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Profit-driven and demand-driven investment growth and fluctuations in different accumulation regimes.

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Abstract

The main task of this work is to develop a model able to encompass, at the same time, Keynesian, demand-driven, and Marxian, profit-driven determinants of fluctuations. Our starting point is the Goodwin model (1967), rephrased in discrete time and extended by means of a coupled dynamics structure. The model entails the combined interaction of a *demand effect*, which resembles a rudimentary first approximation to an accelerator, and of a *hysteresis effect* in wage formation in turn affecting investments. Our model yields *"business cycle"* movements either by means of persistent harmonic oscillations, or chaotic motions. These two different dynamical paths accounting for the behaviour of the system are influenced by its (predominantly) *profit-led* or *wage-led* structures.

Keywords: Endogenous Growth; Business Cycles; Investment; Aggregate Demand; Complex Systems; Nonlinear Dynamics; Chaos Theory.
JEL Classification: E32, E11, E12, E17

1 Introduction

Capitalistic growth proceeds through *"fits and starts"* as Schumpeter (1934) put it. Indeed fluctuations are permanent feature of modern economic dynamics. But what drives them? As known, the theory offers two ensembles of answers depending on

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whether fluctuations are seen as driven by exogenous shocks or by endogenous determinants. In this work we shall focus on the latter explanations. The main task is to develop a model able to encompass, at the same time, Keynesian, demand-driven, and Marxian, profit-driven determinants of fluctuations. Our starting point is the Goodwin model (1967), rephrased in discrete time and extended by means of a coupled dynamics structure. The model entails the combined interaction of a *demand effect*, which resembles a rudimentary first approximation to an accelerator, and of a *hysteresis effect* in wage formation in turn affecting investments. Our model yields business cycle movements either by means of persistent harmonic oscillations, or chaotic motions. These two different dynamical paths accounting for the behaviour of the system are influenced by its (predominantly) *profit-led* or *wage-led* structures (see Bhaduri and Marglin, 1990).
Just to recall, models of exogenously driven fluctuations basically, with roots in the classic Frisch (1933), entail a story of a system in equilibrium, hit by an exogenous (generally supply-side) shock, which adjust to it. The fluctuations are the work of such adjustments and in some versions, such as Real Business Cycle models, are themselves equilibrium phenomena.
Conversely, alternative and diverse stream of interpretations, which will be our concern here, see fluctuations as an intrinsic feature of capital accumulation (as in Marx and in Goodwin), of technological innovation (Marx and Schumpeter) or of aggregate demand formation (Keynes and the Keynesians).
Granted endogeneity, another issue regards the linearity or nonlinearity of the relations and the inadequacy of the former to represent persisting fluctuations, that are not explosive nor damped. The early models by Goodwin (1967), Kalecki (1935), Samuelson (1939a), Goodwin (1951), Kaldor (1940), Harrod (1939), Domar (1946), Pasinetti (1960) among others, belong to the family of explicitly dynamic models. All of them show the business cycles movements as the result of the interactions among aggregate variables, but while the first one is built under classical assumptions that emphasize the role of supply, and in particular profits, in determining capital accumulation, all the others focus on the role of variation in consumption and directly or indirectly investment, hence in demand, in boosting economic growth. Building on these original models a literature has developed (see Medio, 1979; Sordi, 1999; Lorenz, 1993; Puu and Sushko, 2006; Yoshida and Asada, 2007 among others) that tries to interpret observed macrodynamics by means of nonlinear dynamical systems. The notion of nonlinearities is indeed fully embodied in representations of the economy as a *'complex evolving system'* (Arthur et al., 1997; Kirman, 2011). The model which follows contributes to such a stream of analysis.

In section 2 we analyze the dynamical paths generated by Classic-Marxian and Keynesian fluctuations, comparing the behaviour of conservative and dissipative systems in the description of macro phenomena. In section 3 we discuss strengths and limitations of the Goodwin (1967) Growth Cycle which will be the starting point of our analysis. Section 4 presents a discrete time version of Goodwin's model. Its structural instability will push us toward a generalized version, where we introduce, along with the 'classical' elements of a "class struggle" model, also a demand effect which incorporates a Keynesian perspective in interpreting the dynamics in output growth. In section 5 our conclusions.

2 "Classical" and Keynesian fluctuations

As already mentioned, Marxian and Keynesian approaches both highlight the instability of the system and both perspectives attempt to describe booms, stagnations and downturns as direct consequences of the very nature of capitalist dynamics. However, the two families of models differ in both the formal modeling investments – ultimately, conservative vs. dissipative systems – , and in the historical reference to different "archetypes of capitalism" of socio-economics relations and accumulation regimes (in addition to a vast French-language literature on the Regulation approach, see in English Boyer, 1988a and Boyer, 1988b).

So, in terms of historical archetypes, models of Marxian inspiration address a sort of *"Smithian-Victorian"* form of capitalism, are usually supply-led economies, where the process of profits accumulation plays the role of driver of the system. Conversely, Keynesian models are demand-led economies where wages accumulation and expected demand are the main engines of the investment activity. In the former archetype, prices are competitive, wages are pegged down by an infinitely elastic supply of labour – as from the original Marxian notion of the *'industrial reserve army'* – , and possibly dependent, as in Marx and Goodwin, upon unemployment rates affecting the bargaining strength of workers themselves. This is the *competitive wage-labour nexus* which the 'Regulation School' flags as one of the fundamental ingredients of *Victorian* capitalism, with little institutional representation for workers, no explicit indexation mechanism of wages, neither on productivity nor inflation, and the market playing a major role in wage determination. In a complementary process, accumulation is driven by reinvested profits. Individual capitalists are too small to consider individual past demands as indicators of future demand. In fact, a "competitive approximation" is the population of capitalists investing and producing as much as they are allowed by past profits (with 'imperfections' on the financial markets, as we would say nowadays, so massive and so obvious, not to be talked about!). And add to that a propensity to consume at the very least lower than workers: even without Protestant ethic, most models

just need that to yield the desired qualitative properties!
Historically, a deep regime transition occurred roughly from the beginning of the 20th century to its fully completion after WWII, involving permanent changes in the working of the major markets (for labour, goods and finance) as well as in the relations between the state and the private actors (associated also with major changes in production technologies and forms of corporate organizations whose analysis is beyond the scope of this paper. See, however Boyer, 1988a and Boyer, 1988b as a succinct overview). In the new regime, – call it Fordist, or corporatist, or Keynesian – , supply is dominated by mass production within oligopolistic price-making firms. Wages stop being primarily governed by the rates of unemployment but, sustained by collective bargaining, became linked to the rates of inflation and of productivity growth. Moreover, far from being just a cost, they became an essential component of demand which in turn drives investment decisions.

