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Decidability and manipulability in social choice

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Decidability and manipulability in social choice

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Abstract

We present a geometric model of social choice among bundles of interdependent elements, that we will call objects. We show that the outcome of the social choice process is highly dependent on the way these bundles are formed. By bundling and unbundling the same set of constituent elements an authority enjoys a vast power of determining the social outcome, as locally or globally stable social optima can be created or eliminated. At the same time, by bundling and unbundling elements, it is also possible to greatly increase or decrease decidability both because the likelihood of intransitive cycles varies and because the time required to reach a social optimum varies. In this paper we present a rigorous framework which allows us to study this trade-off between decidability and non-manipulability.

Keywords:
Social choice, agenda, object construction,
hyperplane arrangement, directed graphs, algorithm.

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1 Introduction

Social choice theory usually assumes that agents are faced with a set of exogenously given and mutually exclusive alternatives. These alternatives are given in the sense that the pre-choice process through which they are constructed is not analyzed. Moreover, these alternatives are "simple", in the sense that they are one-dimensional objects or, even when they are multidimensional, they are simply points in some portion of the homogeneous $\mathbb{R}^n$ space and they lack any internal structure that limits the set of possible alternatives.

Many choices in real life situations depart substantially from this simple setting. Choices are often made among bundles of interdependent elements. Those bundles may be formed in a variety of ways, which in turn affect the selection process of a social outcome. Let us consider, for instance, a typical textbook example of social choice, i.e. the case of a group of friends deciding how to spend the evening by democratic and sincere pairwise majority voting. The textbook would start from a given and predefined choice set as $X = \{A, B, C, D, \ldots\}$ where $A, B, C, D, \ldots$ could stand for *movie, concert, restaurant, dinner at home, ...*. At closer scrutiny, these alternatives are neither primitive nor exogenously given. Going to the movies or to a restaurant are labels for bundles of elements (e.g. with whom, where, when, movie genre, director, type of food, etc.) and everyone's preference is unlikely to be expressed before these labels are specified in their constituting elements.

Moving on, to more serious examples, candidates and parties in political elections stand for complex bundles of interdependent policies and personality traits. Committees and boards are called to decide upon packages of policies, e.g. a recruitment package that a university governing board has to approve. In principle, any combination of elements (subject to a budget or some other constraint) could be considered and compared (e.g. through majority voting) with any other, but in reality only a relatively small number of packages undergo examination. Typically, the bundling of elements serves the purpose of reducing the number of alternatives to be examined, by decomposing the whole space of alternatives into smaller subspaces.

In this paper we present a model of social choice among bundles of elements, which we call *objects*, and model two non-standard features that objects are likely to have. First, generally objects are not simply aggregations of primitive components but have an internal structure that is likely to determine interdependencies and non-separabilities in individual preferences.

In the *"what shall we do tonight?"* choice setting, my preferences on the *with whom* element is likely to be highly interdependent with the other elements, as I may well find a given person a perfect companion for an evening at the movies but relatively dislike her or his company if we finally decide to go to a restaurant. On the same token I may prefer Italian food as instantiation of the *type of food* item if *dinner at home* is chosen but French cuisine if we opt for *going to the restaurant* and the *where* item takes the value "Paris".

Second, objects provide structure to the choice problem. Consider again the *"what shall we do tonight?"* case. A possible reply to our point on bundles would be that the choice set X is underspecified and that we should start from a choice set formed by all possible combinations of the elements, i.e. that the set X should be properly built in such a way as to include the exhaustive list of all mutually exclusive alternatives. However, for obvious combinatorial arguments, this set, even in this simple example, would be so large that any exhaustive choice procedure, e.g. pairwise majority voting, could not be completed in a feasible time span. In our approach, objects decompose the search space into quasi-separable subspaces (Simon 1982) and simplify the computational task of collective choice, making decisions possible.

There is also another way in which objects can contribute to making the determination of a social outcome easier. We will show that, by appropriate object construction, intransitive cycles that often characterize social decisions can almost always be eliminated. In general, coarse objects, i.e. those made of many elements, tend to produce many cycles, whereas fine objects made of one or few elements do not. However finer objects do so by increasing the number of locally stable social optima and thereby making the social outcome more manipulable through the control of object construction, initial conditions and agendas. We will analyze how different sets of objects strike different balances in the trade-off between decidability and non-manipulability.

The main contribution of this paper is that we offer a rigorous framework in which we can analyze this trade off. In the classical social choice model, whereby choice takes place among an exhaustive set of unstructured and primitive alternatives, decidability tends to be low, both because the likelihood of encountering an intransitive cycle is very high and because an exhaustive comparison among alternatives may take too long. In our framework instead social choice is among structured bundles of elements (that we call objects). The space of alternatives is decomposed and many social optima (that we call local and u-local optima) are generated, while the likelihood of intransitive cycles is sharply reduced. Thus in our framework decidability is greatly enhanced, but at the same time also manipulability is greatly increased.

Because of interdependencies that are likely to characterize individual preferences over objects, the way objects are constructed by bundling (or unbundling) elements can strongly impact on the outcome of social choice. We show that, in general, by appropriately constructing objects, the outcome of a social choice process, e.g. pairwise majority voting, may be heavily manipulated. An authority who has the power to construct objects may obtain a desired outcome even when the latter is chosen democratically. In the paper we will provide a precise formal characterization of the trade-off between decidability and non-manipulability.

In order to formally analyze the properties of a social choice model with object construction power we will use some geometric properties of hyperplane arrangements and link them to graph theoretic representations. We believe that our paper also provides novel analytical tools for modeling choice problems that could be applied to a variety of different settings.

The paper is structured as follows: in section 2 we briefly discuss the similarities and differences between our approach and those already existing in the literature. To our knowledge, the issue of object construction in social choice has never been addressed the way we do. Indeed, our approach has close links with standard results on multidimensional voting and on agenda power, but there are fundamental differences that make our model new and somehow more general.

In section 3 we set a general framework of social choice, then in section 4 we outline our geometric and algebraic model of choice among objects. A key ingredient of our analytic approach is the theory of hyperplane arrangements and the related graphs. Then, in section 5 we draw the main results concerning how objects strike a balance in the trade-off between decidability and non-manipulability. Finally, in section 6 we draw some conclusions.

2 Relation to Literature

To our knowledge, the issue of object construction has not been dealt with by economic models before the recent contributions of Marengo and Pasquali (2011), Marengo and Settepanella (2010), Amendola and Settepanella (2012).

In the first paper the notion of object construction power is presented and discussed by way of examples and agent based simulations, in the second a mathematical model is given, while in the third the problem is tackled using tournament theory and an efficient algorithm which finds local optima is presented.

The literature on multidimensional voting models (Kramer 1972, Shepsle 1979, Denzau and Mackay 1981, Enelow and Hinich 1983) is relatively close to our perspective. In particular, Shepsle (1979) presents a model of majority voting in which institutions play a similar role to the one objects have in our own model, i.e. that of limiting the set of outcomes that undergo examination. Two institutional mechanisms are analyzed: jurisdictional restrictions – especially those induced by decentralization and division of labour among decision making units – and agenda limitations to the possible amendments to the *status quo*. Both limit the set of attainable outcomes and equilibria (called structure-induced equilibria) and can rule out cycles. There are at least two important differences between this perspective and ours. First, the problem tackled by all these papers is essentially the one arising from the sequential interdependency of voting: how we settle an issue today may change how we prefer to settle a related issue tomorrow. In our approach, we instead focus on interdependencies generated by how elements interact within the particular objects we are deliberating upon. Second, in Shepsle (1979), restrictions on attainable outcomes are placed by legal and organizational rules, that limit the set of allowed amendments to the *status quo*. Instead, in our approach restrictions are placed by the object construction process exerted by some agent or institution: once an object has been defined, all its instances are always generated and compared. Related to this is also Enelow and Hinich (1983), that considers a multi-issue case in which each issue is voted sequentially in time and when the agenda induces path-dependency, which might be mitigated by the agents' forecasting abilities.

