{pmatrix} , \quad k, h = 1, \dots, K . \qquad \text{(A.10)}$$

The eigenvalues associated with the price blocks are obtained from the characteristic polynomial defined as the determinant

$$P(\lambda) = \begin{vmatrix} \frac{\partial \mathcal{P}_1}{\partial \mathcal{P}_1} - \lambda I & \dots & \frac{\partial \mathcal{P}_1}{\partial \mathcal{P}_K} & \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{P}_K}{\partial \mathcal{P}_1} & \dots & \frac{\partial \mathcal{P}_K}{\partial \mathcal{P}_K} - \lambda I & \end{vmatrix} ,$$

where $I$ stands for the $(L+1) \times (L+1)$ identity matrix. The last zero columns in each column-block is responsible for a factor $\lambda$. This generates an eigenvalue 0 of multiplicity $K$. Once the associated $K$ columns, and their corresponding rows, have been removed one remains with a residual matrix of dimension $KL$. This matrix has $K$ rows filled with $\bar{\alpha}$s. Each other row is zero but for two elements 1 and $-\lambda$. Using the Laplace formula iteratively, the final expression of the characteristic polynomial of the lower-right block becomes

$$P(\lambda) = \lambda^k \sum_{l_1=1}^{L} \dots \sum_{l_K=1}^{L} \lambda^{LK-\sum_j l_j} \begin{vmatrix} (\bar{\alpha}_1^I)^{1,l_1} - \lambda\delta_{1,l_1} & (\bar{\alpha}_1^I)^{2,l_2} & \dots & (\bar{\alpha}_1^I)^{K,l_K} \\ (\bar{\alpha}_2^I)^{1,l_1} & (\bar{\alpha}_2^I)^{2,l_2} - \lambda\delta_{1,l_2} & \dots & (\bar{\alpha}_2^I)^{K,l_K} \\ \vdots & \vdots & \ddots & \vdots \\ (\bar{\alpha}_K^I)^{1,l_1} & (\bar{\alpha}_K^I)^{2,l_2} & \dots & (\bar{\alpha}_K^I)^{K,l_K} - \lambda\delta_{1,l_K} \end{vmatrix},$$

which, using the Leibniz formula for the computation of the determinant, and dropping the factor $\lambda^K$, reduces to (4.4).

Consider now the linear Random Dynamical System $J^*(T,\omega)$ generated by the Jacobian in $x^*$, $J^*(\cdot)$, and by the shift operator $\theta$

$$J^*(T,\omega) = J^*(\theta^{T-1}\omega)\dots J^*(\theta\omega)J^*(\omega)\,.$$

Applying the Oseledec's multiplicative ergodic theorem (or MET see e.g. Coayla-Teran and Ruffino, 2004, Th. 2.1) the Lyapunov spectrum of $J^*(T,\omega)$ can be used to determine the stability or instability of $x^*$, provided that the integrability condition is satisfied, that is, as long as

$$\mathrm{E}\log^+||J^*|| := \int_\Omega \log^+||J^*(\omega)||\rho(\omega) < \infty\,, \tag{A.11}$$

where $\log^+ a = \mathrm{Max}\{\log a, 0\}$. Due to assumptions **A2-A3** the element of $J^*$ are finite almost surely, so that (A.11) immediately follows.

The MET then states that:

i) there exists a splitting of $R^{(I-1)K(L+1)}$ in $p$ random subspaces $E_1(\omega),\dots,E_p(\omega)$ with non-random dimensions $d_1,\dots,d_p$ such that

$$J^*(T,\omega)\,E_i(\omega) = E_i(\theta^T\omega)\,,\quad i = 1\dots,p\,;$$

ii) there exists a Lyapunov spectrum $\{\eta_1,\dots,\eta_p\}$, $p \le (I-1)K(L+1)$, such that for all $v \in R^{(I-1)K(L+1)}$, $v \neq 0$,

$$\lim_{T\to\infty}\frac{1}{T}\log||J^*(T,\omega)v|| = \eta_i \Leftrightarrow v \in E_i(\omega)\,.$$

The spectrum is related to the stability and instability of $x^*$. When all $\eta_i \neq 0$ for all $i$ we say that $x^*$ is hyperbolic and the stability or instability of $x^*$ carry over locally for the nonlinear map $\varphi$.