The formal representation of the dynamic of the competitive accumulation regime is satisfactorily captured by a fluctuating *conservative* system to which is superimposed an exogenous drift (standing for technical progress). Goodwin's Growth Cycle belongs to this class of models: a Lotka-Volterra type of interaction between workers and capitalists making conflicting claims upon aggregate output induces a cycle in the distributive shares and with that in investment and thus in the rates of growth. The simplest visual image of it is a sort of "wheat-wheat" model wherein the total output is made of wheat which can be appropriated by the workers as wages and consumed, or by the capitalists as profits and invested. In that, no portion of the total amount of wheat is left unused and of course one cannot eat or invest more wheat than it is produced. This is indeed a basic feature of the conservative nature of the system. The property however does not apply to the representation of Keynesian regime wherein it is demand (past and expected) which drives the growth or contraction of the system itself. A more adequate account is in terms of *dissipative* systems which can easily undergo more complex dynamics (such as chaotic ones) and, under certain conditions, can self-organize and generate orderly patterns as emergent from out-of-equilibrium fluctuations (Nicolis and Prigogine, 1977). The term *dissipative* stems from the analysis of physical systems characterized by a permanent input of energy which dissipates over time. If the energy input is interrupted, the system collapses to some equilibrium state. However, as distinct feature, socio-economic systems characterized by endogenous innovation systematically violate any general principle of conservation. In a way you can get out more than you put in or conversely you can get out less than you put in. Evolutionary processes characterized by learning and innovation dynamics, increasing returns, accumulation of knowledge are potential drivers of "getting more out of less", while Keynesian endogenous generation of demand and its associated possibilities of structural crises entail the chance of "getting less" out of the potential resources.

In what follows we are going to start from the Goodwin model with a conservative/ "Marxian" structure and add an endogenous (formally dissipative) mechanism of demand generation in order to account at the same time *Keynesian* and *Classical-Marxian* engines as sources of macro-fluctuations. Analytically, dissipation in continuous time dynamical systems can be formally characterized by the property that the *divergence* (or *Lie derivative*) is lower than zero:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} < 0 \tag{1}$$

A contraction of the phase-space volume occurs over time. These systems are not *area-volume preserving*.

An early example of dissipative systems in economics is the Kaldor (1940) model: being the equilibrium point of this system unstable, there is a tendency away from it; a spiraling flow emerges without closed orbits. Technically, this behaviour is determined by the trace sign when is positive, while a negative trace corresponds to a positive friction, so that the exploding fluctuations are dampened for points sufficiently far away from the equilibrium point. A closed orbit arises when exploding and imploding forces collide, so that the trace will be zero.

Conversely, in *conservative systems* no friction exists because neither inputs nor loss of energy emerge. According to the previous characterization, in conservative systems the trace/*Lie derivative* always equals zero for all points in the phase space. They are *area-volume preserving* systems:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = 0 \tag{2}$$

The zero trace implies the fixed points are centres or saddles. In physical systems, the pendulum motion is a classical example, but the "pendulum metaphor" is often used also for equilibrium models in economics (Louca, 2001).

3 The Goodwin model: strengths and limitations

Goodwin (1967) represents the first formalization of the distributive conflict between profits and wages, based on Say's Law, opposed to a Keynesian, demand-driven perspective with product market-clearing. Capitalists save and immediately reinvest all their profits, without any concern about over-accumulation. Workers spend all income they receive.

According to the Marxian theory, crises can occur at least for two different causes related to capital accumulation namely, overproduction crises or "class struggle" crises.

Overproduction crises occur when a high rate of profit accumulation determines an excess of supply, for a given wage level. In this case, capitalists should suffer some losses because they are not able to sell all their production. "Class struggle" crises happen when capital accumulation determines an increase in labour demand, leading to an employment increase, so strengthening workers' bargaining power, yielding wage increases, a drop of the profit rate and lower capital accumulation. In turn, this drives to lower production, lower employment, lower wages and higher profit rate. This cyclical and opposite movement of wages and profits, present in all classical economists, is indeed central to the Marx theory of capitalist instability. This classical view of capitalist system underlies the model of economic fluctuations driven by the interrelations between profits and wages. And it is the core of Goodwin's Growth Cycle model:

> *It has long seemed to me that Volterra's problem of the symbiosis of two populations, partly complementary, partly hostile, is helpful in the understanding of the dynamical contradictions of capitalism, especially when stated in a more or less Marxian form. (Goodwin, 1967)*

As mentioned, Goodwin (1967) accounts for technical change in the form of a time-dependent drift. Interestingly, the model can be extended to an endogenous – in principle investment embodied – technical change, preserving its basic Lotka-Volterra structure (see Silverberg and Lehnert, 1994).

Although it is particularly insightful from an economic point of view, the model however presents a technical drawback: it is *structurally unstable*. Structural instability entails that every minimal modification of the functional form will destroy its fundamental characteristics, in our case regular and persistent fluctuations of the aggregate variables of interest. (Conversely, a dynamical system is structurally stable if for every sufficiently small perturbation of the vector field the perturbed system is topologically equivalent to the original system). A consequence is that if we try to generalize the original Goodwin (1967) system, the model will lose the possibility of accounting for regular economic cycles. The Growth Cycle model presents an other drawback, again due to the nature of the stationary point: the amplitude of oscillations is entirely due to initial conditions. Trajectories that start from points near to the centre have a limited amplitude. Vice-versa, trajectories starting from points far from it have violent and explosive oscillations. This is the further consequence of the lack of stability of the Goodwin model: nothing ensures that a trajectory starting from acceptable values in the phase space will remain in the same region. In the literature one finds a few attempts to overcome the structural instability, moving from conservative toward dissipative structures of the system (see Desai (1974); Flaschel (1984); Wolfstetter (1982); Velupillai (1979); Pohjola (1981)). Except for the latter, all these contributions involve, adding further determinants to the wage equation, in the continuous time Goodwin model.