Our work is closely linked to the literature on agenda power (McKelvey 1976, Plott and Levine 1978), and we will show that we generalize some of its results in the sense that even agenda power is subject to manipulation through object design. Moreover, our model presents some instances of a wide family of aggregation paradoxes in voting. Saari and Sieberg (2001) discuss the links between aggregation paradoxes in voting and similar aggregation paradoxes arising in statistics such as the so-called Simpson's paradox. Logrolling models (Buchanan and Tullock 1962) discuss some of these paradoxes which are similar to those in the present paper. Bernholz (1974) showed that logrolling implies cycles, therefore our result proving that cycles may be broken or created by appropriate object construction also extends to logrolling.

Our paper is also related to recent literature that has begun to analyze decision-making when agents group states of the world into coarse categories (Mullainathan 2000, Fryer and Jackson 2008). They show, among other things, that in these circumstances agents may be persuaded, meaning that uninformative messages may influence their decisions (Mullainathan, Schwartzstein, and Shleifer 2008). Our perspective is different and complementary: our objects are not categories based on similarities among the states of the world, but are bundles of different and separate elements with an internal structure of interdependencies and not sets of states of the world that agents cannot distinguish from each other.

Context-dependent voting has also been analyzed by some papers (Callander and Wilson 2006). In these papers context-dependency refers to the violation of the axiom of Independence of Irrelevant Alternatives (IIA), i.e. the assumption that the preference expressed by an agent between two outcomes x and y does not depend on the presence or absence of other outcomes in the choice set. Psychologists and marketing scholars have observed systematic violations of IIA (Kahneman and Tversky 2000). In our model we assume a different form of context dependency, meaning that preferences between two instantiations of an element (feature in our terminology) in general depend on the value taken by other traits. In the next section we argue why this form of non-separability is very likely to happen in our context of objects made up of interdependent features.

This paper is also meant to contribute to the development of rigorous analytical tools in social choice models. We provide here a geometric representation based on hyperplanes arrangement theory and algebraic topology. Indeed, geometric approaches have already been used in the literature on social choice. Donald Saari has greatly contributed to establishing general geometric representations of voting models and voting paradoxes (Saari 1994, Saari 2000a, Saari 2000b), and we will argue later that our representation is more general in many respects. Eckmann (1954), Eckmann, Ganea, and Hilton (1962), Chichilnisky (1980), Chichilnisky (1983) study the problem of the existence of a social decision function from a topological point of view and show that the paradoxes of social choice are partly a consequence of the topological structure of the spaces of ordinal preferences. On the other hand, Baryshnikov (1997) discusses the possibility of introducing topological methods in the combinatorial paradigm of social choice theory. Weinberger (2004) and Terao (2007) extend well-known results on social choice functions to, respectively, CW complexes and arrangements, thus obtaining new results for both mathematical objects.

In this respect, our model is a novel contribution to the analysis of the relation between discrete problems of social choice and their topological structure and it provides a bridge between a geometric and topological representation of social choice problems in order to create a more general framework in which the topological space is manipulable through object construction.

3 Social decision rules

Social decision rules Consider a population of ν *agents*. Each agent i is characterized by a *system of transitive preferences* $\succeq_i$ over the set of social outcomes X. The set of systems of transitive preferences $\succeq$ is denoted by $\mathcal{P}$. A *social decision rule* $\mathcal{R}$ is a function:

$$\begin{array}{cccc} \mathcal{R}: & \mathcal{P}^\nu & \longrightarrow & \overline{\mathcal{P}} \\ & (\succeq_1, \dots, \succeq_\nu) & \longmapsto & \succeq_{\mathcal{R}(\succeq_1,\dots,\succeq_\nu)} \end{array}$$

which determines a *system of social preferences* or *social rule* $\succeq_{\mathcal{R}(\succeq_1,\dots,\succeq_\nu)}$ from the preferences of ν individual agents. With $\overline{\mathcal{P}}$ we denote the set of systems of (non-necessarily transitive) social preferences. As well known, the social rule $\succeq_{\mathcal{R}(\succeq_1,\dots,\succeq_\nu)}$ is not, in general, transitive, even when each individual's preferences are indeed transitive.

If Δ is the diagonal of the cartesian product $X \times X$, the element $\succeq_{\mathcal{R}} \in \overline{\mathcal{P}}$ defines a subset

$$\mathcal{Y}_{1,\succeq_{\mathcal{R}}} = \{(x, y) \in X \times X \setminus \Delta \mid x \succeq_{\mathcal{R}} y\}$$

and the set of *relevant* social outcomes

$$\mathcal{Y}_{0,\succeq_{\mathcal{R}}} = \{x \in X \mid \forall y \in X,\ (x, y) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}} \text{ or } (y, x) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}}\}.$$

If $\mathcal{Y}_{0,\succeq_{\mathcal{R}}}$ is the whole X, the social rule is said to be *complete*. If the two conditions $x \succeq_{\mathcal{R}} y$ and $y \succeq_{\mathcal{R}} x$ are mutually exclusive, the social rule is said to be *strict*. For the sake of simplicity *we will mainly focus on complete and strict social preferences* $\succ$, generalization to weak preferences is almost always straightforward.

The graph The sets $\mathcal{Y}_{0,\succ}$ and $\mathcal{Y}_{1,\succ}$ are, respectively, the sets of nodes and arcs of a graph $\mathcal{Y}_{\succ} = (\mathcal{Y}_{0,\succ}, \mathcal{Y}_{1,\succ})$. Two nodes x and y in $\mathcal{Y}_{0,\succ}$ are connected by an arc if $(x, y) \in \mathcal{Y}_{1,\succ}$ or $(y, x) \in \mathcal{Y}_{1,\succ}$; the orientation is from x to y in the former case and from y to x in the latter. Notice that the assumption of complete preferences guarantees that we will deal only with *connected* graphs. For the sake of simplicity, we will use the same symbol x for the nodes of $\mathcal{Y}_{\succ}$ and (x, y) for its arcs.

A cycle

$$(x_1, x_2), (x_2, x_3), \ldots, (x_h, x_1)$$

in the graph $\mathcal{Y}_{\succ}$ corresponds to a cycle *à la* Condorcet-Arrow (Condorcet de Caritat 1785, Arrow 1951), i.e. to the sequence

$$x_1 \succ x_2 \succ \cdots \succ x_h \succ x_1.$$

4 Choices among bundle of elements

Let us now explicitly allow for choice among bundles of interrelated elements, by going back to our *"what shall we do tonigt?"* example. If the friends have simply to decide upon where and when to go, we get, for instance, the following two-dimensional case:

Where? movie (0), restaurant (1), pub (2) First feature f_1

When? 20:00 (0), 22:00 (1) Second feature f_2

There are $3 \times 2 = 6$ possibilities and each alternative is a bundle of interdependent elements. The sub-alternatives are grouped into features and, in each group, denoted by numbers (starting from 0). So, for instance, "movie at 20:00" is preferred to "pub at 22:00", and this preference is denoted by $00 \succ 21$.