The next step is to compute the spectrum of $J^*(T,\omega)$. First, since $J^*(\omega)$ is block triangular for every $\omega$, so it is $J^*(T,\omega)$, which can be written as

$$J^*(T,\omega) = \begin{pmatrix} \left(\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\big|_{x^*}\right)^T & 0 \\ \blacksquare & \left(\frac{\partial \mathcal{P}}{\partial \mathcal{P}}\big|_{x^*}\right)^T \end{pmatrix}, \tag{A.12}$$

where

$$\begin{aligned} \blacksquare &= \sum_{t=0}^{T-1} \left(\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right)^{T-t-1} \frac{\partial \mathcal{P}}{\partial \mathcal{W}} \left(\frac{\partial \mathcal{P}}{\partial \mathcal{P}}\right)^t, \\ &= \sum_{t=0}^{T-1} \frac{\partial \mathcal{W}(\theta^T\omega)}{\partial \mathcal{W}} \cdots \frac{\partial \mathcal{W}(\theta^{t+1}\omega)}{\partial \mathcal{W}} \frac{\partial \mathcal{P}(\theta^t\omega)}{\partial \mathcal{W}} \frac{\partial \mathcal{P}(\theta^{t-1}\omega)}{\partial \mathcal{P}} \cdots \frac{\partial \mathcal{P}(\omega)}{\partial \mathcal{P}}. \end{aligned} \tag{A.13}$$

This implies that the eigenvalues of $J^*(T,\omega)$ are given by the union of the eigenvalues of the $T$-iteration of the 2 diagonal blocks of $J^*(\omega)$.

The left-upper block is random but diagonal. It follows that the first $I-1$ subspaces of the splitting are deterministic and generated by the canonical base. Then, we can compute the first $I-1$ Lyapunov exponents by evaluating

$$\{J^*(T,\omega)\}_{i,i} = \mu_i(\theta^{T-1}\omega)\ldots\mu_i(\theta\omega)\mu_i(\omega)\,, \qquad i = 1\ldots, I-1\,.$$

with $\mu_i()$ as defined in (A.6). Thus

$$\eta_i = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \log||\mu_i(\theta^t\omega)|| = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \log\left(\sum_{k=1}^{K} \frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} d_k(\theta^t\omega)\right)$$

which, because of the assumed ergodicity of the process, reduces to the log of the expression in (4.3).

Concerning the right-lower block, the matrix in (A.10) does not depend upon the realization of $\omega$. This implies that the the subspaces of the splitting of the right-lower block components are just the eigenspaces of $\partial\mathcal{P}/\partial\mathcal{P}$. Moreover, since the eigenvalues of its $T$-product are just the $T$ power of the eigenvalues of $\partial\mathcal{P}/\partial\mathcal{P}$, each Lyapunov exponent $\eta_i$ for $i = I,\ldots,p$ are the roots of (4.4).

Summarizing, the sufficient condition for the asymptotic stability (instability) of $x^*$ is that all the (at least one of the) roots of (4.4) and $\mu_i$ in (4.3) are lower (greater) than one. Moreover, $x^*$ is hyperbolic if all these quantities are different from one.

Finally, since the Random Dynamical System $\varphi$ is $\mathcal{C}^1$ (because $\mathcal{F}$ in (A.1) is $\mathcal{C}^1$) the Local Hartman-Grobman theorem (see Coayla-Teran and Ruffino, 2004, Th. 3.2) ensures that the asymptotic stability results of the linear Random Dynamical System $J^*(T,\omega)$ carry over locally to the Random Dynamical System $\varphi$, and the theorem is proved. $\square$

**Proof of Corollary 4.1** The polynomial (4.4) is heavily simplified when the investment rule of agent $I$ in asset $k$ depends only on current and past prices of asset $k$ itself, the no-cross-dependence condition. In this case all off-diagonal price/price blocks (A.10) have zero entries, and the characteristic polynomial of each diagonal block $k = 1, \ldots, K$ is given by

$$P(\lambda) = \lambda \left( \lambda^L - \sum_{l=1}^{L} \lambda^{L-l} (\bar{\alpha}_k^I)^{(k,l)} \right),$$

that is, one eigenvalue is equal to zero while the other $L$ eigenvalues are the zeros of (4.5). □

**Proof of Theorem 4.5** The proof proceeds along the same lines of that of Theorem 4.3. It is still convenient to omit the state variable $w_t^I$ by using $w_t^I = 1 - \sum_{i=1}^{I-1} w_t^i$. Consider the Jacobian $J^*(\omega)$, of $\mathcal{F}$ computed at the fixed point $x^*$. The components of the off-diagonal wealth/price and price/wealth blocks read