However, a discrete time formulation in our view is more economically appropriate. After all, investment is a time consuming activity: equipments have to be purchased, stocked, introduced in production and so forth. Entrepreneurs usually make investment plans setting today how much to invest tomorrow. A considerable time interval exists between investment decision and capital production/utilization. Investment and disinvestment activities cannot happen in an instantaneous way, as pointed out by Kalecki (1935). Wage bargaining is a process that takes time as well: labour contracts cannot be instantaneously modified. This is the first modification we shall explore in the following, in tune with Pohjola (1981) and Canry (2005). In particular the former reduces the original two dimensional system into a one dimensional logistic equation. An interesting feature of Pohjola's article is the attempt to obtain a chaotic behaviour from the predator-prey model. The model substitutes the original Phillips curve equation with the Kuh (1967) specification, so that it is not the *wage share* but the *level of wage* which depends positively upon employment. This apparent slightly modifications induces dramatic changes in the dynamics of the model. In particular the author obtains a nonlinear first order difference equation, topologically equivalent to the logistic equation. Thus, considering different parameters constellations, the system is able to exhibit simple and chaotic dynamics. Another discrete time version is in Canry (2005) with an attempt to link the Keynesian specification of endogenous fluctuations which bears some analogy with what we shall do in the following. The model tries to combine a "classical" investment equation (involving Say's Law) with a demand-effect, that resembles the Keynesian tradition. The model is able to reproduce the limit cycle behaviour of the continuous time Goodwin model, but this happens by means of somewhat far-fetched change in the Phillips curve, with the rate of wage variation dependent upon the *current* level of unemployment.

4 The model

Let us try to move some steps toward the construction of a model where both Keynesian and Marxian features live together. The first step involves reformulating the Goodwin model in a discrete time version.

4.1 A discrete time version of the Goodwin Growth Cycle model

We build the model, as mentioned, upon a discrete time reformulation of the original one with some modifications. We assume that:

$$Y_t = AK_t \tag{3}$$

i.e. a constant output-capital ratio (Y/K) while the dynamic of capital is similar to the one considered by Pasinetti (1960):

$$K_{t+1} = (1-\delta) K_t + I_{t+1} \tag{4}$$

where $0 < \delta < 1$ is a constant rate of capital depreciation.

$$L_t = \frac{Y_t}{a_t} \tag{5}$$

Labour demand L equals total output over labour productivity a_t. Labour productivity grows at a constant, exogenous rate $\alpha > 0$:

$$a_{t+1} = a_t (1+\alpha) \tag{6}$$

The current wage rate depends on lagged wages plus a correction factor consisting in the difference between the past employment rate and the "equilibrium" value, the zero wage-inflation rate of employment (nowadays one would say the NAIRU with zero inflation):

$$w_{t+1} = w_t (1 + \lambda (v_t - \bar{v})) \tag{7}$$

with λ that parametrizes the strength of workers' reaction to labour market disequilibria, assumed here $\lambda < 1$. Differently from the linearized Phillips curve [1] present in Goodwin (1967), in our equation, the coefficient multiplies the deviation from equilibrium. The employment rate is defined as the ratio of total labour demand over labour supply:

$$v_t = \frac{L_t}{N_t} = \frac{Y_t}{N_t a_t} = \frac{AK_t}{N_t a_t} \tag{8}$$

Population is assumed to grow at a constant rate $\beta > 0$:

$$N_{t+1} = N_t (1+\beta) \tag{9}$$

[1] Indeed, the labour market equation of the Growth Cycle, being expressed in real terms, is not exactly a Phillips curve which is a negative relation between changes in money wages and the unemployment rate. It lies in between the Phillips curve and the so called Wage curve. The last one is a real relation between the *levels* of the wage rate and the unemployment rate (see Blanchflower and Oswald, 1994).

Finally, investments are function of the share of profits gained in the previous period, where $0 < s < 1$ is the capitalists' propensity to save:

$$I_{t+1} = s\pi_t = sY_t\left(1 - u_t\right) \tag{10}$$

where:

$$u_t = \frac{w_t}{a_t} \tag{11}$$

is the workers' share in output. Making the necessary substitutions, we get:

$$\begin{cases} K_{t+1} = \left(1-\delta\right)K_t + sAK_t\left(1-u_t\right) \\ w_{t+1} = w_t\left(1 + \lambda\left(v_t - \bar{v}\right)\right) \end{cases} \tag{12}$$

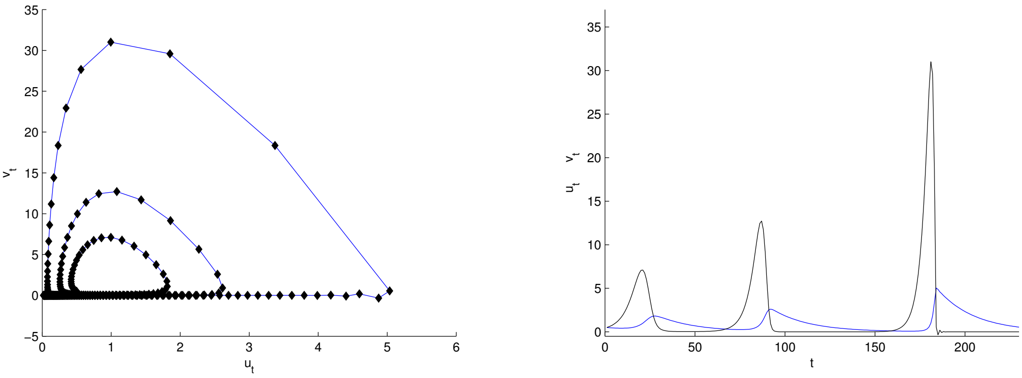
Rewriting the system in terms of v_t and u_t we get a system of two equations in two variables, with seven parameters:

$$\begin{cases} \dfrac{v_{t+1}}{v_t} = \dfrac{K_{t+1}}{K_t}\dfrac{N_t}{N_{t+1}}\dfrac{a_t}{a_{t+1}} = \dfrac{1-\delta+sA\left(1-u_t\right)}{\left(1+\alpha\right)\left(1+\beta\right)} \\ \dfrac{u_{t+1}}{u_t} = \dfrac{w_{t+1}}{w_t}\dfrac{a_t}{a_{t+1}} = \dfrac{\left(1+\lambda\left(v_t-\bar{v}\right)\right)}{1+\alpha} \end{cases} \tag{13}$$