If all the preferences are expressed, one obtains the associated graph, which states the aggregated preferences of the group. In figure 1 an example of such a graph is shown. In order to refer to this example later on, we will call it *the 2d social rule.*

In figure 2 instead we present an hypothetical graph associated to a social rule with three features, each with two sub-alternatives. In order to make the graph readable, we draw only the main edges, i.e. the ones that determine, in our framework, optima and cycles irrespectively of the direction of the remaining edges. We will call this example *the 3d social rule.*

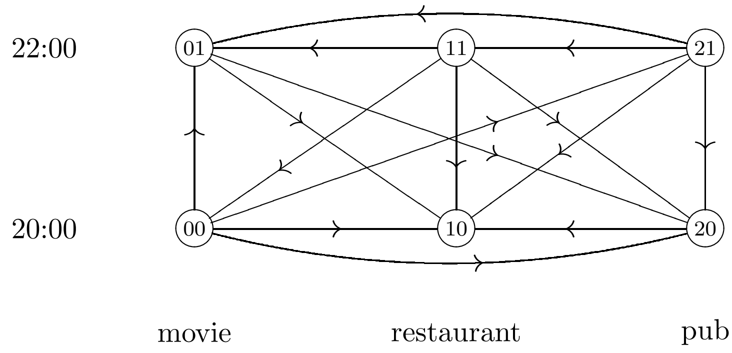

Figure 1: The graph associated to the 2d social rule.

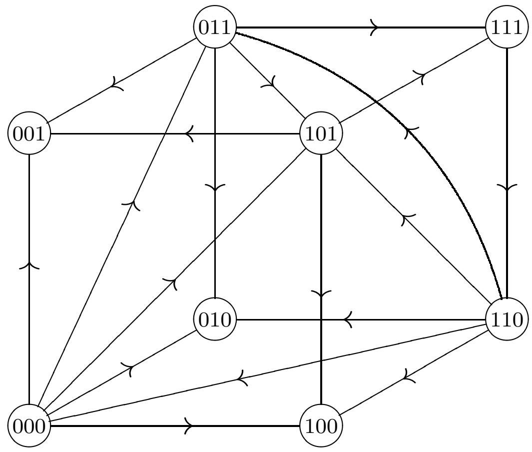

Figure 2: The graph associated to the 3d social rule.

Features Let $F = \{f_1, \ldots, f_n\}$ be a bundle of elements, called *features*, the i-th of which takes m_i values, i.e. $\{0, 1, 2, \ldots, m_i - 1\}$ with $i = 1, \ldots, n$. Denote by $m = (m_1, \ldots, m_n)$ the multi-index of the numbers of values of the features. From now on, a *social outcome* (or *configuration*) will be an n-sequence $v_1 \cdots v_n$ of values such that $0 \leqslant v_i < m_i$. The set of all social outcomes will be denoted by X. The cardinality of X is $\prod_{i=1}^{n} m_i$ and will be denoted by M.

The decision process We suppose that social choice proceeds along the following steps:

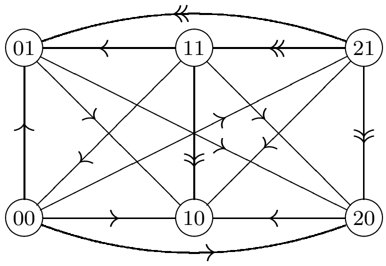

Figure 3: The arcs yielding the decision process "10 $\to$ 11 $\to$ 21 stop" for the 2d social rule.

- start from a *status quo* social outcome,
- take the first object in the agenda, i.e. a subset of features, and find the most preferred configuration of such an object (keeping constant all the other features),
- repeat for all objects in the agenda,
- repeat until either a cycle is encountered or a configuration that cannot be improved (an *optimum*) is reached.

Example 4.1. Three such processes in the 2d social rule are the following:

- $10 \overset{f_2}{\to} 11 \overset{f_1}{\to} 21$ stop,
- $10 \overset{f_1}{\to} 00$ stop,
- $01 \overset{f_1}{\to} 21$ stop;

see figure 3. (In the figures the arcs that are significant are drawn with two arrows.) In any of the three cases the last configuration cannot be improved anymore by considering the objects $\{f_1\}$ and $\{f_2\}$.

Remark 4.2. The classical model corresponds in this approach to considering only one object which contains all the features.

This approach has advantages in terms of decidability, e.g. we can find an "optimum" more often than in the classical model (Marengo and Settepanella 2010, Amendola and Settepanella 2012), but it poses many questions in terms of manipulability such as:

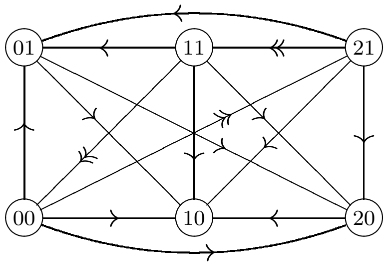

Figure 4: In the 2d social rule the social outcomes 00 and 21 are not optima in the classical meaning.

- Who decides the sets of objects?
- Who decides the order with which objects are considered (i.e. the agenda)?
- Who decides the initial *status quo*?

Example 4.3 (New kinds of optima)**.** In the 2d social rule the social outcomes 00 and 21 are "optima", i.e. there are decision processes that end in them. However, none of them is an optimum in the classical sense, because they are contained in a cycle, as shown in figure 4.

4.1 Hyperplanes and social choice

In this section we present a formalization of the social choice problem outlined above which is based upon hyperplane arrangements.

The hyperplane arrangement In the n-dimensionale space $\mathbb{R}^n$, an *hyperplane* H is a *flat* subset of dimension $n-1$, or equivalently, of codimension 1. Any hyperplane can be given in coordinates as the zero locus of a single degree-1 polynomial $\alpha_H \in \mathbb{R}[\lambda_1, \ldots, \lambda_n]$. An *hyperplane arrangement* is a finite set of hyperplanes.

Consider the hyperplane arrangement defined by:

$$\mathcal{A}_{n,m} = \left\{ H_{i,j} \mid \alpha_{H_{i,j}} = \lambda_i - j \right\}_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j < m_i - 1}} .$$

Note that the hyperplanes $H_{i,*}$ correspond to the i-th feature, and they are one less than the number of values taken by the i-th feature.

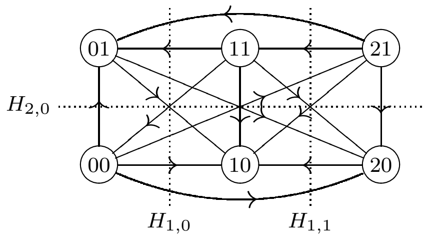

Figure 5: The hyperplane arrangement corresponding to the 2d social rule.

The *complement* of an hyperplane arrangement $\mathcal{A}$ is defined as the whole space minus the hyperplanes in $\mathcal{A}$, i.e.:

$$\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

The complement of $\mathcal{A}$ is disconnected: it is made up of separate pieces (called *chambers*) each of which may be either bounded or unbounded. There is a correspondence (Marengo and Settepanella 2010) between the set X of social outcomes and the set of the chambers of the hyperplane arrangement $\mathcal{A}_{n,m}$. Namely, $x = v_1 \cdots v_n$ corresponds to the chamber that contains the open set

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid v_j - 1 < \lambda_j < v_j,\ j = 1, \ldots, n\}.$$

Example 4.4. The hyperplane arrangement

$$\mathcal{A}_{2,(3,2)} = \{H_{1,0},\ H_{1,1},\ H_{2,0}\}$$

corresponding to the 2d social rule is shown in figure 5. Each chamber corresponds to a vertex (i.e. a social outcome), and vice versa. Moreover, each vertex is connected to any other by an arc that crosses one or more hyperplanes.

Objects schemes Given a non-empty subset $I \subseteq \{1, \ldots, n\}$, the *object* $\mathcal{A}_I$ is the subset

$$\mathcal{A}_I = \{H_{i,j}\}_{\substack{i \in I \\ 0 \leqslant j < m_i - 1}}$$

of the hyperplane arrangement $\mathcal{A}_{n,m}$, i.e. the subset made up of the hyperplanes corresponding to the features belonging to I. The complemental set of a set I in $\{1, \ldots, n\}$ will be denoted by I^c, and corresponds to the complemental hyperplane arrangement $\mathcal{A}_I^c = \mathcal{A}_{n,m} \setminus \mathcal{A}_I$ of the hyperplane arrangement $\mathcal{A}_I$ in $\mathcal{A}_{n,m}$.