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_k}\Big|_{x^*} \right\}_{i,1} = \begin{cases} 0 & i = 1, \ldots, I-M \\ -\frac{w^{i*}}{p_k^*} d_{k,t}(\omega) & i = I-M+1, \ldots, I-1 \end{cases}, \tag{A.14}$$

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_k}\Big|_{x^*} \right\}_{i,j>1} = \begin{cases} 0 & i = 1, \ldots, I-M \\ \sum_{k'} \frac{w^{i*}}{p_{k'}^*} (\alpha_{k'}^i)^{k,j-1} d_{k',t}(\omega) & i = I-M+1, \ldots, I-1 \end{cases}, \tag{A.15}$$

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{W}}\Big|_{x^*} \right\}_{1,j} = \begin{cases} \mu_j(\omega_{t+1})(\alpha_k^j(p^*) - p_k^*) & j = 1, \ldots, I-M \\ 0 & j = I-M+1, \ldots, I-1 \end{cases}, \tag{A.16}$$

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{W}}\Big|_{x^*} \right\}_{i>1,j} = 0 \quad j = 1, ..., I-1\,, \tag{A.17}$$

for $k = 1, \ldots, K$ and where $\mu_j(\omega_{t+1})$ is defined as in (A.6). Diagonal blocks have a similar structure to that found for the single survivor case. In particular the wealth/wealth block is

$$\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\Big|_{x^*} = \begin{pmatrix} \mu_1(\omega_{t+1}) & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \mu_{I-M}(\omega_{t+1}) & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix}, \tag{A.18}$$

where each $\mu_i(\omega_{t+1})$ is defined in (A.6) and 1s comes from the fact that $\mu_i(\omega_{t+1}) = 1$ for all $i = I-M+1, \ldots, I-1$. Price/price blocks are obtained from (A.7) with the substitution of the derivatives of the $I$-th investment rule with the average of the derivative of all surviving rules, weighted with the associated equilibrium wealth shares. Defining $\langle H \rangle$, $\langle M \rangle$, $\langle \bar{\alpha} \rangle$, as in, respectively, (4.1), (A.8) and (A.9) replacing $(\alpha_k^I)^{h,l}$ with $\langle \alpha_k \rangle^{h,l}$ defined in (4.6), each price/price blocks is given by

$$\left.\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\right|_{x^*} = \begin{pmatrix} \langle \bar{\alpha}_k \rangle^{h,1} & \langle \bar{\alpha}_k \rangle^{h,2} & \dots & \langle \bar{\alpha}_k \rangle^{h,L} & 0 \\ \delta_{k,j} & 0 & \dots & 0 & 0 \\ 0 & \delta_{k,j} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_{k,j} & 0 \end{pmatrix}, \quad k, h = 1, \dots, K\,, \qquad \text{(A.19)}$$

The resulting Jacobian matrix has the structure

$$J^*(\omega) = \begin{pmatrix} W & 0 & 0 \\ 0 & I & A \\ B & 0 & P \end{pmatrix}, \qquad \text{(A.20)}$$

where $\left(\begin{smallmatrix} W & 0 \\ 0 & I \end{smallmatrix}\right)$ is the wealth/wealth block (A.18), in particular $W$ is the $(I - M) \times (I-M)$ upper diagonal block and $I$ is the $(M-1)\times(M-1)$ identity matrix, $P$ is the $K(L+1)\times K(L+1)$ price/price block built using (A.19), $A$ is a $(M-1)\times K(L+1)$ matrix with elements defined by (A.14-A.15), $B$ is a $K(L+1) \times (I - M)$ matrix with elements defined by (A.16-A.17), and 0 denotes, case by case, a matching null matrix. It is a trivial algebraic result that the $T$ products of (A.20) possess the structure

$$J^*(T, \omega) = \begin{pmatrix} W^T & 0 & 0 \\ C' & I & A' \\ B' & 0 & P^T \end{pmatrix},$$

where the exact form of the matrices $A'$,$B'$,$C'$ depend on the choice of $T$ and is not relevant for our analysis. It then follows that the determinant of $J^*(T, \omega)$ can be easily computed as the product of the determinants of its diagonal blocks $W^T$ and $P^T$. As a result, sufficient conditions for stability can be derived along the same lines of the proof of Theorem 4.3, where diagonal blocks have changed from (A.5) and (A.10) to (A.18) and (A.19), respectively.