It is formally a predator-prey relation between wage share and employment rate. In order to study the property of the system, let us perform the usual analysis of its fixed points. The system presents two fixed points, a trivial and a non trivial one.

$$\left(v^0, u^0\right) = \left(0, 0\right) \quad \left(v^*, u^*\right) = \left(\frac{\alpha + \lambda\bar{v}}{\lambda}, 1 - \frac{\delta + g}{sA}\right) \tag{14}$$

where $g = \alpha + \beta + \alpha\beta$. It is simple to verify that the trivial fixed point is a saddle node, the other one is an unstable focus. Unfortunately, this discretization has destroyed the topological properties of the continuous time Goodwin model. By performing simulations, we find that variables manifest explosive behaviors. Thus the system left to itself is not able to reproduce any persistent regular cyclicity (see fig. 1.a and 1.b). Indeed, the Goodwin Growth Cycle does not appear to be robust to the discretization, corroborating similar results obtained in Sordi (1999), where a different discretization is explored. Next, we introduce a generalized extension of the system, that recovers richer dynamics. The generalized version is built on a *coupled dynamic model*. We obtain both the existence of a limit cycle and chaotic dynamics that respectively occur by means of a Neimark-Sacker or a period-doubling bifurcation. Furthermore co-existence of several attractors may occur. Indeed, within this last formulation we are able to generate both the Goodwin results as well as those outcomes achieved by its extensions.



(a) *Phase plane: after a finite number of iterations, the trajectories in the state space become unfeasible.*

(b) *Explosive oscillations in u (blue line) and v (black-line).*

Figure 1: Goodwin discrete time model.
Parameters values: $A = 0.5, \alpha = 0.02, \beta = 0.01, s = 0.8, \delta = 0.02, \lambda = 0.03, \bar{v} = 0.95$

4.2 From conservative to dissipative: a generalized version of the Goodwinian Growth Cycle with a Keynesian component

Let us address the structural instability and analyze a generalized discrete time formulation of the original model. As already observed, the original Goodwin system can be expressed by the following discrete formalization [2]:

$$\begin{cases} \dfrac{y_{t+1} - y_t}{y_t} = bw_t + e = f(w_t) \\ \dfrac{w_{t+1} - w_t}{w_t} = dy_t + f = g(y_t) \end{cases} \tag{15}$$

A similar kind of discretization process is used by Goodwin (1989). In the Lotka-Volterra framework, $b < 0$ and $d > 0$. In this specification, the variation rate of each variable depends only upon the other variable. One way to make the system structurally stable in the continuous time formulation has been by creating a dependence of at least one rate of variation not only on the level of other variable but on its own level. Here we analyze the effect of a reciprocal interdependency of both variation rates on the levels of both variables. Thus we consider a discrete time generalized version of the Goodwin

[2] In the current section we replace the usual notation of u and v with w and y respectively, since we do not anymore discuss shares or relative variables.

model:

$$\begin{cases} \dfrac{y_{t+1} - y_t}{y_t} = ay_t + bw_t + e = f(y_t, w_t) \\ \dfrac{w_{t+1} - w_t}{w_t} = cw_t + dy_t + f = g(y_t, w_t) \end{cases} \tag{16}$$

(where $a, c \neq 0$). The first equation describes the dynamics of output growth rates and, the second the dynamic of wage growth rates. To recall, both depend upon the *level of wages* (w_t) and *level of activity* (y_t) of the previous period. Note that our formulation entails some form of path dependence of the process together with the dynamic coupling of the two variables. Keeping the same assumptions of the original model, we continue to assume that $b < 0$ and $d > 0$.

Following Medio (1979), let us consider the possible economic interpretation of the partial derivatives in this framework.

1. $\dfrac{\partial f(y_t, w_t)}{\partial w_t} = b \leq 0$: call it *'profits effect'* since in the "classical" regime the higher the level of wages, the lower will be the output growth rate, because reduction in the profit margin will decrease resources available for investment activity. For any given output-capital ratio, high wages will yield to lower rates of investment. In our formulation the negative parameter b embodies the profit effect and it is defined, recalling the original Goodwin model [3], as:

$$1 < b \simeq \frac{1}{\sigma}(1 + \alpha)(1 + \beta) < 0 \tag{17}$$

   In a "pure Keynesian" regime the effect is nil (or in the real world possibly even negative).

2. $\dfrac{\partial f(y_t, w_t)}{\partial y_t} = a \gtreqless 0$: call it *"demand effect"* which is positive under any form of Keynesian multiplier/accelerator process while it is negative under a "classical" accumulation regime in that it indirectly captures the negative impact that high incomes (and thus a large wage bill) exert on the rates of investment and hence on growth. The *"demand effect"* with $a > 0$ originates from a revised form of the Samuelson (1939b) and Goodwin (1951) accelerator models for investments. In particular we adopt an investment equation where the *profit effect* and the influence of change in aggregate demand coexist and allow to get a simultaneous effect of profits and income levels upon output growth rate:

$$\frac{y_{t+1} - y_t}{y_t} = ay_t + bw_t + e \tag{18}$$

[3] For the parameters specifications we follow the discretization process of the predator-prey dynamics suggested by Sordi (1999).

3. $\dfrac{\partial g\left(y_t, w_t\right)}{\partial y_t} = d > 0$: it stands for the *"employment effect"* captured by the Phillips curve. The higher the *level* of activity, the higher the wage rate variations. It embodies the assumption that workers' contractual power positively depends upon the level of employment. The parameter is:

$$0 < d \simeq \frac{\rho}{1+\alpha} < 1 \tag{19}$$

.

