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A Numerical Revolution: The diffusion of practical mathematics and the growth of pre-modern European economies

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A Numerical Revolution: The diffusion of practical mathematics and the growth of pre-modern European economies.

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Abstract

The accumulation of knowledge and its application to a variety of human needs is a discontinuous process that involves innovation and change. While much has been written on major discontinuities associated, for instance, with the rise of new technologies during industrial revolutions, other phases of economic development are less well understood, even though they might bring into even sharper focus the mechanisms through which growth is generated by the systematic application of human knowledge to practical problems. In this paper, we investigate the transmission of new mathematical knowledge from the 13th to the end of the 16th century in Europe. Using an original dataset of over 1050 manuals of practical arithmetic, we produce new descriptive and quasi-experimental evidence on the economic importance of the European transition from Roman to Hindu-Arabic numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). This numerical revolution laid the foundations for the commercial revolution of the 13th century, and the diffusion of knowledge through organised learning had positive and significant effects on the growth of pre-modern European economies.

Keywords: Human capital; knowledge diffusion; learning; economic growth

JEL codes: O3; O4; N13; N3

1 Introduction

Until the late middle ages people in Europe relied on tallies, finger reckoning and Roman numerals to represent numbers, and on reckoning boards to carry out calculations (Chrisomalis, 2010; Ifrah, 2000). While these systems were effective for calculating addition and subtraction with natural numbers, they were cumbersome for multiplications and divisions, and for handling large and rational numbers. The adoption of Hindu-Arabic numerals in European commercial practices, which began in Italy in the 13th century, not only changed the history of mathematics, but also enabled

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new applications of algebra to practical economic activities. Thanks to the principle of place value and to the symbol for zero, Hindu-Arabic numerals provided a system that was at the same time a reckoning tool and a notation technique to record numbers. The principle of place value, in fact, made it possible to use a limited set of symbols (i.e. just ten figures) to represent any number and to make algorithmic calculations with them (Ifrah, 2000, p. 679).

Moreover, as Hindu-Arabic numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) were introduced into Europe together with the notation for fractions, they made it possible to represent and to calculate with virtually any rational number. This brought about enormous improvements in the calculation of exchange and interest rates, the design of modern insurance contracts, and the ability to manage businesses based on the distribution of profits and losses according to shares – all characteristic features of the 'commercial revolution' of the 13th century, and fundamental stepping stones in the development of modern market institutions. Max Weber was among the first to notice that, while the positional numeral system had been invented in ancient India, it was put at the service – "in den *Dienst*" – of the development of capitalism only in Europe.[1] However, the importance of this transition is seldom discussed in detail in the economic history literature.

In this paper we study the relationship between the diffusion of Hindu-Arabic numerals and the growth of pre-modern European economies. The paper expands upon the broader theme concerning the role of knowledge in the process of economic growth, a well-established principle in both the evolutionary neo-Schumpeterian tradition (Nelson and Winter, 1982; Metcalfe, 2001; Dosi et al., 2000) and mainstream 'new growth' theory (Romer, 1990; Grossman and Helpman, 1993; Aghion and Howitt, 1992; Jones, 1995; Weitzman, 1998). Interestingly, the kind of new knowledge we are dealing with entails a codified component – it is contained and preserved through time in manuscript and printed texts – and a tacit component that resided in the people who adapted and transmitted this knowledge for practical purposes through social interactions (Dasgupta and David, 1994; Cowan et al., 2000; Johnson et al., 2002). In documenting the economic relevance of this new knowledge, we illustrate the characteristics of the texts and how their diffusion took place through organised learning in dedicated institutions, the so-called 'practical arithmetic schools'.

The paper combines qualitative (historiographical) evidence with quantitative (econometric) analyses of the relationship between Hindu-Arabic numerals and city-level growth. The study is based on a new and original dataset of over a thousand manuals of practical arithmetic. The dataset was constructed from existing catalogues as well as from extensive archival research. Contrary to previous studies, this dataset includes both manuscript and printed sources, making it possible to extend the period of analysis before the introduction of the printing press (Dittmar, 2011). After

[1] Weber noted this in the *Vorbemerkung* ('prefatory remark') to his *Die protestantische Ethik und der Geist des Kapitalismus*. The German text reads as follows: "Gerechnet, mit Stellenzahlen gerechnet, Algebra getrieben haben auch die Inder, die Erfinder des Positionszahlensystems, welches erst in den *Dienst* des sich entwickelnden Kapitalismus im Abendland trat, in Indien aber keine moderne Kalkulation und Bilanzierung schuf" (Weber, 2016, p. 115). English versions translate the term 'Stellenzahlen' as 'decimals', making Weber's explicit reference to the positional numeral system more difficult to spot: "Calculation, even with decimals, existed also in the algebra of India, where the decimal system was discovered. Yet in India it never led to modern calculation and accounting methods; this mode of calculation was first placed into *operation* only in the West's developing capitalism" (Weber, 2012, pp. 158–59).

presenting the historiographical framework, firstly, we provide systematic descriptive results by means of pooled and panel OLS regressions with as good a set of controls as can be applied in this empirical context. Secondly, we apply two quasi-experimental designs (an instrumental variable approach and matched-sample estimations, complemented with a placedo test) to show that this 'numerical revolution' had a causal effect on the growth of European cities.

