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Selection in incomplete markets and the CAPM portfolio rule

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Abstract

This paper studies whether, and to what extent, trading in an incomplete competitive market rewards the CAPM portfolio rule over alternative rules. We find that, if a mean-variance trader faces an agent who invests in each asset proportionally to expected relative payoffs, in the long-run only two scenarios are possible: either the mean-variance trader vanishes or both agents survive with fixed and constant wealth shares. In both cases, asymptotic prices are proportional to assets' expected payoff, and the relation between prices and returns implied by the CAPM does not generally hold. Conversely, when a mean-variance trader faces a generic fixed-mix investor, several long-run outcomes are possible, such as dominance of one trader, survival of both, and generic path-dependency. We provide sufficient conditions to assess such outcomes. We find that the different outcomes can be effectively discussed in terms of the *effective risk aversion* of the trading strategies, as implied by their portfolio choices conditional on prevailing market prices. In general, a larger effective risk aversion constitutes a survival advantage.

Keywords: Selection; Evolution; Capital Asset Pricing Model; Incomplete Markets.

JEL Classification: C60, D53, G02, G12, G14

1 Introduction

In an interview with Jason Zweig (1998), Harry Markowitz, the father of portfolio selection by mean-variance optimization (Markowitz, 1952), confessed how, when confronted with retirement investment allocation, he equally split his contributions between bonds and equities instead of computing the historical co-variances of the asset classes and drawing an efficient frontier. Such episode is usually taken as anecdotal evidence of how even economists can behave in a suboptimal way because of behavioral biases (see e.g. Benartzi and Thaler, 2007). Could it be, instead, that investing according to a simple rule, like splitting wealth among asset classes in fixed proportions, provides some advantages in a competitive setting with respect to mean-variance optimization? The empirical analysis of DeMiguel et al. (2009) provides evidence in this direction: an equally weighted investment strategy can outperform portfolios derived from mean-variance optimization. The reason for that lies in the errors one can make in estimating the Variance-Covariance matrix. Sciubba (2006) studies the selection dynamics of a short-lived asset market model where an agent with mean-variance preferences and an agent who follows the *Kelly Criterion* (Kelly, 1956, equivalent to maximizing logarithmic preferences) compete. In the particular framework the author considers (perfect information, complete markets, asset payoffs perfectly anti-correlated), the Kelly agent holds a fixed-mix portfolio and in the long-run is able to accrue all the wealth.

In this paper we extend and complement the analysis by Sciubba studying a short-lived asset market model where a mean-variance optimizer and a fixed-mix trader repeatedly exchange a riskless and a risky asset. Markets can be incomplete and, while the mean-variance trader has perfect information, the fixed-mix investor may hold imperfect information about the economy. We find that, when the fixed-mix rule consists in splitting wealth among assets proportionally to expected relative payoffs (the *Generalized Kelly* rule, Evstigneev et al., 2002; Amir et al., 2005; Evstigneev et al., 2009; Bottazzi et al., 2018), only two market selection outcomes are possible in the long-run. Depending on the mean-variance trader risk aversion level, one has either the dominance of the fixed-mix trader or the long-run coexistence of both agents with asymptotically constant wealth shares. It follows that, when trading against a generalized Kelly trader, a fully informed mean-variance optimizer is never able to accrue all the wealth and, eventually, set prices according to her representative agent levels. This also implies that the Capital Asset Pricing Model (CAPM) does not hold in the long-run when a generalized Kelly investor is trading in the market.

When the fixed-mix rule does not coincide with the Generalized Kelly portfolio, several long-run outcomes are possible and we provide sufficient conditions to assess them. Indeed, depending on the economy's parameters – especially the risk aversion level of the mean-variance trader and the portfolio fraction invested by the fixed-mix trader in the riskless asset – one can generically observe *i*) the dominance of the fixed-mix trader and the vanishing of the mean-variance trader; *ii*) the dominance of the mean-variance trader and the vanishing of the fixed-mix trader; *iii*) the survival of both agents; *iv*) path-dependent cases in which the selection outcome depends on the sequence of states of the world realized.

In the end, we analyze several examples and provide a rationale for the good performance of the equally weighted investment strategy which does not rely on estimation errors on the mean-variance side. Indeed, our analysis is able to highlight the conditions that let the equally weighted portfolio achieve an evolutionary advantage over the mean-variance optimization strategy.

2 The model

Consider an exchange economy in discrete time, indexed by $t \in \mathbb{N}$, where two short-lived assets are traded in each period. Asset 1 is risk-free and pays $d_f > 0$ in every t. Asset 2 is risky and pays a stochastic dividend $d_t \in [0, D]$ in each period. Such dividend depends only on the realized state of the world, i.e. $d_t = d(s_t)$. Without loss of generality, we order states of the world such that $d(s)$ increases in s. That is, we assume that $d(1) = 0$, $d(S) = D$, and $d(s+1) > d(s)$.

Uncertainty is modeled in terms of an i.i.d. stochastic process $(s_t)_{t=1}^{\infty}$ with $s_t \in \mathcal{S} = \{1, 2, \ldots, S\}$, $S > 2$. We call $\sigma_t = (s_1, s_2, \ldots, s_t)$ the partial history until time t such that $\Sigma_t = \times_1^t \mathcal{S}$ is the set of partial histories. An history is indicated with $\sigma = (s_1, s_2, \ldots)$ and the set of all possible histories is $\Sigma = \times_1^{\infty} \mathcal{S}$. Let $\mathcal{C}(\sigma_t) = \{\sigma \in \Sigma | \sigma = (\sigma_t, s_{t+1}, \ldots)\}$ be the cylinder with base σ_t with $\mathcal{F}_t$ indicating the σ-algebra generated by the cylinders. Then, $(\mathcal{F}_t)_{t=1}^{\infty}$ is a filtration and $\mathcal{F}$ is the σ-algebra generated by the union of filtrations. We indicate with P the probability measure on $(\Sigma, \mathcal{F})$ and it is such that $\mathrm{P}\{s_{t+1} = s | \sigma_t\} = \pi_s \; \forall \sigma_t, t$. We indicate the expected dividend and the dividend variance as

$$\bar{d} = \sum_{s=1}^{S} \pi_s d(s) \quad \text{and} \quad v = \sum_{s=1}^{S} \pi_s (d(s) - \bar{d})^2 \,.$$

Moreover, we assume $\bar{d} > d_f$. The economy is populated by two agents. The first one is a myopic Mean-Variance optimizer who, in each period t, derives her optimal holdings $h_{1,t}^{mv}$, $h_{2,t}^{mv}$ solving

$$\max_{h_{1,t}^{mv}, h_{2,t}^{mv}} \; h_{1,t}^{mv} d_f + h_{2,t}^{mv} \bar{d} - \frac{\beta v}{2} \left(h_{2,t}^{mv}\right)^2 \quad \text{s.t.} \quad h_{1,t}^{mv} P_{1,t} + h_{2,t}^{mv} P_{2,t} = W_{t-1}^{mv} \,,$$

with W_{t-1}^{mv} her wealth at the end of period $t-1$, $\beta > 0$ her risk aversion parameter, and $P_{1,t}$, $P_{2,t}$ the time t prices of the two assets. Define the time t Sharpe ratio of the risky security as

$$\rho_t = \frac{\bar{d}/P_{2,t} - d_f/P_{1.t}}{\sqrt{v}/P_{2,t}} \,,$$

solving the problem we obtain

$$h^{mv}_{1,t} = \frac{W^{mv}_{t-1}}{P_1, t} - \frac{P_{2t}\rho_t}{P_{1,t}\beta\sqrt{v}} \,, \qquad h^{mv}_{2,t} = \frac{\rho_t}{\beta\sqrt{v}} \,. \tag{1}$$

The second agent is a fixed-mix trader whose holdings $h^{fm}_{1,t}$, $h^{fm}_{2,t}$ in each t read

$$h^{fm}_{1,t} = \frac{\alpha W^{fm}_{t-1}}{P_{1,t}} \,, \quad h^{fm}_{2,t} = \frac{(1-\alpha)W^{fm}_{t-1}}{P_{2,t}} \,, \tag{2}$$

with $\alpha \in (0,1)$ the portfolio share she allocates to the risk-free asset and W^{fm}_{t-1} her wealth at the end of period $t-1$.

The initial wealth of agent $i \in \{mv, fm\}$ is defined as $W^i_0 > 0$ and, at the end of period t, her wealth reads

$$W^i_t = h^i_{1,t} d_f + h^i_{2,t} d_t \,.$$

We assume that assets are in exogenous unitary supply. Thus, in every t, prices are fixed such as to satisfy the market clearing conditions

$$1 = h^{mv}_{1,t} + h^{fm}_{1,t} \,, \quad 1 = h^{mv}_{2,t} + h^{fm}_{2,t} \,. \tag{3}$$

The total wealth in the economy at the end of time t is

$$W_t = W^{mv}_t + W^{fm}_t = d_f + d_t \,, \tag{4}$$

and in every t we have $P_{1,t} + P_{2,t} = W_{t-1}$. Defining the normalized wealth of the fm agent at time t as $w_t = W^{fm}_t / W_t$ and the normalized price of the risk-free asset at time t as $p_t = P_{1,t}/W_{t-1}$, the wealth dynamics of the model can be studied focusing on

$$w_t = \frac{w_{t-1}}{d_f + d_t}\left(\frac{d_f \alpha}{p_t} + \frac{d_t(1-\alpha)}{1-p_t}\right) \,. \tag{5}$$

Market clearing conditions, instead, become

$$1 = \frac{1-w_{t-1}}{p_t} - \frac{(1-p_t)\rho_t}{p_t\beta\sqrt{v}} + \frac{\alpha w_{t-1}}{p_t} \,, \tag{6}$$

$$1 = \frac{\rho_t}{\beta\sqrt{v}} + \frac{(1-\alpha)w_{t-1}}{1-p_t} \,, \tag{7}$$

where the time t Sharpe ratio turns out to be a function of p_t alone, that is

$$\rho_t = \rho(p_t) = \frac{\bar{d}/(1-p_t) - d_f/p_t}{\sqrt{v}/(1-p_t)}\,. \tag{8}$$

The analysis of the static equilibrium of the economy described in this section is in appendix A. In Proposition 2.1 we report the main implications: in every time step the wealth share of the fm agent is positive, unique equilibrium prices exist, and the equilibrium normalized price of the riskless security depends negatively on the fm's wealth share. Moreover, we report the behavior of p_t when the wealth share of fm reaches its extrema and provide an explicit formula to compute the equilibrium normalized price of the riskless security at time t from the fm's wealth share at $t-1$.

Proposition 2.1. *In every $t \in \mathbb{N}$:*

- *it is $w_t > 0$;*
- *there exists one and only one $p_t \in (0,1)$ such that equations (6)-(7) are satisfied;*
- *$p_t = p(w_{t-1})$ with*

$$\begin{aligned} p(w) =& \frac{2d_f + \bar{d} + ((1-\alpha)w - 1)\beta v}{2(d_f + \bar{d} - \beta v)} + \\ & - \frac{\sqrt{\left(\beta v - \bar{d} - (1-\alpha)w\beta v\right)^2 + 4(1-\alpha)w d_f \beta v}}{2(d_f + \bar{d} - \beta v)}\,; \end{aligned} \tag{9}$$

- *$\partial p_t / \partial w_{t-1} < 0$;*
- *if $w_{t-1} \to +\infty$ then $p_t \to 0$;*
- *if $w_{t-1} \to 0$ and $\beta < \bar{d}/v$ then $p_t \to d_f/(d_f + \bar{d} - \beta v)$;*
- *if $w_{t-1} \to 0$ and $\beta \geq \bar{d}/v$ then $p_t \to 1$.*

Proof. See appendix A. □

The main difference with respect to the framework studied by Sciubba (2006) is that our market structure allows the mv trader to finance her trades by means of short selling. This means that the mv trader may decide to increase the supply of a given asset in order to increase her investment in the other. Thus, in our model the wealth of the mv trader can generically become negative. At the same time, Proposition 2.1 ensures that the mv investing choices do not generate negative equilibrium prices. As a consequence, given eq. 2, the fm agent cannot short any asset and her wealth never attains negative values. On the contrary, the fm's wealth share can grow beyond 1 and asymptotically diverge. Indeed, whenever mv ends up with negative wealth, this means that fm, as well as owing the whole present aggregate endowment of the economy, has claims also on future earnings of the mv trader. In this case one may imagine to set a maximum leverage ratio and force mv to eventually pay back her debts. However, in order to consider the best case scenario for the mv trader, we avoid such limit: the mv wealth can be asymptotically (infinitely) negative.[1]

These considerations, however, suggest that the standard definitions of dominance and vanishing (see e.g. Blume and Easley, 1992; Sciubba, 2006; Bottazzi et al., 2018) have to be amended. In particular, the fact that asymptotically the wealth share of mv is non positive shall be considered as a case in which fm dominates and mv vanishes. Concerning survival, instead, we maintain the standard definition: both agents have to maintain positive wealth shares asymptotically.