4. $\dfrac{\partial g\left(y_t, w_t\right)}{\partial w_t} = c \leq 0$: it represents a *"mark-up"* effect, under a product market regime with price-making firms, which will take value zero under a "pure Classical/Marxian" regime. With mark-up pricing, capitalists index prices on unit labour cost so that when monetary wage growth is higher than labour productivity growth they increase prices, and vice-versa since the increase in prices is not immediately and possibly not fully compensated by an increase in monetary wages of the same magnitude, the real wage growth rate is lower (and of course possibly negative). The prices growth rate depends upon the level of wages weighted by a mark up factor. The mark-up effect comes from the introduction of prices determined by a mark-up equation. In particular we assume that the wage rate variation is affected by the previous price level:

$$\frac{w_{t+1} - w_t}{w_t} = \beta p_t + d y_t + f \tag{20}$$

where:

$$p_t = \left(\frac{1+\mu}{\alpha}\right) w_t \tag{21}$$

$\mu > 0$ is the mark-up and α is the labour productivity, here both considered as constants. This implies that the wage growth depends also upon its previous level:

$$\frac{w_{t+1} - w_t}{w_t} = \beta\left(\frac{1+\mu}{\alpha}\right) w_t + d y_t + f = c w_t + d y_t + f \tag{22}$$

where $c = \beta\left(\dfrac{1+\mu}{\alpha}\right)$. This factor negatively influences the rate of growth of real wages since it erodes workers' purchasing power.

5. The two constants play the same role of the original Goodwin model being two autonomous components of the income and wage growth rate. In particular:

$$0 < e \simeq \frac{1+\sigma}{\sigma\left(1+\alpha\right)\left(1+\beta\right)} < 1 \tag{23}$$

$$f \simeq \frac{1-\gamma}{1+\alpha} \lesseqgtr 0 \tag{24}$$

Summing up, we introduce the following hypotheses on the sign and magnitude of the parameters:

$$-1 < b \leq 0, \quad -1 < c \leq 0, \quad 0 < d < 1, \quad 0 < e < 1, \tag{25}$$

The only two parameters that can both be positive or negative are a and f; for analytical convenience we assume:

$$-1 < a < 1, \quad -2 < f < 2, \tag{26}$$

4.3 System dynamics and simulations results

The analysis of the system dynamics means the study of the map [4]:

$$\begin{cases} y_{t+1} = ay_t^2 + by_t w_t + (e+1)y_t \\ w_{t+1} = cw_t^2 + dy_t w_t + (f+1)w_t \end{cases} \tag{27}$$

Technically the map described by system (27) shows many interesting general properties. First note the map may be non-invertible (see Mira, 2010 for a depth study). Indeed, for a given (y_{t+1}, w_{t+1}) , the rank of the preimage (that is, the backward iterate defined as M^{-1}) may not exist or may be multivalued (considering the degrees of polynomial on the right hand side of (27) we can have 4 different preimages at most). This also implies that the basin of attraction may be disconnected (see fig. 5.b).
Second, for particular parametrizations for which the map is symmetric ($f(y_t, w_t) = g(w_t, t_t)$) synchronization as well as blow-out bifurcations and intermittence may occur (see, among others Bischi et al., 1998 and Fanti et al., 2012). A complete study of the map is beyond the aim of the present work and we only focus on dynamic properties that have interesting implications in terms of Business Cycle phenomena. System (27) has four fixed points:

$$(y_1^*, w_1^*) = (0, 0), \quad (y_2^*, w_2^*) = \left(0, -\frac{f}{c}\right), \quad (y_3^*, w_3^*) = \left(-\frac{e}{a}, 0\right) \tag{28}$$

and

$$(y_4^*, w_4^*) = \left(\frac{bf - ec}{ca - bd}, -\frac{-de + fa}{ca - bd}\right) \tag{29}$$

The study of local stability of equilibrium solutions is based on the Jacobian matrix of the dynamical system. The Jacobian matrix of (27) computed in a generic point has the following form:

$$\mathcal{J} = \begin{pmatrix} 2ay_t + bw_t + e + 1 & by_t \\ dw_t & 2cw_t + dy_t + f + 1 \end{pmatrix} \tag{30}$$

[4]See Bischi et al., 1998 for a similar modeling.

The stability conditions for a two dimensional map follow the usual characterization: a fixed point $(\bar{x}, \bar{y})$ is (locally) asymptotically stable if the eigenvalues λ_1 and λ_2 of the Jacobian matrix, calculated at the fixed point, are less than one in modulus. The necessary and sufficient conditions ensuring that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are:

$$1 + \mathrm{Tr}\,(\mathcal{J}_{\bar{x}}) + \det\,(\mathcal{J}_{\bar{x}}) > 0 \tag{31}$$

$$1 - \mathrm{Tr}\,(\mathcal{J}_{\bar{x}}) + \det\,(\mathcal{J}_{\bar{x}}) > 0 \tag{32}$$

$$1 - \det\,(\mathcal{J}_{\bar{x}}) > 0 \tag{33}$$

The violation of one of these conditions leads to the emergence of a local bifurcation. More specifically we have that:

i A Neimark–Sacker bifurcation occurs when the modulus of a pair of complex, conjugate eigenvalues is equal to 1 $(\det\,(\mathcal{J}_{\bar{x}}) = 1, \mathrm{Tr}\,(\mathcal{J}_{\bar{x}}) \in (-2, 2))$ and other technical conditions that involve approximation of the map of order higher than linear. (Recall that this local bifurcation implies the birth of an invariant curve in the phase plane). It can be considered equivalent to the Hopf bifurcation in continuous time and is indeed the major instrument to prove the existence of quasi-periodic orbits for the map.

ii A flip bifurcation occurs when a single eigenvalue becomes equal to -1 $(1 + \mathrm{Tr}\,(\mathcal{J}_{\bar{x}}) + \det\,(\mathcal{J}_{\bar{x}}) = 0, \mathrm{Tr}\,(\mathcal{J}_{\bar{x}}) \in (0, -2))$. This local bifurcation entails the birth of a period 2-cycle.

iii A transcritical bifurcation occurs when a single eigenvalue becomes equal to 1 $(1 - \mathrm{Tr}\,(\mathcal{J}_{\bar{x}}) + \det\,(\mathcal{J}_{\bar{x}}) = 0, \mathrm{Tr}\,(\mathcal{J}_{\bar{x}}) \in (0, 2))$. This local bifurcation leads to the stability switching between two different steady states.

From the study of the eigenvalues it is simple to prove that the two corner fixed points $(y_2^*, w_2^*) = (0, -\frac{f}{c})$ and $(y_3^*, w_3^*) = (-\frac{e}{a}, 0)$ may undergo a flip or a transcritical bifurcation.