An *objects scheme* is a set of objects $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ such that $\bigcup_{j=1}^k I_j = \{1, \ldots, n\}$. Note that the sets I_j may have non-empty intersections. From now on, unless explicitly stated, A will always denote an objects scheme $\{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$.

Example 4.5. In the 2d social rule there are three objects (see figure 5):

$$\mathcal{A}_{\{1\}} = \{H_{1,0}, H_{1,1}\}, \qquad \mathcal{A}_{\{2\}} = \{H_{2,0}\}, \qquad \mathcal{A}_{\{1,2\}} = \{H_{1,0}, H_{1,1}, H_{2,0}\}.$$

The sets

$$\{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\} \qquad \text{and} \qquad \{\mathcal{A}_{\{1,2\}}\}$$

are two different objects schemes.

Agenda An *agenda α of an objects scheme A* is an ordered t-uple of indices $(h_1, \ldots, h_t)$ with $t \geq k$ such that $\{h_1, \ldots, h_t\} = \{1, \ldots, k\}$. An agenda α states the order in which the objects $\mathcal{A}_{I_i}$ are decided upon. The ordered t-uple of objects $(\mathcal{A}_{I_{h_1}}, \ldots, \mathcal{A}_{I_{h_t}})$ is denoted by A_α. The set of all possible agendas of A is denoted by $\Lambda(A)$. Note that repetitions, in general, may be allowed.

Domination path A *domination path* $DP(x, y, A)$ through an objects scheme A is a sequence of social outcomes

$$x = x_0 \to x_1 \to \cdots \to x_s = y$$

such that x_i is the optimum among the social outcomes that lie on the same side of the hyperplanes in the complement of an object $\mathcal{A}_{I_{h_i}}$ as x_{i-1} (numbers are considered modulo k, i.e. the cardinality of A). The social outcome x_i is called the *best neighbor* of x_{i-1} with respect to the object $\mathcal{A}_{I_{h_i}}$. Note that a social outcome x_{i-1} can be the best neighbor of itself, i.e. it is the preferred choice among the social outcomes that lie on the same side of itself when we consider the hyperplanes in the complement of the object $\mathcal{A}_{I_{h_i}}$. The domination path is said to *end in x_s* if it can be indefinitely extended to

$$x_0 \to x_1 \to \cdots \to x_s \to \cdots \to x_s$$

by considering all the objects in $\mathcal{A}$ at least once, or equivalently if x_s is the best neighbor of itself with respect to each object in $\mathcal{A}$. Note that no assumption on the order of the objects $\mathcal{A}_{I_{h*}}$ is made.

Example 4.6. In the 2d social rule consider the objects scheme $A = \{\mathcal{A}_{\{1\}}, \mathcal{A}_{\{2\}}\}$. The sequence of social outcomes

$$10 \to 11 \to 21$$

is a domination path through A. Indeed, the social outcome 11 is the best neighbor of 10 with respect to the object $\mathcal{A}_{\{2\}}$ (because 11 dominates 10), and the social outcome 21 is the best neighbor of 11 with respect to the object $\mathcal{A}_{\{1\}}$, because 21 dominates 01 and 11 (see figure 3). Moreover, this domination path ends in the social outcome 21.

Example 4.7. In the 3d social rule consider the objects scheme $A = \{\mathcal{A}_{\{1,2\}}, \mathcal{A}_{\{1,3\}}, \mathcal{A}_{\{3\}}\}$. The sequence of social outcomes

$$000 \to 110 \to 111 \to 011$$

is a domination path through A. Indeed (see figure 6), the social outcome 110 is the best neighbor of 000 with respect to the object $\mathcal{A}_{\{1,2\}}$ (because 110 dominates 000, 010 and 100), the social outcome 111 is the best neighbor of 110 with respect to the object $\mathcal{A}_{\{3\}}$ (because 111 dominates 110), and the social outcome 011 is the best neighbor of 111 with respect to the object $\mathcal{A}_{\{1,2\}}$ (because 011 dominates 001, 101 and 111).

Note however that the social outcome 011 admits a best neighbor, 110, with respect to the object $\mathcal{A}_{\{1,3\}}$. Indeed this domination path enters a cycle.

Let $\alpha = (h_1, \ldots, h_t)$ be an agenda of an objects scheme A. A domination path through A

$$x = x_0 \to x_1 \to \cdots \to x_s = y$$

is said to be *ordered along* α if the order of the objects $\mathcal{A}_{I_{h*}}$ is given by α, i.e. if x_i is the best neighbor of x_{i-1} with respect to the object $\mathcal{A}_{I_{h_q+1}}$ where h_q is the remainder of the division of $i-1$ by t. Such a domination path will be denoted by $DP(x, y, A_\alpha)$.

Example 4.8. The domination path described in Example 4.6 is ordered along the agenda $(2, 1)$.

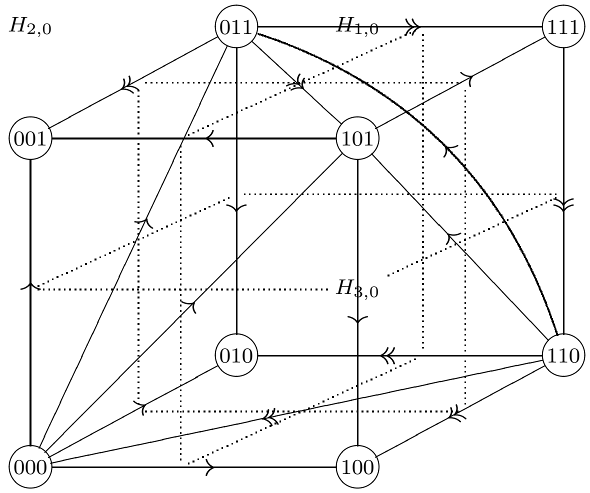

Figure 6: The arcs yielding the domination path 000 → 110 → 111 → 011 for the 3d social rule.

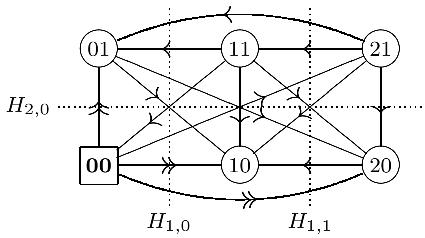

Figure 7: In the 2d social rule the social outcome 00 is a local optimum for the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$.

Local optima A *local optimum* for an objects scheme A is a social outcome such that at least one domination path through A ends in it.

It is obvious that, in general, more than one domination path ends in a local optimum and there may be more than one local optimum.

Example 4.9. It is easy to show that in the 2d social rule there are two different local optima for the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$: 00 and 21. For instance, figure 7 shows that 00 is a local optimum.

Remark 4.10. Consider a domination path ordered along an agenda of an object scheme such that its length is greater than the length of the agenda.

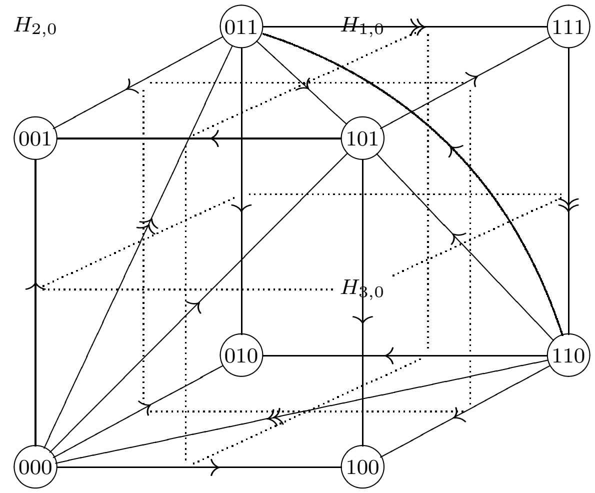

Figure 8: In the 3d social rule a domination path that is a cycle.