Definition 2.1. We say that fm dominates and mv vanishes on a sequence σ if

$$\liminf_{t\to\infty} w_t(\sigma) \geq 1\,. \tag{10}$$

We say that fm vanishes and mv dominates on a sequence σ if

$$\lim_{t\to\infty} w_t(\sigma) = 0\,. \tag{11}$$

We say that fm survives on a sequence σ if

$$\limsup_{t\to\infty} w_t(\sigma) > 0\,. \tag{12}$$

We say that mv survives on a sequence σ if

$$\liminf_{t\to\infty} w_t(\sigma) < 1\,. \tag{13}$$

We say that fm dominates and mv vanishes, fm vanishes and mv dominates, fm survives or mv survives if (10), (11), (12) or (13) holds P almost surely.

Another important feature that differentiates our analysis from previous studies in the field is the possibility of reaching asymptotic homogeneity in terms of

[1]Notice that, according to Proposition 2.1, also in such a case one should asymptotically observe arbitrages. However, here they result from the market interaction of traders and can be excluded only setting a maximum leverage threshold for mv, which we explicitly avoid.

portfolio shares. That is, the imitative nature underlying the Mean-Variance investing approach may generically lead the mv trader to invest in each asset the same wealth fractions of the fm trader. Thus, define the function

$$a^{mv}(w) = 1 - \frac{\rho(p(w))(1-p(w))}{(1-w)\beta\sqrt{v}} \,,$$

such that $a^{mv}(w_{t-1}) = h^{mv}_{1,t} p_t/(1-w_{t-1})$ is the portfolio share of mv at time t, and consider the quantity

$$\tilde{w} = \frac{\alpha(\beta v - \bar{d}) + (1-\alpha) d_f}{\alpha\beta v} \,.$$

Then, one immediately has $a^{mv}(\tilde{w}) = \alpha$. Hence, whenever the wealth share of fm reaches $\tilde{w}$ homogeneity in terms of portfolio shares is obtained. Notice also that in such a case normalized prices become $(\alpha, 1-\alpha)$. From (5) one immediately notices that $\tilde{w}$ is *deterministic fixed point* for w_t: if the wealth share of fm reaches $\tilde{w}$ it will never move away. At the same time, provided an initial wealth share of fm different from $\tilde{w}$ (i.e. $w_0 \neq \tilde{w}$), homogeneity can be attained only asymptotically. Indeed, if w_0 is smaller than $\tilde{w}$ then w_t will always stay below the deterministic fixed point, while, if w_0 is larger than $\tilde{w}$, then w_t will remain above the deterministic fixed point.

Proposition 2.2. *Assume $\tilde{w} > 0$. If $w_t = \tilde{w}$ then $p_{t+\tau} = \alpha$ and $w_{t+\tau} = \tilde{w}$ for all $\tau \in \mathbb{N}$. If $w_0 < \tilde{w}$ then $w_t < \tilde{w}$ for any finite t. If $w_0 > \tilde{w}$ then $w_t > \tilde{w}$ for any finite t.*

Proof. See appendix B. □

3 Long-Run Selection

After having analyzed some important characteristic of the model, we turn to investigate the long-run selection outcomes. First, we focus on a particular fixed-mix portfolio rule, named generalized Kelly (Evstigneev et al., 2002, 2009, 2016), that ensures survival when competing against a mean-variance trader. Then, we analyze long-run outcomes when generic fixed-mix rules are adopted, providing conditions to discriminate among the different selection scenarios.

3.1 A survival fixed-mix rule

In our framework the generalized Kelly rule is a fixed-mix investing strategy that prescribe to invest in each asset a fraction of wealth which is proportional to the asset's expected relative payoff. We indicate such a rule with α^* and it reads

$$\alpha^* = \sum_{s=1}^{S} \pi_s \frac{d_f}{d_f + d(s)} .$$

With respect to the analysis performed by Sciubba (2006), the generalized Kelly rule can be thought as an heuristic extension to incomplete markets of the *bet your beliefs* principle derived from intertemporal log-utility maximization in complete markets of Arrow securities Evstigneev et al. (2009, 2016). We focus on such rule because of the good properties it has when competing in an evolutionary frameworks against other adapted portfolio rules. An adapted portfolio rule prescribes to split wealth among assets according to fractions that depend only on the information revealed until the beginning of the time step. In a short-lived asset market where agents invest according to adapted rules and states of the world follow an i.i.d process, the generalized Kelly rule is a survival strategy: it maintains a strictly positive wealth share in the long-run with probability one (Evstigneev et al., 2016). When competing strategies are basic, i.e. they depend only on the sequence of realized states, the generalized Kelly rule owns asymptotically all the wealth in the market with probability one (Evstigneev et al., 2016). The generalized Kelly rule remains a survival strategy also when competing against adapted rules that can rely on short selling (Belkov et al., 2017). The investing behavior of *mv* in our framework is not adapted: portfolio fractions depend on contemporary prices, a piece of information that is not revealed at the beginning of the period.

Proposition 3.1. *If $\alpha = \alpha^*$ then*

1. *fm dominates and mv vanishes if*

$$\beta \leq \frac{\bar{d}}{v} - \frac{(1-\alpha^*)d_f}{\alpha^* v} ;$$

2. *both fm and mv survive if*

$$\beta \geq \frac{\bar{d}}{(1-w_0)v} - \frac{(1-\alpha^*)d_f}{(1-w_0)\alpha^* v} ;$$

3. *fm dominates and mv vanishes on sequences $\sigma' \in \Sigma' \subset \Sigma$ and both fm and mv survive on sequences $\sigma'' \in \Sigma'' \subset \Sigma$, with $\mathrm{P}\{\Sigma'\} + \mathrm{P}\{\Sigma''\} = 1$, if*

$$\frac{\bar{d}}{v} - \frac{(1-\alpha^*)d_f}{\alpha^* v} < \beta < \frac{\bar{d}}{(1-w_0)v} - \frac{(1-\alpha^*)d_f}{(1-w_0)\alpha^* v} .$$

Proof. See appendix D. □

Proposition 3.1 shows that the generalized Kelly rule is a survival strategy when competing against a mean-variance investor in an incomplete market. Indeed, mv is never able to dominate and asymptotically let fm's wealth share go to zero. Two situations are possible in the long-run: either fm dominates and mv vanishes or both survive. The level of risk aversion of mv plays a fundamental role in discriminating the cases. If mv is not sufficiently risk averse, then $\tilde{w}$ turns out negative and, for any possible initial wealth share, fm is able to dominate. The reason for that lies in the generalized Kelly rule. Such rule, when mv is not able to imitate it, ensures that fm's wealth share has a positive expected growth rate. This explains also how long-run survival of both agent is obtained. When mv is sufficiently risk averse, then $\tilde{w}$ ends up in the $(w_0, 1)$ interval. Thus, the positive expected growth rate of fm's wealth share entails that in the long-run w_t converges to $\tilde{w}$. Asymptotically mv imitates fm and both survive. When the risk aversion of mv is at "intermediate" levels, the situation is more complicated. Indeed, one has that $\tilde{w}$ is locally stable. At the same time, if w_t grows too much, then it will diverge. Hence, we can only say that one case between survival of both agents and dominance of fm may occur depending on the sequence of realized events. Basically, when point *3.* of Proposition 3.1 occurs, we observe a form of path-dependence.

The selection results in Proposition 3.1 has consequences for long-run asset pricing. Indeed, the fact that mv cannot dominate against a generalized Kelly trader implies that in the long-run prices are never set according to the mv prescriptions.

Corollary 3.1. *If* $\alpha = \alpha^*$ *then, with probability one, either* $\lim_{t\to\infty} w_t = \tilde{w} < 1$ *and* $\lim_{t\to\infty} p_t = \alpha^*$ *or* $\lim_{t\to\infty} w_t = +\infty$ *and* $\lim_{t\to\infty} p_t = 0$.

Proof. See appendix E. □

Corollary 3.1 confirms that, when mv competes with a generalized Kelly trader, either she ends up imitating the behavior of the opponent or she is forced to a sort of Ponzi scheme. Moreover, the Corollary shows the long-run pricing outcome. It is immediate to notice that, in the best case scenario for mv (i.e. survival of both agents), mv has no role in setting long-run prices: they are fixed according to the generalized Kelly prescriptions. In the other case, when mv vanishes and fm dominates, prices are quite opposite with respect to mean-variance representative agent levels. This helps to explain how fm forces mv to a Ponzi scheme. If $w_{t-1} \approx 1$ then, the negative dependence of p_t on w_{t-1} together with Proposition 3.2 implies $p_t > \alpha^*$. Thus, the riskless security appears underpriced to mv, while the risky security appears overpriced. Hence, mv starts shorting the risky security in order to invest in the riskless one. This can become a self-defeating strategy. Since the risky security pays more in expectation, mv may end up with negative wealth, the relative price of the riskless security decreases even more and mv shorts more strongly the risky security. A negative spiral is set up and asymptotically fm dominates with divergent wealth share.

The drawback of the generalized Kelly rule is that it requires the knowledge of the true probability distribution to be implemented. When such knowledge is not available to the fm agent, market structure and agents' characteristics play a primary role is driving asymptotic outcomes.

3.2 Generic fixed-mix rules

Suppose that the fm trader chooses a generic portfolio $(\alpha, 1-\alpha)$, which are the long-run selection outcomes of the model? Does fm still maintain an evolutionary advantage on mv or has the latter some chances to asymptotically own all the wealth and, eventually, set prices? These are the questions we try to answer in this section. The picture that will emerge is that several possible outcomes are possible depending on asset structure, the portfolio fractions fm chooses, and how much risk averse mv is. Indeed, depending on the parameter setting, we may observe fm dominating and mv vanishing, fm vanishing and mv dominating, one or both traders surviving. Path dependent cases are also possible: some combinations of parameter make the long-run selection outcome depend on the particular sequence of event realized.

At the technical level, our asymptotic results are inferred studying the properties of stochastic process derived from a transformation of w_t, see appendix C for further detail. In particular, we use the conditions for persistence or transience of stochastic processes provided by Bottazzi and Dindo (2015). Those conditions rely on the sign of asymptotic conditional drifts. In our case, it is enough to study the sign of two quantities,

$$\mu_0 = \sum_{s=1}^{S} \pi_s \log \left(\frac{\alpha(d_f + \bar{d} - \beta v)}{d_f + d(s)} + \frac{d(s)(1-\alpha)(d_f + \bar{d} - \beta v)}{(d_f + d(s))(\bar{d} - \beta v)} \right) \tag{14}$$

and

$$\mu_{\tilde{w}} = \sum_{s=1}^{S} \pi_s \log \left(1 + \left(\frac{d_f}{d_f + d(s)} - \alpha \right) \frac{d_f - \alpha(d_f + \bar{d} - \beta v)}{d_f - \alpha^2(d_f + \bar{d} - \beta v)} \right) . \tag{15}$$

The sign of μ_0 has a clear economic interpretation. Indeed, it can be considered the expected log-growth-rate of fm's wealth share when mv has all the wealth in the market under the assumption $\beta < \bar{d}/v$. If μ_0 is positive then, by continuity, the fm's wealth share grows more in expectation than the mv's one when the latter has almost all the wealth in the economy. If, instead, μ_0 is negative it is the other way round. Thus, given a sequence of events such that w_t ends up sufficiently close to 0, if $\mu_0 > 0$ then w_t is pushed away from 0, while, if $\mu_0 < 0$, then w_t is attracted toward 0. If $\beta \geq \bar{d}/v$, instead, the expected log-growth-rate of fm's wealth share becomes infinite in the limit of mv owing everything. This means that, whenever mv is close to dominate, fm is able to grow more in expectation and w_t is pushed away from 0.