Proposition 1. *The stability conditions for the equilibrium point* $(y_2^*, w_2^*) = (0, -\frac{f}{c})$ *are:*

$$f < min\left[2, \frac{(e+2)c}{b}\right], f > \frac{ec}{b} \tag{34}$$

When $f = \dfrac{ec}{b}$ *the corner point and the fixed point* (y_4^*, w_4^*) *smash together leading to a transcritical bifurcation. When* $f = \dfrac{(e+2)c}{b}$ *the point undergoes a flip bifurcation.*

Proof 1. *The result follows from the analysis of the Jacobian matrix evaluated at the corner equilibrium point* $(y_2^*, w_2^*) = (0, -\frac{f}{c})$*:*

$$\mathcal{J} = \begin{pmatrix} 1 - \frac{bf}{c} + e & 0 \\ -\frac{fd}{c} & -f + 1 \end{pmatrix} \tag{35}$$

Proposition 2. *The stability conditions for the equilibrium point* $(y_3^*, w_3^*) = (-\frac{e}{a}, 0)$ *are:*

$$\frac{de}{a} - 2 < f < \frac{de}{a} \tag{36}$$

When $f = \dfrac{de}{a}$ *the corner point and the fixed point* (y_4^*, w_4^*) *smash together leading to a transcritical bifurcation. When* $f = \dfrac{de}{a} - 2$ *the point undergoes a flip bifurcation.*

Proof 2. *The result follows from the analysis of the Jacobian matrix evaluated at the corner equilibrium point* $(y_3^*, w_3^*) = (-\frac{e}{a}, 0)$*:*

$$\mathcal{J} = \begin{pmatrix} -e + 1 & -\dfrac{eb}{a} \\ 0 & 1 - \dfrac{de}{a} + f \end{pmatrix} \tag{37}$$

Regarding the fixed point (y_4^*, w_4^*), the analysis of stability conditions is not of easy treatment. The requirement for the Neimark-Sacker bifurcation is that the complex conjugate eigenvalues cross the unit circle, i.e., that $|\lambda| = 1$ at the bifurcation point $\mu = \mu_0$. Furthermore, it is required that the roots do not become real when they are iterated on the unit circle: the first four iterations λ^n must also be complex conjugate. Finally, the eigenvalues must cross the unit circle with nonzero speed for varying μ at μ_0.

Proposition 3. *A necessary condition such that the internal equilibrium point* $(y_4^*, w_4^*) = \left(\dfrac{bf - ec}{ca - bd}, -\dfrac{-de + fa}{ca - bd}\right)$ *undergoes a Neimark-Sacker bifurcation is the second degree polynomial:*

$$af^2b + (ca - ebd - ab - ace)f + e^2dc + ace - ced \tag{38}$$

has to vanish with respect to f*.*

Relying on Propositions 1, 2 and 3 and making use of simulations, we are now able to study the main results of the model. Let us consider first a "classic" set-up in terms of profits accumulation and growth ($b < 0$) under different "accelerator" (or "anti-accelerator") set-ups – i.e. positive or negative values of a –, and governance regime

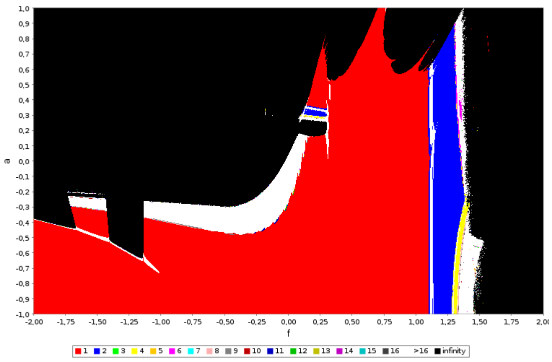
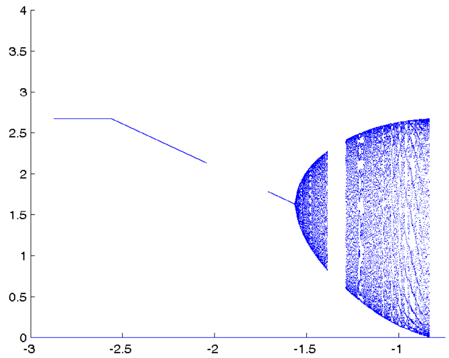


Figure 2: Bifurcation regions.

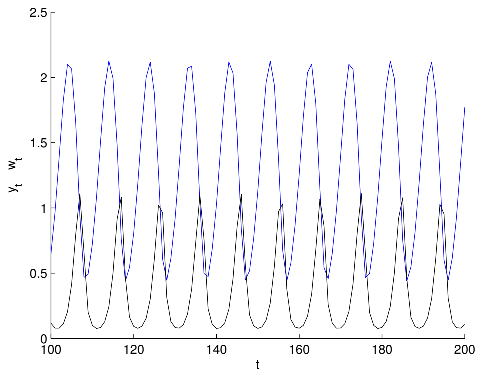
on the labour market – i.e. positive or negative f – . The complexity of the model is well-summarized by the two parameters bifurcation diagram in fig. 2 that shows different periodicity in the long run dynamic of the system, according to a given initial condition. The red area shows the combination of the parameters values where the system is stable; the white area represents the regions of the parameters where occur the passage from stability to instability (possible emergence of the Neimark-Sacker bifurcation and of chaotic dynamic); the blue area the regions of period-2 cycle; the yellow area the parameters regions of period-4 cycles. Finally the black one shows the combination of a and f that determines divergent oscillations.

Let us consider in more detail a "Goodwinian" set-up with classic profit-led accumulation (here $b = -0.92$), an "anti-accelerator" ($a = -0.29$), price-taking firms (c=0), a Phillips mechanism on the labour market ($d = 0.87$), further we parametrize $e = 0.8$, and study its dynamic for different values of f (basically capturing the NAIRU). As shown by the Neimark-Sacker bifurcation diagram (fig. 3.a) the fixed point (y_4^*, w_4^*) is the unique attractor of the system for $f < -1,5$. Trajectories starting in the gray region of fig. 3.d will converge to the invariant curve for $f = -1.5$ (see 3.d). Conversely, trajectories starting in the white region will diverge.