If the agenda is repeated over and over again, then either the ordered domination path ends in a local optimum or a cycle is reached.

Example 4.11. In the 3d social rule the domination path

$$000 \to 110 \to 111 \to 011 \to 000$$

of the object scheme $A = \{\mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{2,3\}},\ \mathcal{A}_{\{3\}}\}$ ordered along the agenda such that A_α is $(\mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{3\}}\ \mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{2,3\}})$ is a cycle (see figure 8).

Basin of attraction The *basin of attraction* $\Psi(x, A)$ of a local optimum x with respect to an objects scheme A is the set of the social outcomes y such that there exists a domination path $DP(y, x, A)$ that ends in x.

Example 4.12. In the 2d social rule the basin of attraction of the social outcome 21 with respect to the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$ is $\{10,\ 20,\ 01,\ 11,\ 21\}$ (see figure 9). Indeed, the following domination paths end in 21:

- $11 \to 21$,
- $10 \to 11 \to 21$,
- $01 \to 21$,
- $20 \to 21$;

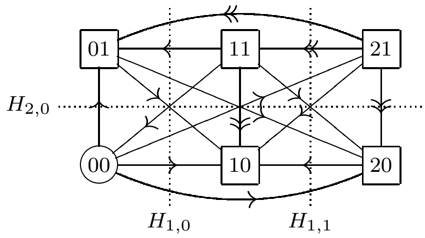

Figure 9: In the 2d social rule the basin of attraction of the social outcome 21 with respect to the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}}\}$ is $\{10,\ 20,\ 01,\ 11,\ 21\}$.

Global optima A *global optimum* of an objects scheme A is a social outcome z whose basin of attraction is the whole set of social outcomes, i.e. $\Psi(z, A) = X$.

Remark 4.13. A *classical optimum* is a global optimum with respect to the objects scheme

$$A = \left\{\mathcal{A}_{\{1,\ldots,n\}}\right\}.$$

As an example, in the 2d social rule there is obviously no global optimum.

Local and global optima strictly depend on the choice of the objects scheme A. If an individual has the right to construct the objects, he or she will enjoy a vast power of influencing the outcome of social choice. This is called "object construction power" (Marengo and Pasquali 2011, Marengo and Settepanella 2010).

To show this let us call *prominent distance* $d_p(x, y)$ between two social outcomes x and y the number of features for which x and y differ. The following result holds.

Theorem 4.14 (Marengo and Settepanella (2010))**.** *Let z be a social outcome. There exists an objects scheme A_z for which z is a local optimum if and only if the inequality $d_p(w, z) > 1$ holds for any social outcome w with $w \succ z$.*

Proof. If z is a social outcome such that all social outcome $x_1, \ldots, x_k$ with $x_j \succ_{\mathcal{R}} z$ verify $d_p(x_j, z) > 1$, then x_j and z are prominently separated at least by two hyperplanes.

It follows that we can build an objects scheme A_z such that $\mathcal{H}_{x_j,z} \nsubseteq \mathcal{A}$ for all $\mathcal{A} \in A_z$ and all $1 \leq j \leq k$. For example, if $H_j^1, H_j^2 \in \mathcal{H}_{x_j,z}$ are two hyperplanes related to different features for $1 \leq j \leq k$, then let us consider an objects scheme A_z such that for any x_j there exist two objects $\mathcal{A}_j^1, \mathcal{A}_j^2$ in A_z with $H_j^1 \in \mathcal{A}_j^1$, $H_j^2 \in \mathcal{A}_j^2$ and $\{H_j^1, H_j^2\} \nsubseteq \mathcal{A}$ for all $\mathcal{A} \in A_z$.

It is obvious that such an objects scheme exists. Moreover z is a local optimum for A_z. Indeed for all $x_j \succ_{\mathcal{R}} z$ and for all $\mathcal{A} \in A_z$ the chambers $C_{x_j}(\mathcal{A}^c)$ and $C_z(\mathcal{A}^c)$ are always separated by H_j^1 or H_j^2. That is $x_j(\mathcal{A}^c) \neq z(\mathcal{A}^c)$ and then $\Phi(z, \mathcal{A}) = \emptyset$ for all $\mathcal{A} \in A_z$. It follows that $z \in \Psi(z, A_z)$ and then $z \in \Psi(z, A_{z,\alpha})$ for all agendas α, i.e. z is a local optimum.

On the other hand if x is a social outcome $x \succ_{\mathcal{R}} z$ such that $d_p(x, z) = 1$ then for any objects scheme A there is at least one object $\mathcal{A}$ such that all hyperplanes H separating x from z are in $\mathcal{A}$. Then, by definition, $x \in \Phi(z, \mathcal{A}) \neq \emptyset$. □

Universal basin of attraction The *universal basin of attraction* of a social outcome z is the union of the basins of attraction of z with respect to each objects scheme, i.e. the set

$$\Psi(z) = \bigcup_{A \in \Pi(\mathcal{A}_{n,m})} \Psi(z, A),$$

where $\Pi(\mathcal{A}_{n,m})$ is the set of all possible objects schemes in $\mathcal{A}_{n,m}$.

Example 4.15. In the 2d social rule the universal basin of attraction of the social outcome 21 is $\{10,\ 20,\ 01,\ 11,\ 21\}$.

U-local optima A social outcome z is said to be a *u-local optimum* if its universal basin of attraction $\Psi(z)$ is the whole set of social outcomes X.

Remark 4.16. A classical optimum is a global optimum that is necessarily a u-local optimum, and a u-local optimum is necessarily a local optimum for at least one objects scheme. In other words, the most demanding (and therefore the least likely to exist) notion of social optimum is the classical one, followed by the global optimum, by the u-local optimum and, finally, by the local optimum, which on the contrary is the least demanding and the most likely to exist.

Example 4.17. In the 3d social rule there are no classical optima, but we can find global optima, u-local optima and local optima. It can also be proved that the 3d social rule is the smallest case with these properties (Amendola and Settepanella 2012). Let us describe all the kinds of optima in detail:

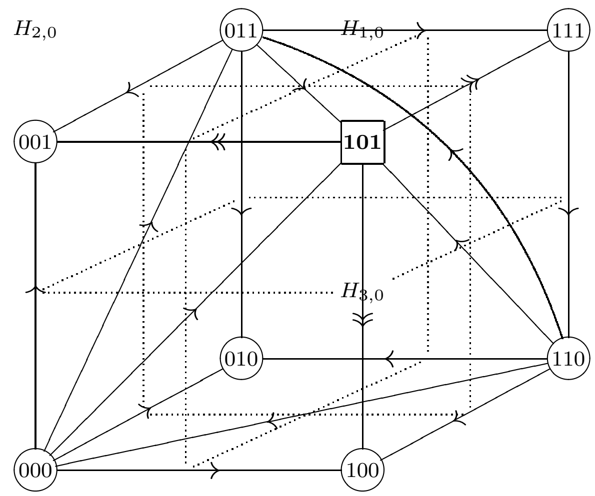

Figure 10: In the 3d social rule the social outcome 101 is a local optimum.