The sign of $\mu_{\tilde{w}}$ is informative of the local stability of the fixed point $\tilde{w}$. Thus, given a sequence of events such that w_t is sufficiently close to $\tilde{w}$, if $\mu_{\tilde{w}}$ is negative we have that $\tilde{w}$ is stable and the fm wealth share is attracted toward $\tilde{w}$. If, instead, $\mu_{\tilde{w}}$ is positive we have that $\tilde{w}$ is unstable and w_t is pushed away from it. Thus, in economic terms, $\tilde{w}$ stable means that, if after a sequence of events agents end up investing in a sufficiently similar way, asymptotically they become identical. If, instead, $\tilde{w}$ is unstable, they tend to differentiate in the long-run.

3.2.1 Long-run dominance

We first focus on those cases in which one of the two traders is able to dominate. In other models belonging to the Evolutionary Finance tradition which exclude short selling (see e.g. Evstigneev et al., 2009; Bottazzi et al., 2018), long-run dominance is associated with a particular trading rule owing unitary wealth share and setting prices as in a representative agent setting. This is only partially true here. It is the case when mv dominates and fm vanishes while it is not always true when fm dominates and mv vanishes. Indeed, in the latter case fm can generically hold a wealth share in the long-run which is larger than one or, eventually, diverge.

Proposition 3.2. *fm dominates and mv vanishes if one of the following conditions is satisfied:*

1. $\mu_0 > 0$, $\alpha > \dfrac{d_f}{d_f + \bar{d}}$, *and* $\beta < \dfrac{\bar{d}}{v} - \dfrac{(1-\alpha)d_f}{\alpha v}$;
2. $\mu_{\tilde{w}} > 0$, $\alpha > \dfrac{d_f}{d_f + \bar{d}}$, *and* $\dfrac{\bar{d}}{v} - \dfrac{(1-\alpha)d_f}{\alpha v} < \beta < \dfrac{\bar{d}}{(1-w_0)v} - \dfrac{(1-\alpha)d_f}{(1-w_0)\alpha v}$;
3. $\mu_0 > 0$, $\mu_{\tilde{w}} < 0$, $\alpha \leq \dfrac{d_f}{d_f + \bar{d}}$, *and* $\beta > \dfrac{\bar{d}}{(1-w_0)v} - \dfrac{(1-\alpha)d_f}{(1-w_0)\alpha v}$.

Proof. See appendix F. □

Proposition 3.2 shows that fm dominates and mv vanishes in three cases. In *1.* we have that the conditions on α and β ensure $\tilde{w} \leq 0$, while the expected growth rate of fm's wealth share when mv has (almost) all the wealth is positive ($\mu_0 > 0$). Thus, for any possible choice of initial wealth shares, we have $w_t > \tilde{w}\ \forall t$. Moreover, whenever the fm's wealth share approaches 0, it is pushed away because fm grows more in expectation than mv when her wealth share is low. Then, sooner or later, w_t shall become so large that it starts to grow with probability one and, asymptotically, diverges. In *2.* the condition on β implies $w_0 > \tilde{w}$ while the condition on α ensures $\tilde{w} < 1$. Hence, asymptotically w_t can either diverge or converge to $\tilde{w}$. However, the condition $\mu_{\tilde{w}} > 0$ ensures that the fm's wealth share cannot converge to $\tilde{w}$. Thus, it asymptotically diverges. In *3.* we have that the condition on β delivers $w_0 < \tilde{w}$, while the condition on α implies $\tilde{w} > 1$. Thus, since w_t cannot converge to 0 ($\mu_0 > 0$) while it is attracted toward $\tilde{w}$ ($\mu_{\tilde{w}} < 0$), we obtain that asymptotically $\tilde{w}$ is reached and mv holds a negative wealth share in the long-run.

Corollary 3.2. *If condition 1. or 2. of Proposition 3.2 is satisfied then, with probability one,* $\lim_{t\to\infty} w_t = +\infty$ *and* $\lim_{t\to\infty} p_t = 0$*. If condition 3. of Proposition 3.2 is satisfied then, with probability one,* $\lim_{t\to\infty} w_t = \tilde{w} > 1$ *and* $\lim_{t\to\infty} p_t = \alpha$*.*

Proof. The statements follow from the arguments in appendix F about the asymptotic behavior of w_t coupled with Proposition 2.1. □

Concerning prices, Corollary 3.2 highlights that, when the sufficient conditions for fm to dominate are met, one shall observe the riskless security's price converge either to zero or to α. The former case is verified when condition *1.* or *2.* is satisfied while the latter occurs when *3.* is met. This result clearly proves that in our model the dominance of fm does not automatically imply that long-run prices are set according to her representative agent levels. This is due to the short selling behavior of mv, who continues to trade even with a negative wealth share.

Proposition 3.3. *fm vanishes and mv dominates if* $\mu(0) < 0$*,* $\mu_{\tilde{w}} > 0$*,* $\alpha < \dfrac{d_f}{d_f + w_0\bar{d}}$*, and* $\dfrac{\bar{d}}{(1-w_0)v} - \dfrac{(1-\alpha)d_f}{(1-w_0)\alpha v} < \beta < \dfrac{\bar{d}}{v}$*.*

Proof. See appendix G □

The conditions that ensure mv to dominate while fm vanishes are more strict. As shown in Proposition 3.3, the risk aversion parameter of mv has to be in a precise interval. The lower bound ensures $w_0 \in (0, \bar{w})$. That is needed in order to prevent w_t from becoming large and, eventually, diverging. The upper bound, instead, prevents the price of the risky security from approaching 0 when mv has almost all the wealth in the economy. In such a case the fm's wealth share has an infinite expected growth rate when mv is close to own everything: fm cannot vanish. The condition on α provides that the interval to which β must belong is not empty. On top of the requirements on β and α, one has also to check that, on the one hand, the fixed point $\tilde{w}$ is unstable and, on the other, 0 is attracting for fm's wealth share. In this way, whenever mv invests similarly to fm, it is profitable for mv to differentiate her portfolio strategy with respect to the fm's one. Hence, mv increases her wealth share and w_t decreases with respect to $\tilde{w}$. Whenever, instead, mv is close to have all the wealth in the economy her wealth share grows more in expectation than fm's one. Thus, asymptotically mv owns a unitary wealth share and prices converge to the mean-variance representative agent levels.

3.2.2 Survival and path dependency

We focus now on the conditions that deliver agents' survival. These conditions are more general than the previous ones delivering dominance since, in order to dominate, the agent needs to survive. As we shall see, the survival of one agent does not necessary exclude the survival of the other. Indeed, intersecting the set of parameter values that let fm survive with the one which let mv survive, we obtain generic cases in which both agents achieve long-run survival.

Proposition 3.4. *fm survives if one of the following conditions is satisfied:*

1. $\mu_0 > 0$ *and* $\beta < \dfrac{\bar{d}}{v} - \dfrac{(1-\alpha)d_f}{\alpha v}$;
2. $\dfrac{\bar{d}}{v} - \dfrac{(1-\alpha)d_f}{\alpha v} < \beta \leq \dfrac{\bar{d}}{(1-w_0)v} - \dfrac{(1-\alpha)d_f}{(1-w_0)\alpha v}$;
3. $\mu_0 > 0$ *and* $\beta > \dfrac{\bar{d}}{(1-w_0)v} - \dfrac{(1-\alpha)d_f}{(1-w_0)\alpha v}$.

mv survives if one of the following conditions is satisfied:

1. $\alpha > \dfrac{d_f}{d_f + \bar{d}}$ *and* $\beta \geq \dfrac{\bar{d}}{(1-w_0)v} - \dfrac{(1-\alpha)d_f}{(1-w_0)\alpha v}$;
2. $\mu_{\tilde{w}} > 0$ *and* $\alpha \leq \dfrac{d_f}{d_f + \bar{d}}$.

Proof. See appendix H. □

Proposition 3.4 shows that fm can survive for any level of mv risk aversion. When the condition on β in *1.* is satisfied, one has that $\tilde{w}$ is negative. In such a case w_t can generically approach zero. Thus, a sufficient condition to observe the survival of fm is that she grows more in expectation than mv when mv is close to own all the wealth, $\mu_0 > 0$. When β is such that the conditions in *2.* are satisfied, we have $w_t > \tilde{w} > 0\ \forall t$, hence, by Proposition 2.2, the survival of fm is ensured. When β satisfies the condition in *3.* one obtains that $w_t < \tilde{w}\ \forall t$, hence also in this case w_t can approach zero. Thus, also here, in order to let fm survive one should ensure that she grows more than mv when the latter has (almost) unitary wealth share.

The conditions that ensure the survival of mv appear somehow more strict. Indeed, now both α and β play an important role. First of all, notice that we have to ensure $w_t < \tilde{w}$. If $\alpha > d_f/(d_f + \bar{d})$ then it is $\tilde{w} < 1$, thus the condition on β is crucial to permit the survival of mv, since it ensures $w_t < \tilde{w}\ \forall t$. Then, no matter whether $\tilde{w}$ is stable or unstable, the wealth share of mw remains bounded away from zero. If instead α is smaller than or equal to $d_f/(d_f + \bar{d})$, then we have $w_t \geq \tilde{w}$. First, we need to ensure that w_t is reflected away when approaches $\tilde{w}$. This is provided by the condition $\mu_{\tilde{w}} > 0$. Then, we can exploit the fact that in every period w_t can decrease with strictly positive probability to prove that values smaller than 1 are attained almost surely. The survival of mv follows by definition.

Another peculiar feature that characterizes our model is the generic occurrence of path dependent scenarios. In such cases we may observe one between two cases depending on the realized sequence of events.

Proposition 3.5. *fm dominates and mv vanishes on sequences $\sigma' \in \Sigma' \subset \Sigma$ while fm vanishes and mv dominates on sequences $\sigma'' \in \Sigma'' \subset \Sigma$, with* $\mathrm{P}\{\Sigma'\} + \mathrm{P}\{\Sigma''\} = 1$*, if $\mu_0 < 0$ and one of the following conditions is satisfied:*

1. $\beta < \dfrac{\bar{d}}{v} - \dfrac{(1-\alpha)d_f}{\alpha v}$;

2. $\mu_{\tilde{w}} < 0$, $\alpha \leq \dfrac{d_f}{d_f + \bar{d}}$, *and* $\beta < \dfrac{\bar{d}}{v}$.

Proof. See appendix I □

Proposition 3.5 shows the conditions that deliver a form of *extreme* path dependence. Indeed, when the conditions are satisfied, we may observe either the dominance of fm and the vanishing of mv or vice-versa. The condition on β in *1.* causes $\tilde{w} < 0$, thus we have $w_t \in (0, +\infty)\ \forall t$. It follows that, if mv grows more in expectation than fm when fm has (almost) nothing, then w_t can converge to zero. At the same time, if w_t becomes large enough, then it will asymptotically diverge. When, instead, the conditions in *2.* are verified, then we have $\tilde{w} \geq 1$ (provided by the condition on α) and $\tilde{w}$ stable (provided by $\mu_{\tilde{w}} < 0$). Thus, in such a case, we observe either the converge of w_t to zero or its convergence to a value greater than (or equal to) one. Again, depending on the realized sequence of events, the dominance of one agent and the vanishing of the other is observed, but we cannot establish their identity *ex-ante*.