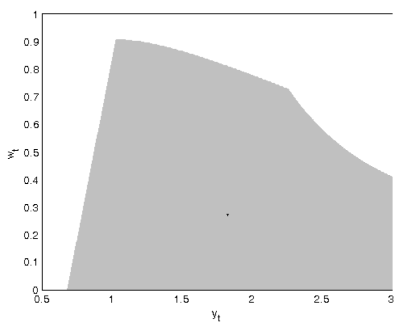
Interestingly, the relative order in the dynamics implied by the quasi-periodic fluctuations (see fig. 3.b) of the "classic" set-up tends to be disrupted when labour and product markets depart from competition. Consider, for example, parametrization $a = -0.6, b = -0.92, c = -0.37, d = 0.84, e = 0.9$. For these values of parameters, below a threshold value of f, the fixed point (y_2^*, w_2^*) is stable, above which, the fixed point undergoes a flip bifurcation (see 4.a). This is followed by a sequence of period doubling bifurcations that eventually generates a chaotic attractor (see fig. 4.b and 4.c). Chaotic orbits never converge to a stable fixed or periodic point, but exhibit sustained



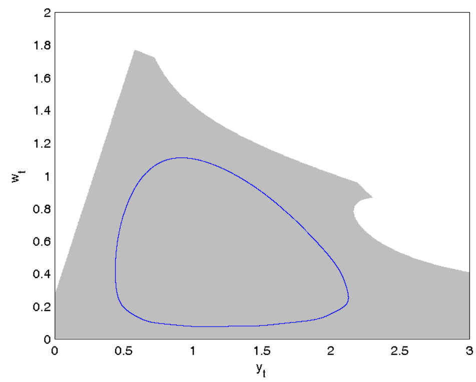
(a) *Neimark-Sacker bifurcation on w_t. The bifurcation diagram shows that increasing values of f may destabilize the internal point. After the occurrence of the bifurcation quasi-periodic orbits describe the long run behaviour of the system. Further increases in f make the stationary state stable again.*



(b) *Periodic oscillations of w (black line) and y (blue line).*

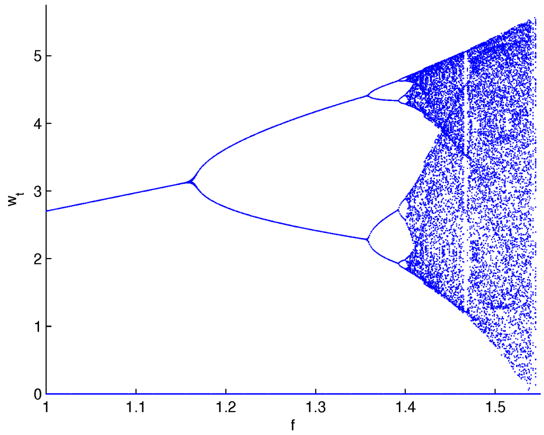


(c) *The unique stable fixed point*

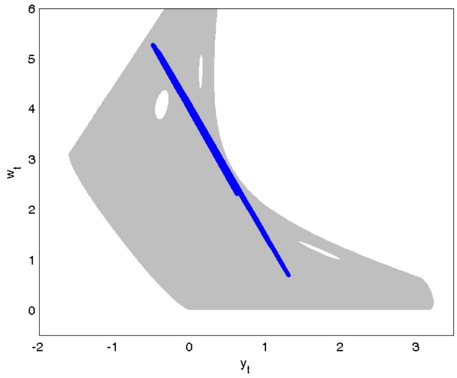


(d) *The limit cycle relative to the Neimark-Sacker bifurcation. The gray region depicts its basin of attraction. Trajectories in the white region are divergent.*

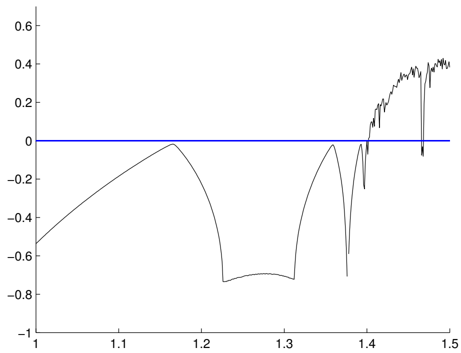
Figure 3: The "quasi-Goodwinian" phase of the system.
Parameters values: $a = -0.29, b = -0.92, c = 0, d = 0.87, e = 0.8, f = -1.5$.



(a) *Period doubling bifurcations: routes to chaos.*



(b) *Henon-like attractor. Chaotic regime.*



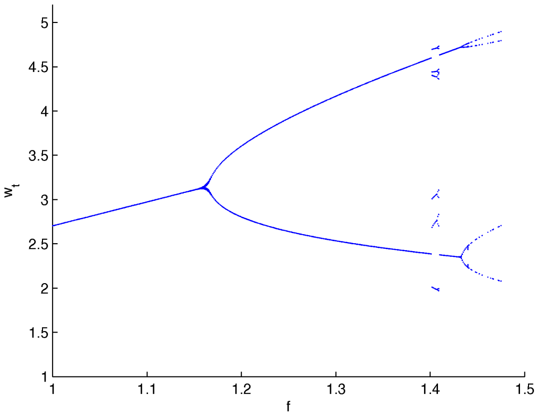
(c) *The Maximal Lyapunov exponent shows the emergence of a chaotic attractor for f>1.4.*

Figure 4: "Classic" investment dynamics with path-dependent wages and price making firms. Parameters values: $a = -0.6, b = -0.92, c = -0.37, d = 0.84, e = 0.9, f = 1.41$.

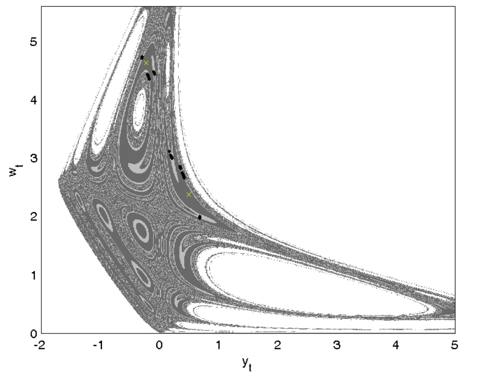
instability, while remaining forever in a bounded region of the state space. They are, as it were, trapped unstable orbits. The chaotic attractor presented in fig. 4.b origins from the merger of a period-4 cycle into two pieces of chaotic attractors, leading to a chaotic regime. The form of the attractor resembles the Henon attractor.