Notice that, also in the case of multiple survivors, the stochastic component enters only in the diagonal wealth/wealth block. For multiple survivors, however, the characteristic polynomial of the wealth/wealth block possesses a unit root with multiplicity $M - 1$. Consequently, the fixed point is non-hyperbolic, and thus not asymptotically stable. We shall show that each fixed point $x^* = (w^*, p^*)$ belonging to the manifold where

$$\sum_{m=1}^{M} w^{(I-M+m)*} = 1$$

is nevertheless stable. For any realization $\omega$ of the process, the direct sum of the eigenspaces associated with each unitary eigenvalue is the linear space $V_I$ spanned by the $M-1$ vectors $\mathbf{e}_m$, $m = I-M+1,\dots,I-1$ of the canonical base of $\mathbb{R}^{I-1+K(L+1)}$. As the direction of each vector $\mathbf{e}_m$ corresponds to a change in the relative wealth of the m-th and I-th survivor, each small enough perturbation $v \in V_I$ away from $x^*$ push the dynamics to a new point $x'^* = x^* + v = (w'^*, p^*)$ where the wealth distribution $w'^*$ differs from $w^*$ for the reallocation of wealth among the $M$ surviving agents corresponding to $v^*$. Since $x'^*$ is a deterministic fixed point, when perturbations are restricted to $V_I$ the original point $x^*$ is stable. For the more general case notice that any perturbation $h$ can be written as $h = h' + h^\perp$ with $h' \in V_I$, $h^\perp \in V_I^\perp$ and that $x'^*$ is asymptotically stable for perturbations $h^\perp$ along the stable manifold and stable for perturbations $h'$ along the center manifold. The fixed point is hence stable, but not asymptotically stable. □

## A.2 Section 5

**Proof of Theorem 5.1** Since $I_{\rho,d}(\alpha, \mathbf{p})$ is defined only for vectors $\mathbf{p} \in \Delta_+^K$, we can change variables from $\alpha_k$ to $x_k = \frac{\alpha_k}{p_k}$ for every $k = 1,\dots,K$. Solving (5.2) is thus equivalent to solving

$$\min_{\mathbf{x} \in B_+(\mathbf{p})} \{I_{\rho,d}(\mathbf{x})\}\,. \tag{A.21}$$

where $B_+(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^K |\, \sum_k x_k d_k(\omega) > 0 \text{ a.s. and } \mathbf{x}\cdot\mathbf{p} \leq 1\}$. Since the lower bound is never reached for those $\mathbf{x}$ where $\sum_k x_k d_k(\omega) = 0$ with positive probability, we can equally solve $\min\{\exp I_{\rho,d}(\mathbf{x})\}$ on $B(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^K |\, \sum_k x_k d_k(\omega) \geq 0 \text{ a.s. and } \mathbf{x}\cdot\mathbf{p} \leq 1\}$.

For this purpose we first show that when there are no arbitrage opportunities the set $B(\mathbf{p})$ is compact. $B(\mathbf{p})$ is the intersection of (possibly infinite) closed halfspaces and thus a convex and closed polyhedron. The set $B(\mathbf{p})$ is not empty because it contains the non empty set $\{\mathbf{x} \in \mathbb{R}^K | x_k \geq 0 \text{ for all } k \text{ and } \mathbf{x}\cdot\mathbf{p} \leq 1\}$. Assume now by absurd that $B(\mathbf{p})$ is unbounded. Then it must exist a direction $a \in \mathbb{R}^K$, $a \neq 0$, such that for every $x \in B(\mathbf{p})$

$$x + \lambda a \in B(\mathbf{p})\,, \quad \forall \lambda \geq 0\,,.$$

This is equivalent to state that for all $\lambda \geq 0$

$$\sum_k (x_k + \lambda a_k) d_k(\omega) \geq 0 \text{ a.s.}$$

and

$$\sum_k (x_k + \lambda a_k) p_k \leq 1\,.$$

Since $\lambda$ is an arbitrary non negative constant then it must hold

$$\sum_k a_k d_k(\omega) \geq 0 \text{ a.s.}$$

and

$$\sum_k a_k p_k \leq 0\,.$$

Adding to this the fact $ad(\omega) = 0$ a.s. if and only if $a = 0$, which we have excluded, implies that $a$ is an arbitrage. Since the latter contradicts our hypotheses we conclude that $B(\mathbf{p})$ is bounded for all $\mathbf{p}$ where arbitrages are not possible. Being also closed, $B(\mathbf{p})$ is compact.