Proposition 3.6. *fm dominates and mv vanishes on sequences $\sigma' \in \Sigma' \subset \Sigma$ while both fm and mv survive on sequences $\sigma'' \in \Sigma'' \subset \Sigma$, with $\mathrm{P}\{\Sigma'\} + \mathrm{P}\{\Sigma''\} = 1$, if $\mu_{\tilde{w}} < 0$, $\alpha > \frac{d_f}{d_f + \bar{d}}$, and $\frac{\bar{d}}{v} - \frac{(1-\alpha)d_f}{\alpha v} < \beta < \frac{\bar{d}}{(1-w_0)v} - \frac{(1-\alpha)d_f}{(1-w_0)\alpha v}$.*

Proof. See appendix J ◻

Proposition 3.6 shows the sufficient conditions to have a form of path dependence that favors fm. Indeed, when the condition in the Proposition is satisfied, then we shall observe either the dominance of fm, with the consequent vanishing of mv, or the survival of both. An important condition that delivers such scenario is $w_t > \tilde{w}\ \forall t$. This is provided by the conditions on α and β, ensuring, respectively, $\tilde{w} < 1$ and $\tilde{w} < w_0$. However, this is not enough. Indeed, we also need to ensure that mv tends to imitate fm even more when agents' investment fractions are similar. This means that $\tilde{w}$ has to be stable and it is provided by $\mu_{\tilde{w}} < 0$. Then, depending of the sequence of realized state of the world, we can have that agents become asymptotically equal or that fm has, asymptotically, infinite wealth.

Proposition 3.7. *fm vanishes and mv dominates on sequences $\sigma' \in \Sigma' \subset \Sigma$ while both fm and mv survive on sequences $\sigma'' \in \Sigma'' \subset \Sigma$, with $\mathrm{P}\{\Sigma'\} + \mathrm{P}\{\Sigma''\} = 1$, if $\mu_0 < 0$, $\mu_{\tilde{w}} < 0$, $\alpha > \frac{d_f}{d_f + \bar{d}}$, and $\frac{\bar{d}}{(1-w_0)v} - \frac{(1-\alpha)d_f}{(1-w_0)\alpha v} < \beta < \frac{\bar{d}}{v}$.*

Proof. See appendix K ◻

Finally, we analyze the last scenario, the one in which the form of path dependence obtained favors mv. Indeed, when the condition in Proposition 3.7 holds, then we shall observe either the dominance of mv, with the consequent vanishing of fm, or the survival of both. Again, β and α play a fundamental role. Indeed, we have to ensure $\tilde{w} < 1$ and $w_t < \tilde{w}\ \forall t$. The former is delivered by the condition on α while the latter is entailed by the condition on β. Next we have to ensure that w_t can converge either to zero or to $\tilde{w}$. This is delivered, respectively, by $\mu_0 < 0$ and $\mu_{\tilde{w}} < 0$. When those conditions are verified, mv grows more than fm when the former has almost everything while mv tends to imitate fm when they are similar. Hence, the sequence of realized events becomes crucial in establishing which long-run outcome is observed between mv owing everything or the two agent being equal.

3.3 Discussion and examples

Here we provide some intuition about the underlying mechanism that produces the results and analyze some specific example. Our discussion revolves around the idea of fm's *effective risk aversion*. That is, we can interpret the fm's trading behavior as a particular instance of mean-variance investing with time-dependent risk aversion. Such risk aversion is somehow *implied* by the wealth allocation choices of fm: it is the one an external observer believing that fm has mean-variance preferences would infer. Then, we proceed to study some examples imposing specific values to the model's parameter. In particular, we shall choose a probability distribution for the risky asset dividends and fix a value for the riskless payoff. Our study shall focus on applying the conditions we derived in the previous sections and discussing how the selection outcomes change for different values of α and β.

3.3.1 Effective risk aversion

Comparing and discussing the investing choices derived from a given behavioral rule (like the fixed-mix one) with those generated by (myopic) preference maximization is not trivial. Indeed, it requires a meaningful and synthetic measure that allows us to understand how one behavior differentiates from the other. Following the example of *effective beliefs* employed by Bottazzi et al. (2018) and Dindo (2019), we introduce the idea of fm's *effective risk aversion*. As briefly summarized in advance, it is possible to map the fm portfolio in one derived from the mean-variance maximization procedure by means of a time-dependent risk aversion coefficient. That is, we basically infer the risk aversion a mean-variance trader should show in order to invest as like as the fm agent. Since, under this interpretation, such time-varying coefficient is the only feature that distinguish the two traders, the fm's effective risk aversion is key to explain long-run selection outcomes.

In formal terms, we define the fm's effective risk aversion coefficient as

$$b_t = \frac{(1-p_t)\rho_t}{(1-\alpha)w_{t-1}\sqrt{v}}$$

such that it is

$$h_{1,t}^{fm} = \frac{w_{t-1}}{p_t} - \frac{(1-p_t)\rho_t}{p_t b_t \sqrt{v}}\,, \quad h_{2,t}^{fm} = \frac{\rho_t}{b_t\sqrt{v}}\,.$$

Imposing the market clearing conditions in (3), we can obtain an *equilibrium* expression of b_t which has an interesting economic interpretation. Indeed, we have

$$b_t = b(w_{t-1}) = \beta\frac{(1-p(w_{t-1})) - (1-\alpha)w_{t-1}}{(1-\alpha)w_{t-1}} = \beta\frac{(1-a(w_{t-1}))(1-w_{t-1})}{(1-\alpha)w_{t-1}} \tag{16}$$

which means that the effective risk aversion coefficient of fm matches the mv risk aversion coefficient rescaled by the ratio between the amounts of wealth agents invest in the risky asset. In particular, the higher the wealth invested by mv (fm) in the risky asset, the larger (smaller) the fm's effective risk aversion. Moreover, one generically has that fm's effective risk aversion is larger (smaller) than β if mv is investing in the risky asset more (less) than fm. The fm's effective risk aversion decreases when her wealth share increases,

$$\frac{\partial b(w)}{\partial w} = -\frac{(1-p(w))^3 d_f \beta}{w^2(1-\alpha)((1-p(w))^2 d_f + p(w)^2(1-\alpha)w\beta v)} < 0\,,$$

and in the limit of an infinite wealth share $b(w)$ converges to $-\beta$: $\lim_{w\to+\infty} b(w) = -\beta$. If, instead, the fm's wealth share goes to zero we have two possible cases depending on the value of β. If $\beta < \bar{d}/v$ then $\lim_{w\to 0} b(w) = +\infty$, while, if $\beta \geq \bar{d}/v$, it is

$$\lim_{w\to 0} b(w) = \beta\frac{\alpha(\beta v - \bar{d}) + \bar{d}}{(1-\alpha)(\beta v - \bar{d})} > 0\,.$$

Hence, the fm's portfolio appears risk averse to mv (i.e. evaluated according to mv preferences) when mv sets the prices and such implied risk aversion decreases as evaluations move away from mv ones, that is, as fm increases her wealth share. As fm becomes richer, her effective risk aversion approaches negative values, meaning that fm appears risk prone to mv when the former has large wealth. Indeed, according to (16), fm seems risk prone to mv whenever the latter shorts the risky asset. This is reasonable: fm always holds a long position in both assets, hence, whenever mv – who has $\beta > 0$ by definition – believes that the risky asset should be sold short, any long position in the risky asset is considered a consequence of risk loving.

Given a certain level of wealth share, instead, the effective risk aversion of fm increases with α. This is intuitively correct: an agent that invests a relatively larger share of her wealth in the riskless asset should show a higher risk aversion. This argument can be formalized looking at the derivative of $b(w)$ with respect to α given a wealth share level w. Indeed, we have[2]

$$\left.\frac{\partial b(w)}{\partial \alpha}\right|_w = \frac{\beta(1-p(w))^3 d_f}{w(1-\alpha)^2((1-p(w))^2 d_f + p(w)^2(1-\alpha)w\beta v)} > 0$$

and such positive dependence of the fm's effective risk aversion upon α let us consider the fraction of wealth fm invests in the riskless security as a rough measure of her (implicit) risk aversion conditional on a given wealth share level.

[2]To compute such derivative we exploit our result on the derivative of $p(w)$ with respect to α for a given wealth share w in Fact A.4, appendix A.

Finally, it is interesting to notice that the case of long-run homogeneity in terms of portfolios (i.e. $w_t \to \tilde{w}$) does not automatically imply homogeneity in terms of effective risk aversions. Indeed, we have

$$\lim_{w \to \tilde{w}} b(w) = \beta \frac{1 - \alpha(\beta v - \bar{d}) - (1-\alpha) d_f}{\alpha(\beta v - \bar{d}) + (1-\alpha) d_f}$$

which is generically different from β.

Concerning the selection outcomes, we shall link agents' behavior and (implied) preferences with evolutionary fitness in the next section by means of specific examples. In particular, we will exploit the relationship between α and fm's effective risk aversion in order to understand the relationship between risk preferences and selection outcomes.

3.3.2 Examples

We assume that there are 1000 different states of the world, $S = 1000$, the riskless payoff is normalized to 1, $d_f = 1$, the maximum dividend the risky asset pays is 10, $D = 10$, and initial wealth is evenly shared, $w_0 = 0.5$. Then, without loss of generality, we assign dividends to states of the world in ascending order: we assume $d(s) = (s-1)D/(S-1)$. Probabilities are assigned to states according to a Boltzmann probability distribution that takes into account the risky asset's dividend levels. That is

$$\pi_s = \frac{\exp\{-\lambda d(s)\}}{\sum_{j=1}^{S} \exp\{-\lambda d(j)\}} \quad \forall s \in \mathcal{S}\,.$$

For the moment, we set $\lambda = 0.5$, such that we obtain $\bar{d} = 1.9274$,[3] $v = 3.3191$, and $\alpha^* = 0.4654$.

In Figure 1 we provide an example of the behavior of $b(w)$ for different values of w and α under the assumption $\beta = 1 > \bar{d}/v$. This exercise confirms the result discussed in the previous section: fm tends to be more risk averse than mv when her wealth share is low and becomes less and less risk averse as she accumulates wealth. Indeed, as her wealth share increases, fm attains negative values of risk aversion, meaning that fm seems to show a form of risk proneness when she is sufficiently rich. One can also notice how the positive relation between $b(w)$ and α emerges, moreover as w increases the relation becomes stronger.

In Figure 2 we present the selection outcomes of the model for different combinations of α and β. As already argued, this can be roughly understood as a study of selection outcomes for different levels of agents' risk aversion.

[3]Notice that the condition $d_f < \bar{d}$ is respected.

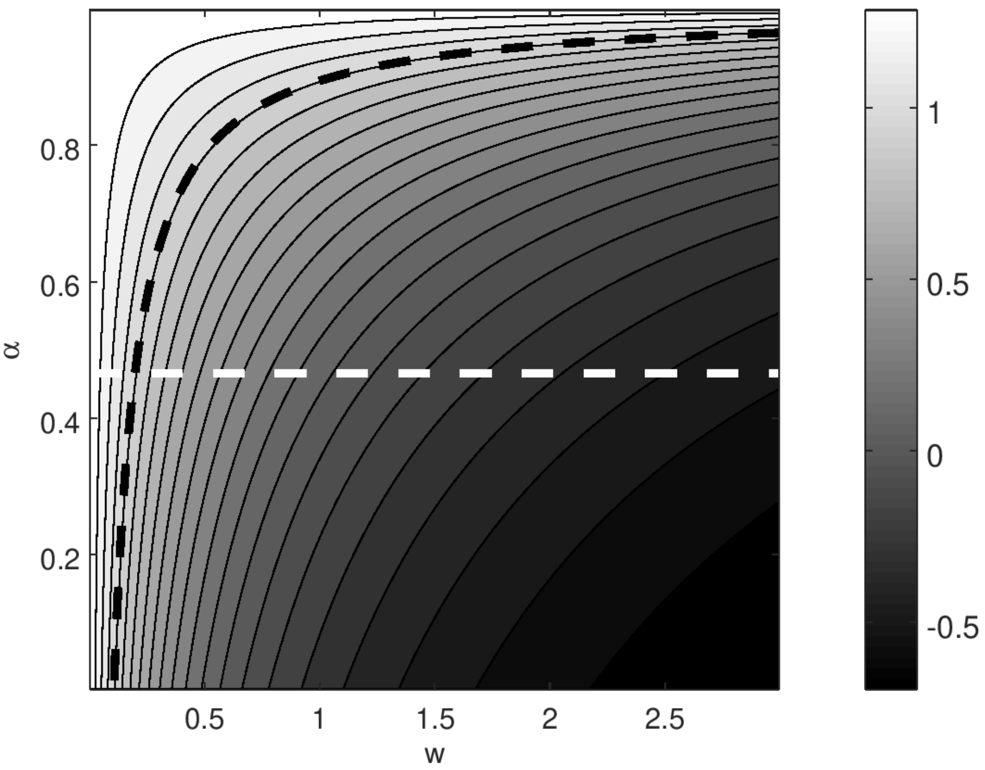

Figure 1: Values of $b(w)$ for different values of w and α assuming $\beta = 1$. The white dashed line represents the generalized Kelly rule, while the black dashed line represents $b(w) = \beta$.