- The social outcome 101 is a local optimum for the objects scheme $A = \{\mathcal{A}_{\{1\}},\ \mathcal{A}_{\{2\}},\ \mathcal{A}_{\{3\}}\}$. Recall that $\mathcal{A}_{\{1\}} = \{H_{1,0}\}$, $\mathcal{A}_{\{2\}} = \{H_{2,0}\}$ and $\mathcal{A}_{\{3\}} = \{H_{3,0}\}$ and see figure 10.

- The social outcome 011 is a global optimum for any agenda of the objects scheme $A = \{\mathcal{A}_{\{1,2\}},\ \mathcal{A}_{\{3\}}\}$. Recall that $\mathcal{A}_{\{1,2\}} = \{H_{1,0},\ H_{2,0}\}$ and $\mathcal{A}_{\{3\}} = \{H_{3,0}\}$ and see figure 11.

- The social outcome 000 is a u-local optimum. Recall that we can consider different objects schemes and see figure 12. For the two-arrow arcs that have 000 as an endpoint we have used the objects (with two hyperplanes) $\mathcal{A}_{\{1,3\}}$ and $\mathcal{A}_{\{2,3\}}$. For the other two-arrow arcs we have used the objects (with one hyperplane) $\mathcal{A}_{\{2\}}$ and $\mathcal{A}_{\{3\}}$. Note that in the latter case we have to add another object, say $\mathcal{A}_{\{1,3\}}$, to have an object scheme.

- Finally, in this social decision rule there is no classical optimum. Indeed there is a cycle (the two-arrow arcs in figure 13), and the three remaining social outcomes, which do not belong to this cycle, are dominated (as also shown in figure 13).

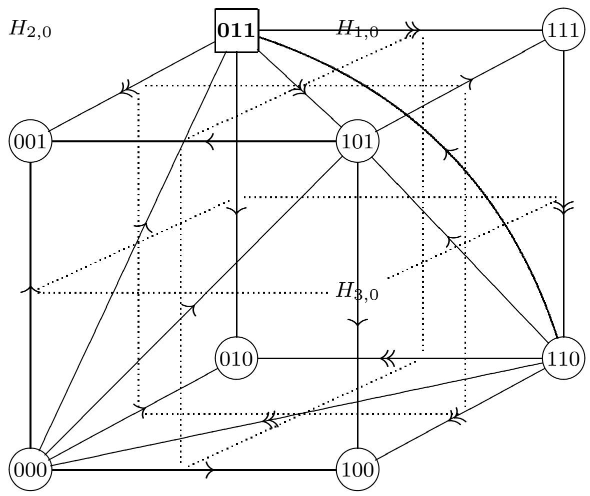

Figure 11: In the 3d social rule the social outcome 011 is a global optimum.

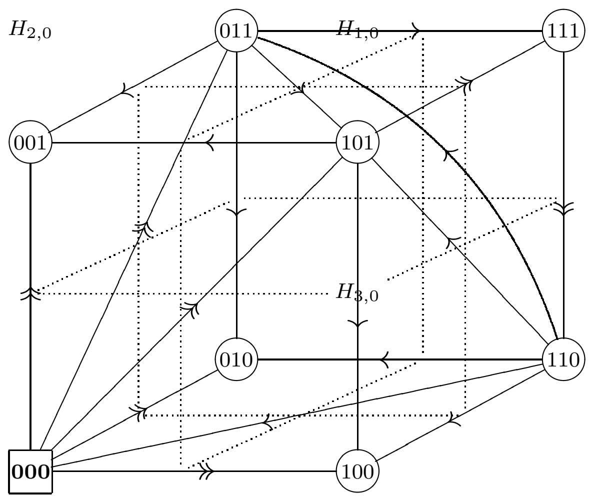

Figure 12: In the 3d social rule the social outcome 000 is a u-local optimum.

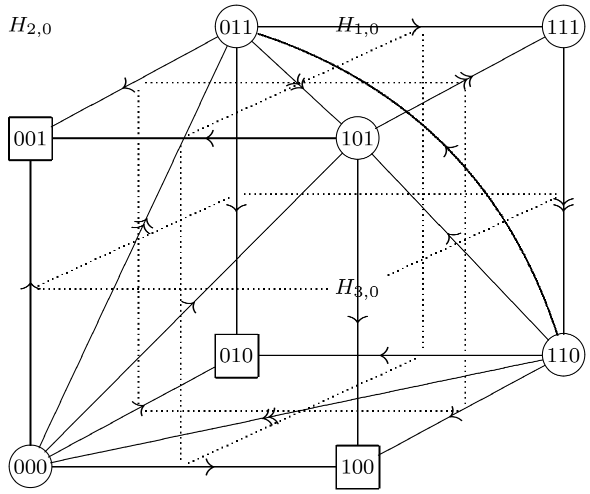

Figure 13: In the 3d social rule there is no classical optimum.

5 Decidability and manipulability

In the framework of classical social choice, a given social outcome z is an optimum if and only if it dominates all the other social outcomes. Therefore, the probability $P(z)$ that a given social outcome z is an optimum for a social rule on M social outcomes is given by the quotient between the number of graphs with $M-1$ nodes and the number of graphs with M nodes, i.e.

$$P(z) = \frac{2^{\binom{M-1}{2}}}{2^{\binom{M}{2}}} = \frac{1}{2^{M-1}}.$$

In our model, global optima play the role of optima in the classical framework, but also a local optimum can be an optimum if the agents vote starting from a particular social outcome. The probability $P(z)$ that a given social outcome z be a local optimum is given by the quotient between the number of the graphs with M nodes and with $\sum_{i=1}^{n} m_i - n$ fixed arcs, and the number of all the graphs with M nodes, i.e.:

$$P(z) = \frac{2^{\binom{M}{2}-(\sum_{i=1}^{n} m_i - n)}}{2^{\binom{M}{2}}} = \frac{1}{2^{\sum_{i=1}^{n} m_i - n}} = \frac{2^n}{2^{\sum_{i=1}^{n} m_i}}.$$

It is clear that, if n is greater than 1, the probability that z be a local optimum is far greater than the probability that z be an optimum in the classical framework. Let us define a function $F : \mathbb{N}^3 \longrightarrow \mathbb{Q}$, depending on n, $M = \prod_{i=1}^n m_i$ and $\sigma = \sum_{i=1}^n m_i$, as the quotient between the probability of a social outcome to be an optimum in the classical framework and that to be a local optimum in the new model, i.e.:

$$F(n, M, \sigma) = \frac{2^n}{2^{\sum_{i=1}^n m_i}} 2^{M-1} = 2^{n+M-(\sigma-1)}.$$

Remark 5.1. The inequality

$$F(n, M, \sigma) \geqslant 1$$

holds. However, the inequality becomes strict,

$$F(n, M, \sigma) > 1,$$

if and only if n is greater than 1.

The function F provides a precise characterization of the trade off between decidability and non-manipulability of choice.

The algorithm ComputeUniversalBasin It is worth noting that finding optima (or, equivalently, basins of attraction) is not straightforward. Given the combinatorial nature of our problem, the number of possible objects and agendas is in general very high and a simple brute-force algorithm would take far more than exponential time.