We have shown that in absence of arbitrage solving (A.21) is equivalent to the minimization of a continuous function on a compact set, whish has always a solution. Moreover, since we have already argued that such minima will never be attained for those $\mathbf{x}$ where $\sum_k x_k d_k(\omega) = 0$ with positive probability, we can go back to solve the original minimization problem (A.21).

Computing the Hessian matrix $H$ of $I_{\rho,d}(\mathbf{x})$ one finds

$$\{H\}_{n,m} = \int_\Omega d\rho(\omega) \frac{d_m(\omega) d_n(\omega)}{\left(\sum_{k=1}^K x_k d_k(\omega)\right)^2}\,, \tag{A.22}$$

so that $\mathbf{y} \cdot H\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^K$, is equal to

$$\mathbf{y} \cdot H\mathbf{y} = \sum_{n,m} y_n y_m \int_\Omega d\rho(\omega) \frac{d_n(\omega) d_m(\omega)}{\left(\sum_{k=1}^K x_k d_k(\omega)\right)^2} = \int_\Omega d\rho(\omega) \frac{\left(\sum_n y_n d_n(\omega)\right)^2}{\left(\sum_{k=1}^K x_k d_k(\omega)\right)^2}\,. \tag{A.23}$$

The former expression is always positive for non-trivial payoff processes $d$, so that $I_{\rho,d}(\mathbf{x})$ is strongly convex for all vectors $\mathbf{x} \in B_+(\mathbf{p})$. Adding to this continuity and non-satiation, which are trivially proved, standard consumer theory theorems, see e.g. Proposition 2.8 in Ginsburgh and Keyzer (1997), can be used to show that $\mathbf{x}^\star(\mathbf{p}) \cdot \mathbf{p} = 1$, which implies $\alpha_0^\star(\mathbf{p}) = 0$, and that $x^\star(\mathbf{p})$ (and thus $\alpha^\star(\mathbf{p})$) is a well defined function of class $\mathcal{C}^1$.

Regarding the equilibria of $\alpha^\star$, by deriving the first order conditions of the minimization problem (A.21), it is immediate to check that $p_k = \int_\Omega d\rho(\omega) d_k(\omega)$ for all $k = 1, \ldots, K$ is the unique vector of prices where $\mathbf{x}^\star(\mathbf{p}) = \mathbf{1}$, and thus where $\alpha^\star(\mathbf{p}) = \mathbf{p}$.

□

**Proof of Theorem 5.2** Given that $\alpha^\star \in \mathcal{E}$, when $\alpha^\star$ is trading equilibrium prices do not allow for arbitrage, otherwise an equilibrium would not exist. It then follows from Th. 5.1 that $\alpha^\star$ is well defined and of class $\mathcal{C}^1$ in $\mathbf{p}$. Let now $w_t$ be the wealth share of the S-rule and assume that there exist s trajectory $\varphi_t$ where $\lim_{t\to\infty} w_t = 0$. Then asymptotic prices converge toward a single survivor equilibrium where the rule $\alpha$ dominates, that is where $\lim_{t\to\infty} p_{t,k} - \alpha_k(p_t) = 0$ for any $k = 1, \dots, K$. It follows that $\lim_{t\to\infty} I_{\rho,d}(\alpha_t, \mathbf{p}_t) = 0$. Since for construction $\alpha^\star$ minimizes $I_\pi(\alpha, \mathbf{p})$ for all $\mathbf{p}$ it holds

$$\lim_{t\to\infty} I_{\rho,d}(\alpha^\star_t, \mathbf{p}_t) = \lim_{t\to\infty} I_{\rho,d}(\alpha^\star_t, \alpha_t) \leq 0 \; .$$

This implies that the quantity $\mu$ defined in (4.3) is never lower than one. When $\mu$ is greater than one, the trajectory $w_t$ converges towards an unstable deterministic fixed point. When $\mu$ is equal to one, the long-run prices are also an equilibrium of $\alpha^\star$. In both cases events in $\omega$ that generate these trajectory $w_t$ on which the S-rule vanishes are of measure zero in $\rho$. $\square$

**Proof of Theorem 5.3** The proof replicate that of Theorem 5.2 at each deterministic fixed point where an agent, or a set of agents, survives. In particular when prices do not converge to the equilibrium of $\alpha^\star$ the corresponding deterministic fixed point is unstable. Thus the only possible stable deterministic fixed points have prices fixed by $\alpha^\star(\mathbf{p}) = \mathbf{p}$ whose unique solution is $p_k = \int_\Omega d\rho(\omega) d_k(\omega)$ for all $k = 1, \dots, K$ as shown in Th. 5.1. Obviously at all these fixed points $\alpha^\star$ survives and at least one of such fixed points exists, namely, the one where $\alpha^\star$ dominates. $\square$