When α and β are low, mv is favored and she dominates in the long-run. Hence, when fm is heavily investing in the risky asset, building a portfolio according to the mean-variance strategy turns out evolutionary fit and allows to make profits in the long-run. Notice that the point in which the boundary between D and E intersects $\beta = 0$ in Figure 2 is $d_f/(d_f + \bar{d}) = 0.3416$. This means that mv is able to dominate over sequences of events in which her wealth has been negative sometimes. Thus, mv is resorting on short-selling from time to time and this behavior may have evolutionary fitness depending on the environment. This complements the analysis of evolutionary finance models with short-selling in Belkov et al. (2017). Indeed, in our case selling short may result effective when driven from a mean-variance investing strategy and other traders in the market are heavily investing in risky assets. In terms of effective risk aversion, we can conclude that a mv investment strategy results evolutionary fit when other traders in the market show a relatively low level of risk aversion (conditional on their wealth share).

When mv presents a low level of risk aversion, fm can dominate investing a large share of her wealth in the riskless security. To provide an intuition about the mechanism underlying such long-run outcomes, consider the derivative of the riskless security's price with respect to α and conditional on a given wealth level. One obtains[4]

$$\left.\frac{\partial p(w)}{\partial \alpha}\right|_{w} = \frac{(1-p(w))p(w)^2 w\beta v}{(1-p(w))^2 d_f + p(w)^2(1-\alpha)w\beta v} > 0\,.$$

Thus, we can say that fm, investing more heavily on the riskless security, makes it relatively more expensive for mv. Hence, mv tends to invest more in the risky asset and this translates in an evolutionary advantage for fm. In terms of effective risk aversion, we have that high α induces relatively higher $b(w)$ levels (conditional on wealth share). Hence, conditionally larger values of effective risk aversion may produce a selection advantage for fm. This recalls the results in Bottazzi and Giachini (2019) and Bottazzi and Dindo (2013) about the higher evolutionary fitness of relatively more risk averse traders.

For intermediate values of mv risk aversion – for instance, β around 0.5 in our example – we shall observe survival of both agents if α is small and a form of path dependence if α is large. Indeed, when fm does not invest very much in the riskless security (i.e. conditionally low level of effective risk aversion) then $\tilde{w}$ is above one and, at the same time, the deterministic fixed point is unstable. Thus, whenever the two agents start to behave in a similar way, mv finds profitable to differentiate and her wealth share grows in expectation. However, mv is not able to dominate: when fm has almost nothing she is able to achieve a positive expected growth rate. When, instead, fm presents a large risk aversion $\tilde{w}$ is below one and stable. Hence, it is profitable for mv to copy fm whenever they end up investing in a similar way. The level of mv risk aversion discriminates between an extreme form of path dependency (i.e. either fm dominates and mv vanishes or vice-versa) or a milder version that favors fm (i.e. fm dominates and mv vanishes or both survives). It is interesting to notice that a more risk averse mv trader seems to decrease the chances she has to dominate. Notice also that no parameter combination allows for the form of path dependency implied by Proposition 3.7, the one which favors mv.

When mv presents an high level of risk aversion, then only one selection scenario is observed: the survival of both agents. Indeed, fm always survive because when $\beta \geq \bar{d}/v$ she achieves high growth rate in expectation whenever her wealth share is small. At the same time, also mv is able to survive. First of all, consider that for large β it is $w_0 < \tilde{w}$. Indeed, $\lim_{\beta \to +\infty} \tilde{w} = 1^-$ if $\alpha > d_f/(d_f + \bar{d})$ while

[4] See Fact A.4 in appendix A.

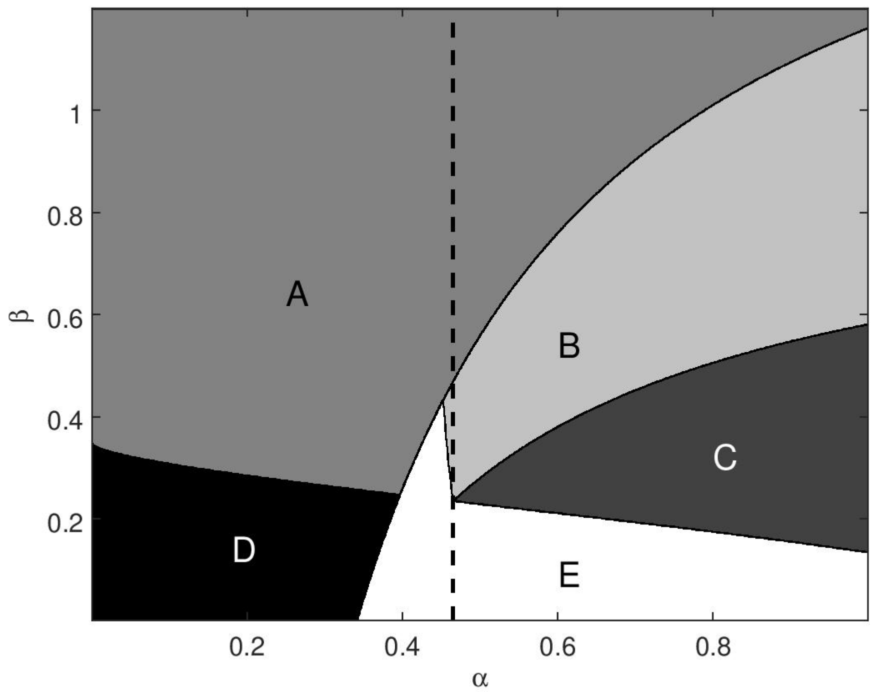

Figure 2: Asymptotic selection outcomes depending on α and β. A: both survive. B: path dependency, either fixed-mix dominates and mean-variance vanishes or both survive. C: path dependency, either fixed-mix dominates and mean-variance vanishes or vice-versa. D: mean-variance dominates and fixed-mix vanishes. E: fixed-mix dominates and mean-variance vanishes. The dashed line represents α^*.

$\lim_{\beta\to+\infty}\tilde{w} = 1^+$ if $\alpha < d_f/(d_f+\bar{d})$. Moreover, it is

$$\lim_{\beta\to+\infty}\mu_{\tilde{w}} = \log d_f - \log\alpha - \sum_{s=1}^{S}\pi_s \log(d_f + d(s))\,,$$

and, by Jensen's inequality, one has

$$\lim_{\beta\to+\infty}\mu_{\tilde{w}} > \log\frac{d_f}{d_f+\bar{d}} - \log\alpha\,.$$

Hence, if $\alpha \le d_f/(d_f+\bar{d})$ then $\lim_{\beta\to+\infty}\mu_{\tilde{w}} > 0$. This means that, for large values of β, if $\alpha > d_f/(d_f+\bar{d})$ then w_t is always lower than one and mv survives. If, instead, $\alpha \le d_f/(d_f+\bar{d})$ then $\tilde{w}$ is unstable and mv is, again, able to survive. Concerning the effective risk aversion of fm, high levels of mv's risk aversion makes the implied risk preferences of fm more extreme.

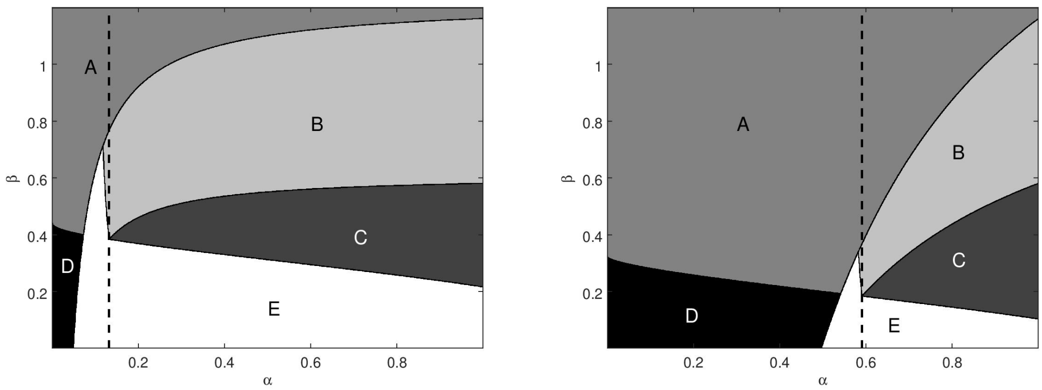

Figure 3: Asymptotic selection outcomes depending on α and β. Left: $d_f = 0.1$. Right: $d_f = 1.9$. A: both survive. B: path dependency, either fixed-mix dominates and mean-variance vanishes or both survive. C: path dependency, either fixed-mix dominates and mean-variance vanishes or vice-versa. D: mean-variance dominates and fixed-mix vanishes. E: fixed-mix dominates and mean-variance vanishes. The dashed line represents α^*.

Notice that, in this particular example, the generalized Kelly portfolio is not very far away from an equally balanced one. Indeed, splitting wealth (roughly) equally between the two assets prevents mv from dominating almost surely. This may provide an evolutionary rationale to the naive diversification strategy.

Next, we investigate the effects on long-run selection outcomes of the asset structure. In particular, we study how our selection is affected by changes in the riskless payoff and in the characteristics of the risky asset payoff distribution.

In Figure 3 we show the selection outcomes for $d_f = 0.1$ (left) and $d_f = 1.9$ (right). As one can notice, a riskless payoff significantly smaller than the expected payoff of the risky asset makes harder for mv to survive asymptotically and the combinations of parameter values that let mv dominate shrinks significantly. Conversely, lowering the riskless payoff makes the set of α and β that entail the dominance of fm expand, together with the areas of path dependence, especially the one that favors fm. In Figure 4, instead, we report the selection outcomes for different levels of average payoff $\bar{d}$ and payoff variance v. In particular, we set the values of λ and D in such a way to obtain a case in which the risky security is very appealing for mv (right) and a case in which it is less (left). As one can notice, The combinations of parameter values that let mv survive almost surely decrease when the risky asset presents a higher $\bar{d}/v$ ratio. However, the effect on the area in which mv dominates is ambiguous: it shrinks in the direction of larger α but squeezes in the direction of larger β. Concerning fm, when $\bar{d}/v$ increases the com-

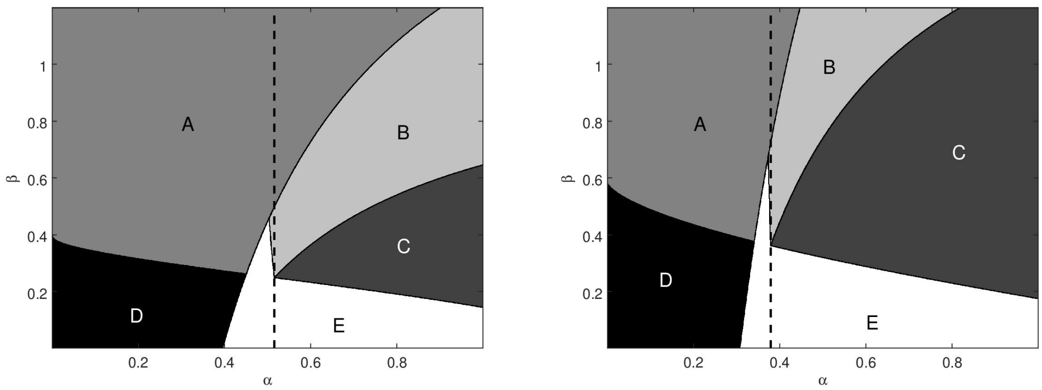

Figure 4: Asymptotic selection outcomes depending on α and β. Left: $D = 20$, $\lambda = 0.65$, such that $\bar{d} = 1.5284$ and $v = 2.3659$. Right: $D = 4.5$, $\lambda = 0$, such that $\bar{d} = 2.25$ and $v = 1.6909$. A: both survive. B: path dependency, either fixed-mix dominates and mean-variance vanishes or both survive. C: path dependency, either fixed-mix dominates and mean-variance vanishes or vice-versa. D: mean-variance dominates and fixed-mix vanishes. E: fixed-mix dominates and mean-variance vanishes. The dashed line represents α^*.

binations that let her dominate expand. At the same time, it is not clear whether her chances to survive may overall decrease, especially because of the expansion of the area in which the extreme form of path dependence occurs (C).