The algorithm ComputeUniversalBasin (Amendola and Settepanella 2012) computes the universal basin of attraction of a social outcome z for a social rule $\succ$. If the social rule $\succ$ is defined on M social outcomes, the algorithm ComputeUniversalBasin computes the universal basin of attraction of z in $O(M^3 \log M)$ time. The algorithm has been implemented in the computer program `FOSoR` (Amendola 2011a). `FOSoR` reads a social rule and

- computes the universal basin of attractions,
- checks whether a social outcome is a local (or an u-local) optimum,
- checks whether a social outcome is in the universal basin of attraction of another one,

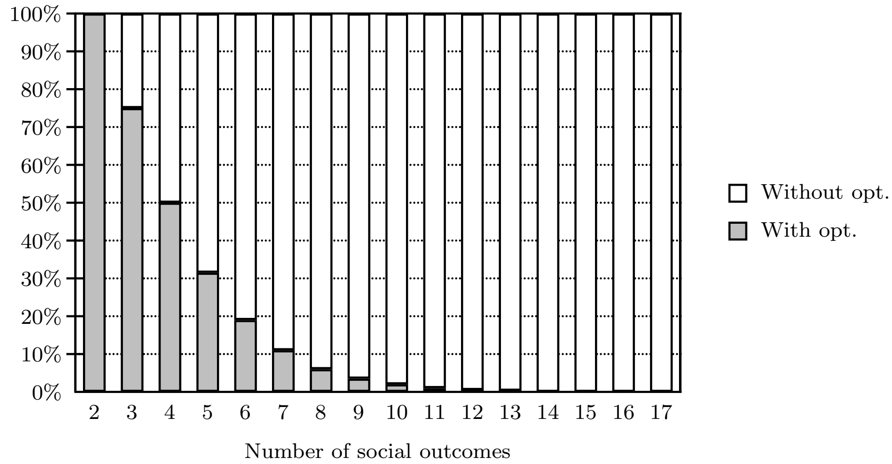

Figure 14: The probability that a social rule has an optimum in the classical case.

- checks whether there is a local (or an u-local) optimum,
- finds the number of local (or u-local) optima,
- given two social outcomes, finds an objects scheme (if any) for which there is a domination path from one to the other.

5.1 Numerical results

In this section we will give numerical results obtained by means of the computer program FOSoRStat (Amendola 2011b), which repeatedly (in this case $1,000,000$ times) generates a random social rule and applies the algorithm ComputeUniversalBasin to find all optima, collecting the results.

The classical case We start by showing in figure 14 the likelihood that an optimum in the classical case exists. As already mentioned, this can be computed with the formula

$$\frac{M}{2^{M-1}}.$$

The figure shows that the probability of finding a classical optimum quickly vanishes as the number of social outcomes increases and that for as little as 10 social outcomes it is already practically zero.

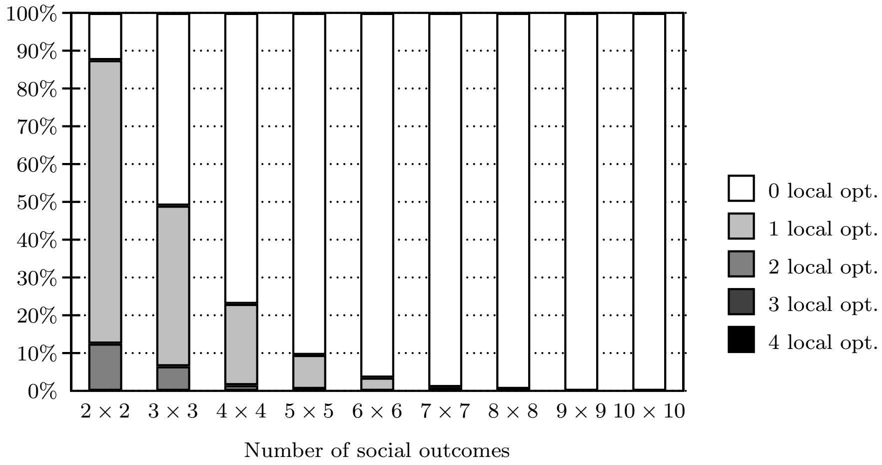

Figure 15: The probability that a social rule with two features has a fixed number of local optima.

Social rules with two features In figure 15 we show the probability that a social rule with two features has a given number of local optima. Compare it with figure 16, where the probability that a social rule with the same number of social outcomes has an optimum in the classical case. We can deduce that local optima are much more likely to exist but in general there may exist more than one of them (we counted up to four of them).

In figure 17 we have shown the probability that a social rule with two features has a given number of u-local optima.

Binary features An important case is when we consider only *binary features*, that could for instance model cases in which agents must take a set of interrelated yes/no decisions. In figure 18 we plot the probability that a social rule with binary features has a given number of local optima, depending on the number of features. We point out that the number of social outcomes, that is 2^n, grows very fast as the number n of features increases. The probability of finding local optima is of course one if there is only one feature (and therefore 2 social outcomes) and decreases slowly as the number of features increases, and seemingly stabilizes just above 60%. Up to 9 different local optima may be found with 9 binary features.

An analogous behaviour occurs in the case of u-local optima, as shown in figure 19.

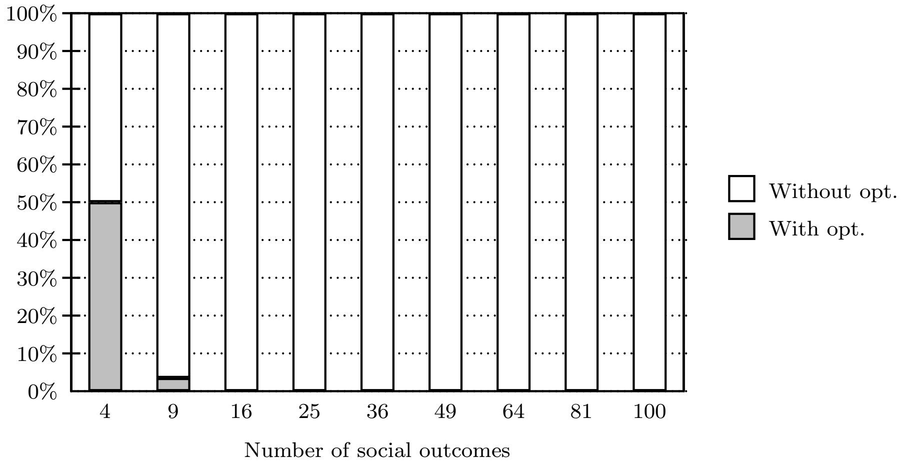

Figure 16: The probability that a social rule with two features has an optimum in the classical case.

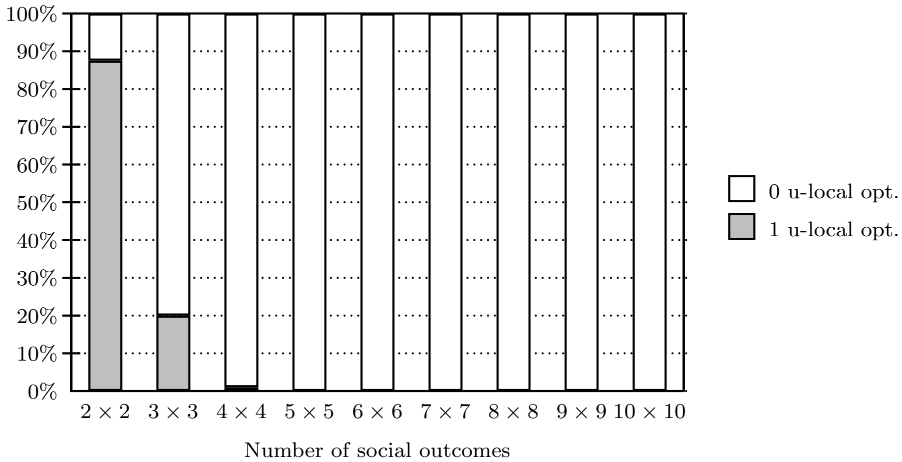

Figure 17: The probability that a social rule with two features has a fixed number of u-local optima.