**Proof of Theorem 5.4** Since by hypothesis the price learner rule $\alpha^L$ does not depend on contemporaneous prices and satisfies both the no-cross dependence condition and (5.4), the characteristic polynomial (4.5) reduces to

$$P(\lambda) = \prod_{k=1}^{K} \left( \lambda^L - (\alpha^L_k)_{x^*} \sum_{l=1}^{L} \lambda^{L-l} \right) \; .$$

$P(\lambda)$ is thus the product of $K$ polynomials having all one zero root and the same form namely

$$P(x; \alpha) = x^L - \alpha \sum_{l=0}^{L-1} x^l \; .$$

The problem of determining whether the roots of $P(\lambda)$ are all inside the unit circle can thus be solved by looking at $P(x; \alpha)$.

If $\alpha = 0$ all roots are inside the unite circle. Assume that $\alpha > 0$. On the unit complex circle, $|z| = 1$, it holds

$$|z^L - P(z;\alpha)| = |\alpha \sum_{l=0}^{L-1} z^l| \leq \alpha \sum_{l=0}^{L-1} |z^l| = L\alpha \, .$$

It follows that if $\alpha < 1/L$, $|z^L - P(z;\alpha)| < 1 = |z^L|$ for $|z| = 1$. The latter inequality together with Rouchè's Theorem (see e.g. Lang, 1993) imply that the polynomial $P(z;\alpha)$ and $z^L$ have has the same number of roots inside the unit circle. Moreover notice that if $\alpha \geq 1/L$, it holds both $P(1;\alpha) \leq 0$ and $\lim_{x\to+\infty} P(x;\alpha) = +\infty$, implying the existence of a root greater or equal to one. Provided that $\alpha$ is positive, we have proved that $\alpha < 1/L$ is both a necessary and sufficient condition for $P(x;\alpha)$ having all the roots inside the unit circle.

Take now $\alpha < 0$. The complex polynomial $P(z;\alpha)$ can be rewritten as

$$P(z;\alpha) = \sum_{l=0}^{L} z^l - (1 - |\alpha|) \sum_{l=0}^{L-1} z^l \, .$$

and its roots are the solutions of

$$\sum_{l=0}^{L} z^l = (1 - |\alpha|) \sum_{l=0}^{L-1} z^l \, .$$

Multiplying the left and right hand side by $z - 1$ (remembering we are adding the root $z = 1$) and rearranging the terms leads to

$$|z - (1 - |\alpha|)| = \frac{|\alpha|}{|z|^L} \, ,$$

provided $z \neq 0$, which we can always assume since zero is never a root. Assume now $|\alpha| < 1$. If a root with modulus bigger or equal than one, but different from $z = 1$, exists, one could write

$$|\alpha| < |z - (1 - |\alpha|)| = \frac{|\alpha|}{|z|^L} \leq |\alpha| \, ,$$

which is a contradiction. We have proved that $|\alpha| < 1$ is a sufficient condition for all roots being inside the unit circle. The condition is also necessary. Indeed, since the modulus of the constant in $P(z;\alpha)$, $|\alpha|$, is given by the product of the moduli of all the roots, when $|\alpha| \geq 1$ there must exist at least a root with modulus bigger or equal to 1.

Interestingly, the role of the memory parameter $L$ is different in the case of positive and negative prices feedbacks. In general, for consistent estimators, partial derivatives depend on the number of lags considered and scale with $1/L$: the longer the agent's memory, the lower the partial derivative. Then if $(\alpha_k^L)_{x^*} < -1$, by increasing the number of past observation, that is, the memory, it is always possible to cross the bound of $-1$ and thus stabilize the fixed point. Conversely, if $(\alpha_k^L)_{x^*} > 1/L$, an increase in the memory of the strategy does not improve the stability of the fixed point because the bound scales with $1/L$ as well. $\square$

**Proof of Corollary 5.1** The corollary is easily proved by using results from Theorem 5.4 and upon realizing that the characteristic polynomial now depends on the convex combination of partial derivatives, that is, $\langle\alpha\rangle_k = (1-w^*)(\alpha_k^L)_{x^*}\ k = 1, \ldots, K$, rather than on $(\alpha_k)_{x^*}\ k = 1, \ldots, K$, because all the partial derivatives of the S-rule are zero. $\square$