Thus, in general, it seems that when the risky security is very attractive for mv – either because d_f is too small or because $\bar{d}/v$ is high – then fm accrues an evolutionary advantage: the combinations of α and β that allow her to dominate increase. Another clear pattern that emerges is that survival of both agents is favored by having the risky security not very attractive for mv. Indeed, considering also that the dominance of one agent is observed only for relatively low level of mv risk aversion, it is possible to argue that when mv holds riskier portfolios then we shall observe one trader dominating the market in the long-run.

4 Conclusion

In this paper we investigate the evolutionary dynamics of CAPM investment rules in an incomplete short-lived asset market. We focus on the case in which one trader chooses her portfolio such as to maximize mean-variance preferences and we show that, if an opponent invests in each asset proportionally to expected relative payoffs, then the mean-variance investor cannot dominate in the long-run. Indeed, only two scenarios are possible: either the mean-variance trader vanishes or both agents survive with constant wealth shares. In both cases long-run prices are proportional to assets' expected relative payoff, hence CAPM cannot hold in the long-run. Such investment rule, that invalidates the asymptotic validity of CAPM, matches the generalized Kelly one, whose evolutionary properties have been analyzed by Evstigneev et al. (2002, 2009, 2016) focusing on different ecologies of rules. When a mean-variance trader faces a generic fixed-mix investor, we propose sufficient conditions to assess long-run selection outcomes that rely only on exogenous parameters of the economy. Our analysis shows that many long-run scenarios are possible. Indeed, as well as the dominance of one of the two traders, one may observe the survival of both agents and path dependent cases in which it is the sequence of realized events to determine the asymptotic selection outcome. Finally, the different outcomes can be discussed in terms of the *effective risk aversion* of the trading strategies: the risk aversion implied by their portfolio choices conditional on prevailing market prices. In general, a larger effective risk aversion constitutes a survival advantage.

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A Static Equilibrium – Proof of Proposition 2.1

Here we prove that the fm wealth share is positive in every t and unique and positive market clearing prices exist. Moreover, we provide a closed-form formula for p_t and show that it negatively depends on w_{t-1}.

Notice that, under normalization, we can focus without loss of generality on one between the two market clearing conditions. Consider, for instance, eq. (7). Defining the function

$$f(p,w)=\frac{(1-p)\left(p(\beta v-\bar{d})+(1-p)d_f\right)}{(1-\alpha)w\beta v} \tag{17}$$

and assuming $0<p_t<1$, eq. (7) is equivalent to $p_t=f(p_t,w_{t-1})$. Then, the following holds

Lemma A.1. *If $w>0$ there is one and only one $p^*\in(0,1)$ such that $f(p^*,w)=p^*$.*

Proof. For the existence of p^*, notice that for $w>0$ it is $f(1,w)=0$ and $f(0,w)=d_f/((1-\alpha)\beta wv)>0$, hence, because of the continuity of $f(p,w)$, a solution in $(0,1)$ exists. For uniqueness, it is enough consider that $\partial^2 f(p,w)/\partial p^2$ does not depend on p. □

Hence, the first and second points of Proposition 2.1 follows from a straightforward application of Lemma A.1 combined with eq. (5) and the fact that eq. (7) is equivalent to $p_t=f(p_t,w_{t-1})$. Indeed, provided a positive initial wealth share, by induction we have $w_t>0$ and $0<p_t<1$ in any finite t.

From the relation $p_t=f(p_t,w_{t-1})$ it is possible to get a closed-form formula for relative prices. To do that, one has to rearrange terms in order to obtain a quadratic equation. Then, solving and noticing that only one root belongs to $(0,1)$, one obtains $p_t=p(w_{t-1})$ with $p(w)$ as in (9).

We prove three facts about equilibrium pricing which directly deliver the last four point of the Proposition. First, we observe that the normalized price $p(w)$ is negatively related to the wealth share w. Second, we show that if the wealth share w goes to $+\infty$ then the normalized price $p(w)$ goes to 0. Third, we show that if the wealth share w goes to 0 then the normalized price $p(w)$ goes to 1 if $\beta\geq\bar{d}/v$ while it goes to $d_f/(d_f+\bar{d}-\beta v)$ if $\beta<\bar{d}/v$.

Fact A.1. $\dfrac{\partial p(w)}{\partial w}=-\dfrac{(1-p(w))p(w)^2(1-\alpha)\beta v}{(1-p(w))^2 d_f+p(w)^2(1-\alpha)w\beta v}<0.$

Proof. Consider $p(w)=f(p(w),w)$ and differentiate both sides by w. Rearranging terms and substituting one gets

$$\frac{\partial p(w)}{\partial w}\left(1+\frac{p(w)(\beta v-\bar{d})+(1-p(w))d_f}{(1-\alpha)w\beta v}-\frac{(1-p(w))(\beta v-\bar{d}-d_f)}{(1-\alpha)w\beta v}\right)=-\frac{p(w)}{w}\,.$$

Exploiting $p(w)=f(p(w),w)$, the first fraction in parenthesis is equivalent to $p(w)/(1-p(w))$ while the second is equivalent to $1-d_f(1-p(w))/(p(w)(1-\alpha)w\beta v)$. Substituting and rearranging terms, the formula in the statement follows. The sign is a straightforward consequence of the results in Proposition 2.1. □

Fact A.2. $\lim_{w\to+\infty} p(w) = 0$.

Proof. Consider $p(w) = f(p(w), w)$ and notice that $f(0, w) = d_f/((1-\alpha)\beta w v)$. The statement simply follows from $\lim_{w\to+\infty} f(0, w) = 0$. □

Fact A.3. *If* $\beta < \bar{d}/v$ *then* $\lim_{w\to 0} p(w) = d_f/(d_f + \bar{d} + \beta v)$. *If* $\beta \geq \bar{d}/v$ *then* $\lim_{w\to 0} p(w) = 1$.

Proof. The statements follow directly from the expression in (9) taking the limit for $w \to 0$ and noticing that the condition on β sets the sign of the second fraction. □

Finally, we prove a fourth fact about pricing that shall be used to provide an intuition about the relation between the portfolio choices of fm and the price of the riskless security.

Fact A.4. $\left.\dfrac{\partial p(w)}{\partial \alpha}\right|_w = \dfrac{(1-p(w))p(w)^2 w\beta v}{(1-p(w))^2 d_f + p(w)^2(1-\alpha)w\beta v} > 0$

Proof. Along the same lines of the proof of Fact A.1, consider $p(w) = f(p(w), w)$ and differentiate both side with respect to α. Exploiting the same equivalences and arguments used in the proof of of Fact A.1, the statement follows. □

B Proof of Proposition 2.2

For the first statement, consider the function $f(p, w)$ defined in (17) and notice that $f(\alpha, \tilde{w}) = \alpha$. By lemma A.1 it is $p_{t+1} = \alpha$ if and only if $w_t = \tilde{w}$. Thus, if $w_t = \tilde{w}$ then $p_{t+1} = \alpha$ and this implies from equation (5) that $w_{t+1} = \tilde{w}$. By induction the statement follows.

For the second statement assume $w_{t-1} < \tilde{w}$. Suppose, by contradiction, that $w_t \geq \tilde{w}$. The inequality is equivalent to

$$\frac{d_f \alpha}{p_t(d_f + d_t)} + \frac{d_t(1-\alpha)}{(1-p_t)(d_f + d_t)} \geq \frac{\tilde{w}}{w_{t-1}} .$$

It follows from the previous results that a necessary condition for having $w_t \geq \tilde{w}$ when $w_{t-1} < \tilde{w}$ is

$$\frac{1-\alpha}{1-p_t} \geq \frac{\tilde{w}}{w_{t-1}} .$$

Substituting from $p_t = f(p_t, w_{t-1})$ and the definition of $\tilde{w}$ one gets

$$\frac{(1-\alpha)(p_t(\beta v - \bar{d}) + (1-p_t)d_f)}{(1-\alpha)w_{t-1}\beta v p_t} \geq \frac{\alpha(\beta v - \bar{d}) + (1-\alpha)d_f}{\alpha \beta v w_{t-1}} ,$$

which is equivalent to $\alpha \geq p_t$. This implies $w_{t-1} \geq \tilde{w}$, a contradiction. By induction the statement follow. A symmetric argument holds for the third statement.

C Auxiliary processes

Here we define and study some stochastic processes that shall be used to prove asymptotic selection results. All of them follow from a transformation of $\{w_t\}$ obtained using the function

$$z(\zeta_1, \zeta_2, \zeta_3) = \zeta_1 \log \frac{\zeta_2}{\zeta_3 - \zeta_2} . \tag{18}$$

The first process we consider is $\{z_t\}$ with $z_t = z(1, w_t, \tilde{w})$. If $w_0 < \tilde{w}$ we have by Proposition 2.2 that $w_t \in (0, \tilde{w})$ $\forall t$ and this immediately implies $z_t \in (-\infty, +\infty)$, with $z_t \to -\infty$ if and only if $w_t \to 0$ and $z_t \to +\infty$ if and only inf $w_t \to \tilde{w}$. Define

$$g(s, w) = \frac{d_f \alpha}{(d_f + d(s))p(w)} + \frac{d(s)(1-\alpha)}{(d_f + d(s))(1 - p(w))} , \tag{19}$$

such that the increment of the process can be written as

$$z_t - z_{t-1} = \log \frac{g(s_t, w_{t-1})(\tilde{w} - w_{t-1})}{\tilde{w} - g(s_t, w_{t-1})w_{t-1}} .$$

If $\beta < \bar{d}/v$ holds, then $\{z_t\}$ has bounded increments with both finite positive and finite negative increments.[5] The process' conditional drift reads

$$\mathrm{E}[z_t - z_{t-1} | z_{t-1} = \log(\tilde{w}/(\tilde{w} - w))] = \sum_{s=1}^{S} \pi_s \log \frac{g(s, w)(\tilde{w} - w)}{\tilde{w} - g(s, w)w}$$

and, computing the limits, one obtains

$$\lim_{z \to +\infty} \mathrm{E}[z_t - z_{t-1} | z_{t-1} = z] = -\mu_{\tilde{w}} ,$$
$$\lim_{z \to -\infty} \mathrm{E}[z_t - z_{t-1} | z_{t-1} = z] = \mu_0 ,$$

with μ_0 and $\mu_{\tilde{w}}$ as in equations (14) and (15). If $\mu_{\tilde{w}} > 0$ and $\mu_0 > 0$ then, by continuity, Theorem 3.2 of Bottazzi and Dindo (2015) applies and $\{z_t\}$ is persistent as in Definition 2.1 of Bottazzi and Dindo (2015). Notice that

$$\lim_{z \to \pm\infty} \mathrm{E}\left[(z_t - z_{t-1})^2 \middle| z_{t-1} = z \right] < \infty ,$$

thus, if $\mu_{\tilde{w}} < 0$ and $\mu_0 > 0$ Theorem 4.1 of Bottazzi and Dindo (2015) applies and $\lim_{t \to \infty} z_t = +\infty$ with full probability. If, instead, $\mu_{\tilde{w}} > 0$ and $\mu_0 < 0$ Corollary 4.1

[5] See Bottazzi and Dindo (2015) for the formal definitions of bounded increments (Def. 2.2), finite positive increments and finite negative increments (Def. 4.1).

of Bottazzi and Dindo (2015) applies and $\lim_{t\to\infty} z_t = -\infty$ almost surely. Finally, if $\mu_{\tilde{w}} < 0$ and $\mu_0 < 0$ Theorem 4.2 of Bottazzi and Dindo (2015) applies and, with probability one, either $\lim_{t\to\infty} z(1, w_t, \tilde{w}) = +\infty$ or $\lim_{t\to\infty} z_t = -\infty$.