In fact, chaotic dynamics emerge in one region: in presence of positive, high level of f. Notice the meaning of the parameter f. The magnitude and the sign of $f = (1 - \gamma)$ [5] obviously depend upon γ, i.e. the vertical intercept of the linearized Phillips curve. The region in which a cycle emerges (the white area in the left hand side of fig. 2), where

[5] As from the original Goodwin model, one would get $f = (1 - \gamma) / (1 + \alpha)$, with α the rate of growth of productivity which we have set here for simplicity, equal to zero

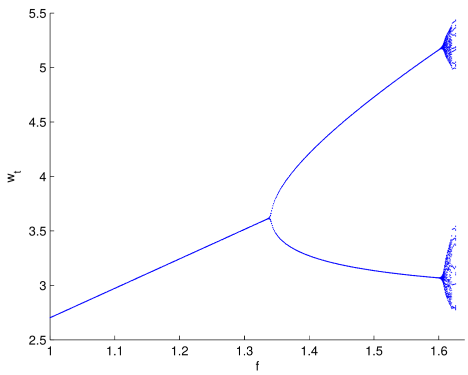


(a) *Discontinuity in the bifurcation diagram shows the coexistence of multiple attractors: the same initial condition previously captured by a period 2-cycle, for $f \in [1.39, 1.42]$ enters the basin of an other attractor.*

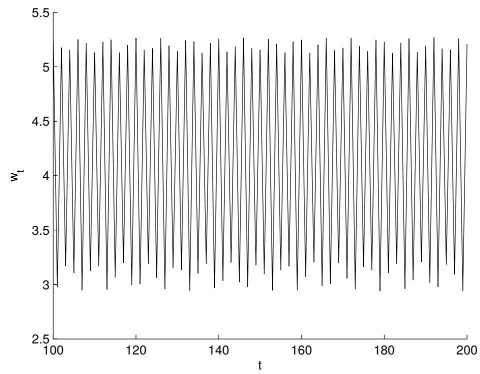


(b) *Basin of coexisting attractors. The stucture of the basins is tangled.*

Figure 5: Coexistence of attractors
Parameters values: $a = -0.3, b = -0.92, c = -0.37, d = 0.84, e = 0.9, f = 1.41$.



(a) *Flip bifurcation ending in a Neimark-Sacker: a good and a bad regime.*



(b) *Quasi-periodic orbits in the Neimark-Sacker phase of the system.*

Figure 6: "Keynesian regime".
Parameters values: $a = 0.45, b = -0.8, c = -0.37, d = 0.84, e = 0.9$.

a form of stability (even though fluctuating) exists, is the region where $\gamma > 1$, and thus is f negative. Since the parameter enters the linearized Phillips curve with a negative specification, it relates to the limit above which the monetary wage growth rate will become inflationary after having passed the NAIRU threshold. We are exactly in the area where the main properties of the *profit-led* economy described by Goodwin are valid. The economy is *profit-led* and wages negatively affect output growth. Conversely, the system enters into a period-doubling bifurcation when f is higher than unity: for this parameter's range the value of γ is lower than one. In this region the value of the parameter f entails a form of hysteresis whereby past wage level affects current one. Interestingly, note also that for higher values of a, (i.e. a milder "anti-accelerator", $a = -0.3$), coexisting attractors emerge (see fig. 5.b), obviously entailing path dependence.
What happens on the contrary to a "predominantly Keynesian" system, displaying accelerator-led growth and price-making firms? We study its dynamic properties under parametrization $a = 0.45, b = -0.8, c = -0.37, d = 0.84, e = 0.9$. The fixed point undergoes a flip bifurcation when $f = 1.34$, ending in a Neimark-Sacker bifurcation in both branches when $f = 1.61$ (see fig. 6.a), involving again quasi-periodic orbits (see fig. 6.b). Indeed, the "Keynesian regime" recovers a relative (fluctuating) order, with multiple basins of attraction, subject to path dependent selection.

5 Conclusions

Profit-led and demand-led models of endogenous fluctuations and growth tend to historically capture two distinct regimes of capital accumulation and of governance of the major markets (for labour, products, etc.). The former models – of which Goodwin's (1967) Growth Cycle is a seminal example – address a "Classic/Marxian" form of capitalism, grounded on competitive price-taking firms, which reinvest their profits, and draw upon an equally competitive labour market.

Conversely Keynesian accelerator/multiplier models find their empirical reference in regimes of capitalist organization characterized by price-making oligopolistic firms which invest according to the demand for their product and draw upon a supply of collectively organized labour, able to some extent to bargain wages independently from labour market conditions.

What happens, however, if one considers at the same time profit-related and demand-related drivers of accumulation and growth?
This is what we have tried to do in this work, formally moving beyond representations of the economic system as a conservative one and studying for properties of dissipative

ones (even if of a particular kind, in that, modern economics, one can, loosely speaking, "get-out", in terms of growth, more than one "puts in", or less). Here, we propose a formalization centered on the coupled dynamics between wages and aggregate income, whose parametrization in turn captures the double nature of wages themselves as element of costs and as a fundamental component of effective demand.
The model exhibits a rich dynamic behaviour and in specific parameters regions yields quasi-periodic orbits, bifurcations and chaos. At a finer analysis, the system tends to be relatively orderly, for example exhibiting quasi-periodic orbits, whenever some consistency conditions between the patterns of accumulation and the form of organization of the major markets are satisfied. It is a vindication of the notion of *discrete regime of socio-economic regulation* (Boyer, 1988b) whose inner "matching" or "mismatching" determines their dynamic stability. Goodwin cycles appear under a profit-led ("anti-accelerator") accumulation regime, whenever this matches with price-taking on the product market and unemployment-driven wage dynamics. And, conversely, a relatively orderly dynamic appears again in a phase of the system characterized by accelerator-driven investment (and, thus, demand-driven growth) matched by price-setting in the product market and hysteresis in wage determination partly independent from labour market conditions. Moreover, interestingly, such a Keynesian set-up exhibits the coexistence of multiple attractors hinting at the multiplicity of growth paths whose selection plausibly depends on history and on public policies.

Where does one go from here? If one considers models such as that presented here as sort of deterministic skeletons addressing some "laws of motion" linking aggregate variables (e.g. income, investment, unemployment, etc.), the challenge is to whether such "laws of motion" emerge out of micro-founded agent-based models (an example of the *genre* is in Dosi et al., 2010). This is one of our tasks ahead.

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