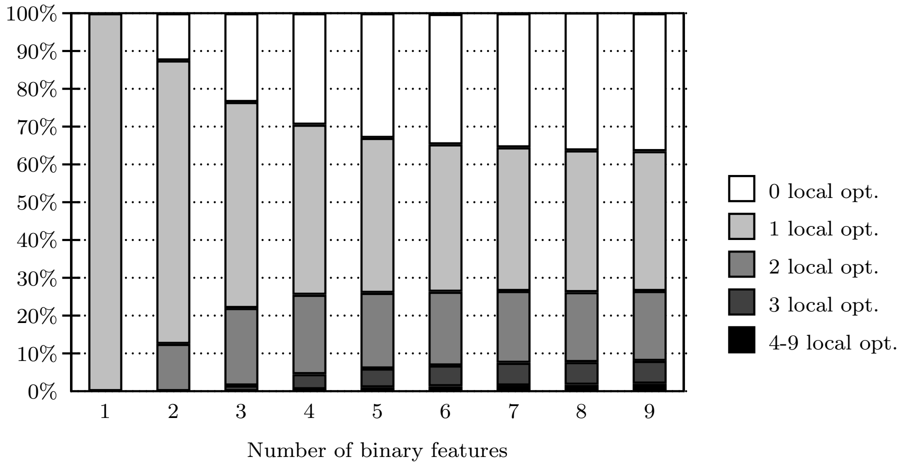

Figure 18: The probability that a social rule with binary features has a given number of local optima, depending on the number of features.

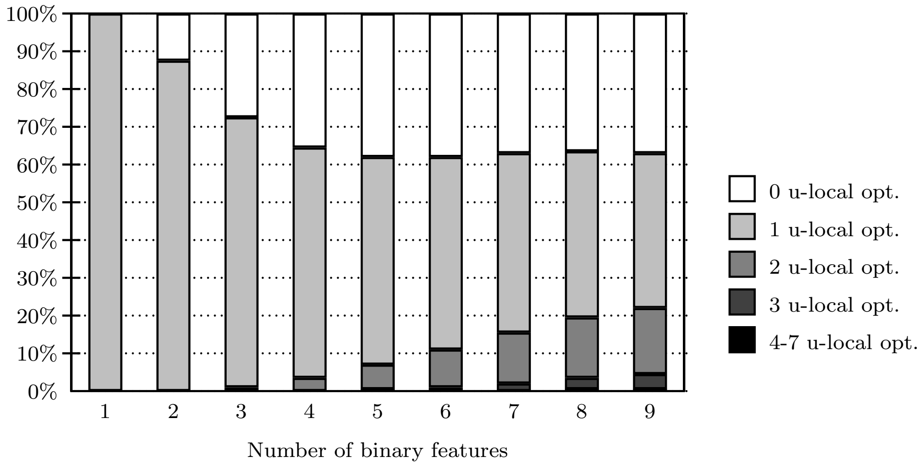

Figure 19: The probability that a social rule with binary features has a given number of u-local optima, depending on the number of features.

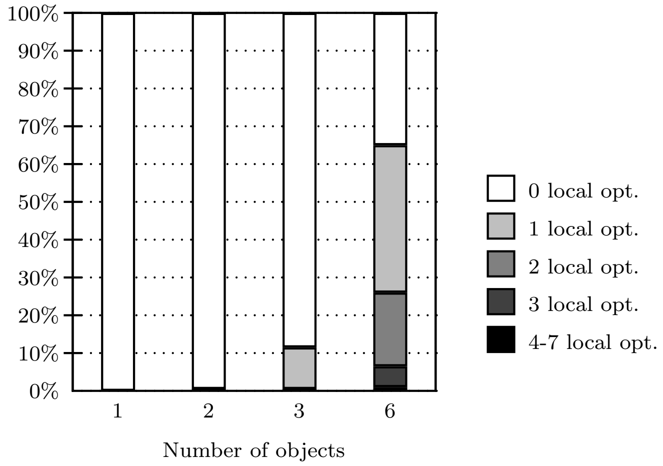

Figure 20: The probability that a social rule with 6 binary features has a given number of local optima, depending on the objects scheme.

We have not shown what happens in the classical case because, as shown in figure 14, if the number of features is greater than 4 (and hence the number of social outcomes is greater that $2^4 = 16$) the probability that a social rule has an optimum in the classical case is almost zero.

Objects schemes Figure 20 plots the probability that a social rule with 6 binary features (hence, with $2^6 = 64$ social outcomes) has a given number of local optima, depending on how such features are bundled together into different objects schemes. In particular, we consider the case in which, respectively, there is only one object of 6 features (i.e. the classical case), two objects of 3 features each, three objects of 2 features each, or, finally, six objects of 1 feature each. Note that if the number of objects increases (and hence the number of features in each object decreases) the probability that a social rule has a fixed number of local optima increases dramatically, up to a maximum of 7 different local optima.

Cardinality of features Figure 20 can be regarded as the plot of the probability that a social rule defined on 64 social outcomes has a given number of local optima, depending on how they are represented by more or less features taking a smaller or larger number of values. In particular, we consider the case in which, respectively, there is only one feature taking 64 different val-

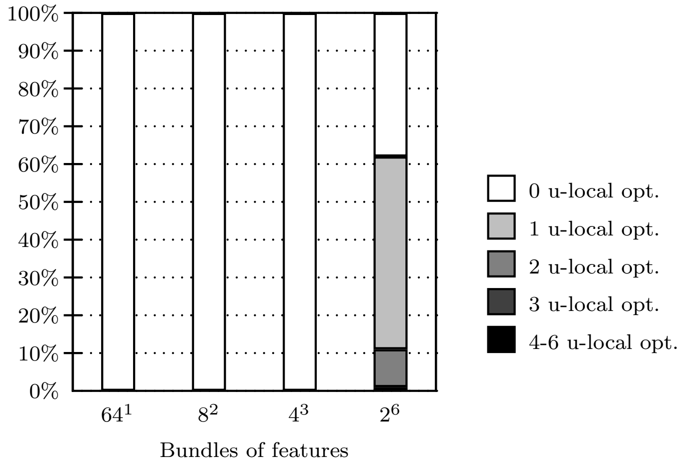

Figure 21: The probability that a social rule defined on 64 social outcomes has a fixed number of u-local optima, depending on the subdivision in features.

ues (i.e. the classical case), two features taking 8 values each, three features taking 4 values each, or, finally, six binary features.

In figure 21 we do the same for case of u-local optima. Although the notion of u-local optimum is more restrictive than that of local optimum, also in this case if the number of features increases (and hence the number of values taken by each feature decreases) the probability that a social rule has a fixed number of u-local optima increases up to 6 different u-local optima.

6 Conclusions

In this paper we have generalized the classical social choice model to the case in which choice takes place among bundles of interdependent elements. We show that in this more general framework we can define new types of social optima in addition to the classical ones, i.e. global, u-local and local optima, which form a hierarchy of optima from the most to the least stringent in terms of conditions to be satisfied.

More importantly, we show that in this framework there is a trade-off between decidability and non-manipulability: more stringent kinds of optima (the classical one) are less likely to exist and, even when they exist, finding them with some algorithmic procedure (e.g. pairwise majority voting) may require an unfeasible number of steps (e.g. pairwise comparisons). On the other hand, less stringent kinds of optima (u-local and, even more, local optima) are much more likely to exist, can be found through a much faster procedure, but are subject to manipulability. Since, in general, many different such optima exist, it will be possible for an authority to select one of them by changing the way features are bundled together in what we call the objects of choice, or by controlling the initial condition and/or the agenda.

Our model is meant to be a first step in the direction of modelling social choice when the alternatives are not given a priori, but the model explicitly addresses the fundamental pre-choice problem of the construction of alternatives. It shows that such construction of alternatives strikes a balance in the trade-off between decidability and non-manipulability and that, in general, the former can be achieved only by increasing the latter.

Although quite general, an important limitation of our model is that it does not allow for strategic misrepresentation of preferences. A further generalization of our model which accounts for the latter is left to future investigation.

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