The next process we consider still applies to the case in which $w_t \in (0, \tilde{w})\ \forall t$, but it is tailored to maintain bounded increments when $\beta \geq \bar{d}/v$. Indeed, in such a case one has

$$\lim_{w\to 0} \log \frac{g(s,w)(\tilde{w}-w)}{\tilde{w}-g(s,w)w} = +\infty \quad \forall s \in \{2,\dots,S\}\,.$$

Hence, for any state different from 1, increments become unbounded as w_t approaches zero. Thus, we define

$$x_t = \begin{cases} z(1, w_t, \tilde{w}) & \text{if } w_{t-1} \in [\varepsilon, \tilde{w}]\,, \\[2ex] x_{t-1} + \log \dfrac{g(s_t,\varepsilon)(\tilde{w}-\varepsilon)}{\tilde{w}-g(s_t,\varepsilon)\varepsilon} & \text{if } w_{t-1} \in [0,\varepsilon) \text{ and } s_t \neq 1\,, \\[2ex] x_{t-1} + \log \dfrac{\alpha(\tilde{w}-\varepsilon)}{\tilde{w}-\alpha\varepsilon} & \text{if } w_{t-1} \in [0,\varepsilon) \text{ and } s_t = 1\,, \end{cases} \qquad t \in \mathbb{N}\,,$$

where ε is strictly positive and small enough such that

$$\begin{aligned} &\pi_2 \log \frac{g(2,\varepsilon)(\tilde{w}-\varepsilon)}{\tilde{w}-g(2,\varepsilon)\varepsilon} > -\pi_1 \log \frac{\alpha(\tilde{w}-\varepsilon)}{\tilde{w}-\alpha\varepsilon} \quad \text{and} \\ &\log \frac{g(s,w)(\tilde{w}-w)}{\tilde{w}-g(s,w)w} > \log \frac{g(s,\varepsilon)(\tilde{w}-\varepsilon)}{\tilde{w}-g(s,\varepsilon)\varepsilon} \quad \forall w \in (0,\varepsilon),\ \forall s \in \{2,\dots,S\}\,. \end{aligned}$$

Notice that a number ε that respects such requirements exists because the increments for states different from 1 become unbounded for asymptotically zero wealth and

$$\lim_{w\to 0} \frac{\partial}{\partial w} \log \frac{g(s,w)(\tilde{w}-w)}{\tilde{w}-g(s,w)w} = -\infty \quad \forall s \in \{2,\dots,S\}\,.$$

By definition we have $x_t \leq z(1, w_t, \tilde{w})\ \forall t \in \mathbb{N}$. The equality trivially holds for any $w_{t-1} \in [\varepsilon, \tilde{w}]$ while the strict inequality holds for any $w_{t-1} \in [0,\varepsilon)$ with $s_t \neq 1$ as a result of the particular choice of ε. For the case $w_{t-1} \in [0,\varepsilon)$ and $s_t = 1$, notice that $g(1,w) \geq \alpha$ and

$$\frac{\partial}{\partial w} \log \frac{\alpha(\tilde{w}-w)}{\tilde{w}-\alpha w} < 0\,,$$

thus, $x_t \leq z(1, w_t, \tilde{w})$ is also ensured for $w_{t-1} \in [0,\varepsilon)$ and $s_t = 1$. Hence, it immediately follows that if $\limsup_{t\to\infty} x_t > -\infty$ then $\limsup_{t\to\infty} z(1, w_t, \tilde{w}) >$

$-\infty$ and $\limsup_{t\to\infty} w_t > 0$. Since

$$\lim_{w_{t-1}\to 0} w_t = \frac{d(s_t)(\beta v - \bar{d})}{(d_f + d(s_t))\beta v} < \tilde{w}\,,$$

it is $x_t \to +\infty$ if and only if $z(1, w_t, \tilde{w}) \to +\infty$ and $w_t \to \tilde{w}$. The process $\{x_t\}$ has bounded increments, with both positive and negative finite increments. Computing the asymptotic conditional drifts, one has

$$\lim_{x\to+\infty} \mathrm{E}[x_t - x_{t-1}|x_{t-1} = x] = -\mu_{\tilde{w}}\,,$$
$$\lim_{x\to-\infty} \mathrm{E}[x_t - x_{t-1}|x_{t-1} = x] = \pi_1 \log\frac{\alpha(\tilde{w}-\varepsilon)}{\tilde{w}-\alpha\varepsilon} + \sum_{s=2}^{S} \pi_s \log\frac{g(s,\varepsilon)(\tilde{w}-\varepsilon)}{\tilde{w}-g(s,\varepsilon)\varepsilon} > 0\,.$$

Thus, if $\mu_{\tilde{w}} > 0$, Theorem 3.2 of Bottazzi and Dindo (2015) applies and $\{x_t\}$ is persistent as in Definition 2.1 of Bottazzi and Dindo (2015). One trivially has $\lim_{x\to\pm\infty} \mathrm{E}\left[(x_t - x_{t-1})^2 \,\middle|\, x_{t-1} = x\right] < \infty$, hence, if $\mu_{\tilde{w}} < 0$ Theorem 4.1 of Bottazzi and Dindo (2015) applies and $\lim_{t\to\infty} x_t = +\infty$ with full probability.

Next, we define a stochastic process suited for analyzing the cases in which $w_t \in [\tilde{w}, +\infty)$. Our proposal derives from $z(-1, \tilde{w}, w_t)$ and it is adapted in order to maintain bounded increments. Indeed, the increment reads

$$z(-1, \tilde{w}, w_t) - (-1, \tilde{w}, w_{t-1}) = \log\frac{g(s_t, w_{t-1})w_{t-1} - \tilde{w}}{w_{t-1} - \tilde{w}}$$

and, since $\lim_{w\to+\infty} g(s, w) = +\infty\ \forall s \in \mathcal{S}$, it is

$$\lim_{w\to+\infty} \log\frac{g(s,w)w - \tilde{w}}{w - \tilde{w}} = +\infty \quad \forall s \in \mathcal{S}\,.$$

Notice that, given Facts A.1 and A.2, there exists a $k > 0$ and large enough such that $g(s, k) > 1\ \forall s \in \mathcal{S}$ and $g(s, w) \geq g(s, k)\ \forall w \in (k, +\infty)$ and $\forall s \in \mathcal{S}$. Hence, we have

$$\log\frac{g(s,w)w - \tilde{w}}{w - \tilde{w}} \geq \log\frac{g(s,k)w - \tilde{w}}{w - \tilde{w}} \geq \log g(s,k)$$

and the process

$$u_t = \begin{cases} z(-1, \tilde{w}, w_t) & \text{if } w_{t-1} \in [\tilde{w}, k] \\ u_{t-1} + \log g(s_t, k) & \text{if } w_{t-1} \in (k, +\infty)\,, \end{cases} \qquad t \in \mathbb{N}\,,$$

is such that $u_t \leq z(-1, \tilde{w}, w_t)\ \forall t \in \mathbb{N}$. Thus, if $u_t \to +\infty$ then $z(-1, \tilde{w}, w_t) \to +\infty$ and $w_t \to +\infty$. Moreover, $u_t \to -\infty$ if and only if $z(-1, \tilde{w}, w_t) \to -\infty$ and

$w_t \to \tilde{w}$. The process $\{u_t\}$ has bounded increments with finite positive increments. Computing the asymptotic conditional drifts, one has

$$\lim_{u\to+\infty} \mathrm{E}[u_t - u_{t-1}|u_{t-1} = u] = \sum_{s=1}^{S} \log g(s,k) > 0\,,$$
$$\lim_{u\to-\infty} \mathrm{E}[u_t - u_{t-1}|u_{t-1} = u] = \mu_{\tilde{w}}\,.$$

Thus, since one has $\lim_{u\to\pm\infty} \mathrm{E}\left[(u_t - u_{t-1})^2 \,\middle|u_{t-1} = u\right] < \infty$, if $\mu_{\tilde{w}} > 0$ Theorem 4.1 of Bottazzi and Dindo (2015) applies and $\lim_{t\to\infty} u_t = +\infty$ with full probability. If, instead, $\mu_{\tilde{w}} < 0$ Theorem 4.2 of Bottazzi and Dindo (2015) can be adapted and $\mathrm{P}\{\lim_{t\to\infty} u_t = -\infty\} + \mathrm{P}\{\lim_{t\to\infty} u_t = +\infty\} = 1$.

The last stochastic process we consider is tailored to study the cases in which $\tilde{w} < 0$. Such condition implies $w_t \in [0, +\infty)\ \forall t$, hence we adapt $z(1, w_t, 1 + w_t) = \log w_t$ in order to maintain bounded increments for $w_t \to +\infty$. Following the reasoning lines proposed in advance, we consider

$$y_t = \begin{cases} z(1, w_t, 1 + w_t) & \text{if } w_{t-1} \in [0, k]\,, \\ y_{t-1} + \log g(s, k) & \text{if } w_{t-1} \in (k, +\infty)\,, \end{cases} \qquad t \in \mathbb{N}\,,$$

such that $y_t \le z(1, w_t, 1 + w_t)\ \forall t \in \mathbb{N}$. By definition, we have $w_t \to +\infty$ only if $y_t \to +\infty$ and $w_t \to 0$ if and only if $y_t \to -\infty$.

The process $\{y_t\}$ has bounded increments with finite positive increments and, computing the asymptotic conditional drifts, one has

$$\lim_{y\to+\infty} \mathrm{E}[y_t - y_{t-1}|y_{t-1} = y] = \sum_{s=1}^{S} \log g(s,k) > 0\,,$$
$$\lim_{y\to-\infty} \mathrm{E}[y_t - y_{t-1}|y_{t-1} = y] = \mu_0\,.$$

Since it is $\lim_{y\to\pm\infty} \mathrm{E}\left[(y_t - y_{t-1})^2 \,\middle|y_{t-1} = y\right] < \infty$, if $\mu_0 > 0$ then Theorem 4.1 of Bottazzi and Dindo (2015) applies and $\lim_{t\to\infty} y_t = +\infty$ with full probability. If, instead, $\mu_0 < 0$ then Theorem 4.2 of Bottazzi and Dindo (2015) can be adapted and $\mathrm{P}\{\lim_{t\to\infty} y_t = -\infty\} + \mathrm{P}\{\lim_{t\to\infty} y_t = +\infty\} = 1$.

D Proof of Proposition 3.1

For the statement in *1.* notice that $\beta \le \bar{d}/v - (1 - \alpha^*)d_f/(\alpha^* v)$ directly implies $\tilde{w} \le 0$. Since by Jensen's inequality one has $\alpha^* > d_f/(d_f + \bar{d})$, the set of cases is not empty: $\bar{d}/v - (1-\alpha^*)d_f/(\alpha^* v) > 0$. Consider the process $\{\log w_t\}$ and notice that is a semimartingale. Indeed, one has

$$\mathrm{E}[\log w_t - \log w_{t-1} | w_{t-1} = w] = \sum_{s=1}^{S} \pi_s \log \left(\frac{d_f \alpha^*}{(d_f + d(s))p(w)} + \frac{d(s)(1-\alpha^*)}{(d_f + d(s))(1-p(w))} \right)$$

and, by means of Jensen's inequality, the definition of α^*, and the properties of the relative entropy, it is

$$\mathrm{E}[\log w_t - \log w_{t-1} | w_{t-1} = w] \geq \alpha^* \log \frac{\alpha^*}{p(w)} + (1-\alpha^*) \log \frac{1-\alpha^*}{1-p(w)} > 0\,.$$

Fact A.2 implies that there exists a $k > 1$ such that

$$\mathrm{Prob}\{w_t > w_{t-1} | w_{t-1} > k\} = 1\,.$$

Hence, consider the process

$$l_t = \begin{cases} k & \text{if } w_t \geq k \\ \log w_t & \text{otherwise}\,. \end{cases}$$

$\{l_t\}$ is a semimartingale bounded from above, thus the semimartingale convergence theorem (Lamperti, 1960) implies that $\lim_{t\to\infty} l_t = k$ almost surely and we have $\liminf_{t\to\infty} w_t > k > 1$ almost surely. Moreover, given the fact that if $w_{t-1} > k$ then $w_t > w_{t-1}$ with probability 1, it is $\lim_{t\to\infty} w_t = +\infty$. The statement directly follows from Definition 2.1.
For the statement in *2.*, notice that the condition on β implies $\tilde{w} \geq w_0$. Hence, for $w_0 < \tilde{w}$, $\{\log w_t\}$ is a semimartingale bounded from above by $\tilde{w}$. The semimartingale convergence theorem (Lamperti, 1960) directly implies $\lim_{t\to\infty} \log w_t = \log \tilde{w}$, hence $\lim_{t\to\infty} w_t = \tilde{w}$. The statement follows from Definition 2.1 noticing that $\alpha^* > d_f/(d_f + \bar{d})$ implies $\tilde{w} < 1$. For the case $w_0 = \tilde{w}$ it is enough to invoke Proposition 2.2.
Concerning *3.*, consider the process $\{u_t\}$ defined in appendix C and notice that the condition on β implies $0 < \tilde{w} < w_0$. Thus, $w_t > \tilde{w}$ $\forall t$ and $u_t \in (-\infty, +\infty)$ $\forall t$. Exploiting de l'Hôpital Theorem and Jensen's inequality, one has

$$\mu_{\tilde{w}} = \sum_{s=1}^{S} \pi_s \log \left(1 + \left(\frac{d_f}{d_f + d(s)} - \alpha^* \right) \frac{d_f - \alpha^*(d_f + \bar{d} - \beta v)}{d_f - (\alpha^*)^2 (d_f + \bar{d} - \beta v)} \right) < 0\,.$$

Hence, $\{u_t\}$ either diverges toward $-\infty$ or toward $+\infty$. Depending on the sequence of realized events, we have either $w_t \to \tilde{w}$ or $w_t \to +\infty$ with probability one. The statement follows from Definition 2.1 defining $\Sigma' \subset \Sigma$ as the set of sequences in which $w_t \to +\infty$ and $\Sigma'' \subset \Sigma$ as the set of sequences in which $w_t \to \tilde{w}$. Notice also that $\alpha^* > d_f/(d_f + \bar{d})$ implies that the set $\{w_0 \in (0,1) | w_0 > \tilde{w}\}$ is not empty.

E Proof of Corollary 3.1

If *1.* of Proposition 3.1 occurs, then, from the proof of the Proposition in appendix D, it is $\lim_{t\to\infty} w_t = +\infty$ almost surely. Fact A.2 implies $\lim_{t\to\infty} p_t = 0$. If *2.* of Proposition 3.1 occurs, then it is $\lim_{t\to\infty} w_t = \tilde{w}$ almost surely and Proposition 2.2 implies $\lim_{t\to\infty} p_t = \alpha^*$. Moreover, $\alpha^* < d_f/(d_f + \bar{d})$ implies $\tilde{w} < 1$. If *3.* of Proposition 3.1 occurs, with probability one either $\lim_{t\to\infty} w_t = +\infty$ and $\lim_{t\to\infty} p_t = 0$ or $\lim_{t\to\infty} w_t = \tilde{w}$ and $\lim_{t\to\infty} p_t = \alpha$ almost surely. Since no other case is possible, the statement follows.

F Proof of Proposition 3.2

Consider the conditions in *1.*, $\beta \leq \bar{d}/v - d_f(1-\alpha)/(\alpha v)$, implies $\tilde{w} \leq 0$ and $\beta < \bar{d}/v$. $\alpha > d_f/(d_f + \bar{d})$, instead, lets $\bar{d}/v - d_f(1-\alpha)/(\alpha v) > 0$, such that the set of cases considered is not empty. Hence, the process $\{y_t\}$ in appendix C can be used to understand long-run outcomes. Since $\mu_0 > 0$, we have $\lim_{t\to\infty} y_t = +\infty$ almost surely, which implies $\lim_{t\to\infty} w_t = +\infty$ with probability one and the statement follows.

Concerning *2.*, notice that the condition on β implies $0 < \tilde{w} < w_0$. Thus, the process $\{u_t\}$ in appendix C can be used to assess long-run outcomes. In particular, the condition $\mu_{\tilde{w}} > 0$ ensures that $\lim_{t\to\infty} u_t = +\infty$ almost surely. Thus, $w_t \to +\infty$ with probability one and the statement follows.

For *3.*, the condition on α implies $\tilde{w} \geq 1$ while the condition on β ensures $w_0 < \tilde{w}$. Then, if $\beta < \bar{d}/v$ the process $\{z_t\}$ is suited to study long-run outcomes. The conditions $\mu_0 > 0$ and $\mu_{\tilde{w}} < 0$ imply $\lim_{t\to\infty} z_t = +\infty$ with probability one, hence $\lim_{t\to\infty} w_t = \tilde{w} \geq 1$ almost surely and the statement follows. If, instead, it is $\beta \geq \bar{d}/v$, then we have to use the process $\{x_t\}$ in appendix C. The condition $\mu_{\tilde{w}} < 0$ is sufficient to ensure $\lim_{t\to\infty} x_t = +\infty$ almost surely, which implies $\lim_{t\to\infty} w_t = \tilde{w} \geq 1$ with full probability and the statement is proven.

G Proof of Proposition 3.3

First of all, notice that the condition on α ensures

$$\frac{\bar{d}}{(1-w_0)v} - \frac{(1-\alpha)d_f}{(1-w_0)\alpha v} < \frac{\bar{d}}{v},$$

hence the condition on β does not generate an empty set. The lower bound for β ensures $w_0 < \tilde{w}$. The upper bound on β allows us to use the process $\{z_t\}$ in appendix C to study long-run outcomes. Since it is $\mu_0 < 0$ and $\mu_{\tilde{w}} > 0$, we have

$\lim_{t\to\infty} z_t = -\infty$ with full probability. This implies $\lim_{t\to\infty} w_t = 0$ almost surely and the statement is proved.

H Proof of Proposition 3.4

Consider the conditions for the survival of fm. The condition on β in *1.* implies $\tilde{w} < 0$. Thus, consider the process $\{y_t\}$ in appendix C and notice that the condition $\mu_0 > 0$ in *1.* directly implies $\limsup_{t\to\infty} y_t > -\infty$ by means of Theorem 3.1 of Bottazzi and Dindo (2015). Hence, $\limsup_{t\to\infty} w_t > 0$ and fm survives as in Definition 2.1. Concerning *2.*, the conditions on β imply $0 < \tilde{w} < w_0$, thus $\limsup_{t\to\infty} w_t > 0$ as a consequence of Proposition 2.2 and fm survives as in Definition 2.1. For *3.*, notice that the condition on β implies $w_0 < \tilde{w}$. If $\beta < \bar{d}/v$ we can use the process $\{z_t\}$ in appendix C. The condition $\mu_0 > 0$, by means of Theorem 3.1 of Bottazzi and Dindo (2015), entails $\limsup_{t\to\infty} z_t > -\infty$. This implies $\limsup_{t\to\infty} w_t > 0$ and fm survives according to Definition 2.1. If $\beta \geq \bar{d}/v$, instead, we can use the process $\{x_t\}$ in appendix C. Since $\lim_{x\to-\infty} \mathrm{E}[x_t - x_{t-1}|x_{t-1} = x] > 0$, Theorem 3.1 of Bottazzi and Dindo (2015) delivers $\limsup_{t\to\infty} x_t > -\infty$, which implies $\limsup_{t\to\infty} w_t > 0$. Thus, fm survives according to Definition 2.1.

Consider the conditions in *1.* If $\beta = \bar{d}/((1-w_0)v) - (1-\alpha)d_f/((1-w_0)\alpha v)$ and $\alpha > d_f/(d_f + \bar{d})$ then $w_0 = \tilde{w} < 1$ and both agents survive as a consequence of Proposition 2.2. If instead the condition on β is satisfied with strict inequality, then $0 < w_0 < \tilde{w} < 1$, where the last inequality follows from the condition on α. Thus, it is $\liminf_{t\to\infty} w_t < 1$ on every possible sequence because of Proposition 2.2 and mv survives as in Definition 2.1. Consider, instead, the conditions in *2.*. The condition on α implies $\tilde{w} \geq 1$ while the condition on β ensures $w_0 < \tilde{w}$. Thus, if $\beta < \bar{d}/v$, we can consider the process $\{z_t\}$ in appendix C. The fact that it has finite negative increments directly implies $\liminf_{t\to\infty} z_t = -\infty$ almost surely. It follows that $\liminf_{t\to\infty} w_t < \tilde{w} < 1$ with probability one and mv survives according to Definition 2.1. If, instead, $\beta \geq \bar{d}/v$, we can use the process $\{x_t\}$ in appendix C. Also in this case the process has finite negative increments, hence $\liminf_{t\to\infty} x_t = -\infty$. Thus, $\liminf_{t\to\infty} w_t < \tilde{w} < 1$ and mv survives as in Definition 2.1.

I Proof of Proposition 3.5

Since the condition on β in *1.* ensures $\tilde{w} < 0$, we can use the process $\{y_t\}$ in appendix C to study long-run outcomes. The condition $\mu_0 < 0$ implies $\mathrm{P}\{\lim_{t\to\infty} y_t = -\infty\} + \mathrm{P}\{\lim_{t\to\infty} y_t = +\infty\} = 1$, thus $\mathrm{Prob}\{w_t \to +\infty\} + \mathrm{Prob}\{w_t \to 0\} = 1$. The statement follows calling $\sigma' \in \Sigma'$ the sequences where $w_t \to +\infty$ and $\sigma'' \in \Sigma''$ the sequences where $w_t \to 0$.

For *2.*, consider that the condition on α ensures $\tilde{w} \geq 1$. Hence, $w_t \in [0, \tilde{w}]$ $\forall t \in \mathbb{N}$ and, since $\beta < \bar{d}/v$, we can use the process $\{z_t\}$ in appendix C to study long-run outcomes. The conditions $\mu_0 < 0$ and $\mu_{\tilde{w}} < 0$ imply $\mathrm{P}\{\lim_{t\to\infty} z_t = -\infty\} + \mathrm{P}\{\lim_{t\to\infty} z_t = +\infty\} = 1$, thus $\mathrm{Prob}\{w_t \to \tilde{w}\} + \mathrm{Prob}\{w_t \to 0\} = 1$. The statement follows calling $\sigma' \in \Sigma'$ the sequences where $w_t \to \tilde{w}$ and $\sigma'' \in \Sigma''$ the sequences where $w_t \to 0$.

J Proof of Proposition 3.6

Notice that the condition on α ensures $\tilde{w} < 1$, while the condition on β ensures $w_0 > \tilde{w}$. Then, the process $\{u_t\}$ in appendix C can be used to study long-run outcomes. The condition $\mu_{\tilde{w}} < 0$ implies $\mathrm{P}\{\lim_{t\to\infty} u_t = -\infty\} + \mathrm{P}\{\lim_{t\to\infty} u_t = +\infty\} = 1$. Thus, $\mathrm{Prob}\{w_t \to +\infty\} + \mathrm{Prob}\{w_t \to \tilde{w}\} = 1$ and the statement follows calling $\sigma' \in \Sigma'$ the sequences where $w_t \to +\infty$ and $\sigma'' \in \Sigma''$ the sequences where $w_t \to \tilde{w}$.

K Proof of Proposition 3.7

Notice that the condition on α ensures $\tilde{w} < 1$, while the lower bound on β entails $w_0 < \tilde{w}$. The upper bound on β allows us to use the process $\{z_t\}$ in appendix C to study long-run outcomes. Then, the conditions $\mu_0 < 0$ and $\mu_{\tilde{w}} < 0$ imply $\mathrm{P}\{\lim_{t\to\infty} z_t = -\infty\} + \mathrm{P}\{\lim_{t\to\infty} z_t = +\infty\} = 1$. This, $\mathrm{Prob}\{w_t \to 0\} + \mathrm{Prob}\{w_t \to \tilde{w}\} = 1$. The statement follows calling $\sigma' \in \Sigma'$ the sequences where $w_t \to 0$ and $\sigma'' \in \Sigma''$ the sequences where $w_t \to \tilde{w}$.