2 Literature review and historiographical framework

2.1 Hindu-Arabic numerals and radical commercial innovations

There is a growing interest in investigating the role played by useful knowledge and skills in the pre-industrial period. Van Zanden and others started to explore this theme from a variety of perspectives, suggesting that 'human capital' played a relevant role in European growth (Baten and Van Zanden, 2008; Van Zanden, 2009). Buringh and Van Zanden (2009) estimated that Europe developed a consistently growing production of texts – both manuscript and printed – since the middle ages, raising the possibility that Europe developed a 'knowledge economy' starting from that period. More recently, it has been argued that 'human capital formation' was 'the driver of growth' in the pre-industrial period, and that diverging economic outcomes of different European areas (the so-called 'little divergence') can be accounted for on the basis of the accumulation of human capital (De Pleijt and Van Zanden, 2016). Focussing on the relationship between institutional and economic change, Cantoni and Yuchtman (2014) argued that it is possible to find a causal relationship between the presence of a university and the establishment of new markets in 14th-century Germany. Dittmar (2011) investigated the role of the printing press in the process of European growth, arguing that the introduction of this new general purpose (information) technology had a causal effect on the growth of cities.

An interesting aspect of this debate is arguably the spread of advanced commercial skills since the onset of the so-called 'commercial revolution' of the late middle ages. As it marked the origins of European economic expansion after the collapse of the western Roman Empire, the commercial revolution was a turning point in European economic history (Lopez, 1976). De Roover was the first to identify a 'commercial revolution of the 13th century' focussing on the wave of financial and organisational innovations that were introduced during that period. He argued that these innovations marked a 'complete or drastic change' in European commercial practices, which he compared in scope to the industrial revolution (De Roover, 1953).

During this period, the first international commercial-banking companies appeared, based on a new kind of partnership contract – the *compagnia* (Sapori, 1955; Goldthwaite, 2009; Tognetti, 2015; Tanzini and Tognetti, 2012; Padgett and McLean, 2006). In order to manage these complex organisations, international merchant-bankers invented the foundational principle of modern accounting, i.e. double-entry bookkeeping (De Roover, 1956; Goldthwaite, 2015; Nobes, 1984; Parker and Yamey, 1994; Melis, 1950; Melis, 1972). As they relied heavily on long-distance trade, these merchants also introduced the first insurance contracts (Ceccarelli, 2020; Ceccarelli, 2012). Thanks to an accounting system which was integrated across different cities, it was possible to make international transfers by means of interlocking accounting operations registered in the ledgers of two branches. This is the origin of the bill of exchange, a fundamental financial innovation which made it possible to both move money internationally without moving specie, and to extend short-term credit under the guise of a double operation of exchange (De Roover, 1953; Lane and Mueller, 1985; Chaudhury and Denzel, 2008; Bell et al., 2017). While the amount of precious metal shipped across the main European trading centres did not significantly change, the bill of exchange made it possible to drastically expand the monetary supply (Spufford, 1988, 254–55; Lopez, 1976, p. 72; Gelderblom and Jonker, 2018; Bolton and Guidi-Bruscoli, 2021).

These innovations characterised European business practices over the long run. The bill of exchange remained the most important means for international exchange well into the 19th century; insurance contracts evolved until reaching today's modern form; and double-entry bookkeeping is still today practiced following the principles developed in the 13th and 14th centuries. Yet, despite the abundant historiographical evidence concerning the importance of these innovations in European economic history, a quantitative assessment of their impact on the European economy is still lacking. Moreover, as these commercial innovations required little capital investment apart from the training necessary to handle them, studying their diffusion is a highly appropriate angle to address the broader question of the role of knowledge as an engine of growth in pre-industrial Europe.

The adoption of Hindu-Arabic numerals was a relevant factor for the development of these commercial innovations. The positional numeral system reached Europe relatively late, i.e. in the late middle ages (Djebbar, 2003; Katz et al., 2016; Folkerts and Kunitzsch, 1997; Folkerts, 2001). Together with developing international merchant-banking, double-entry bookkeeping, insurance contracts, and the bill of exchange, the merchants of the commercial revolution were also the first European economic agents to adopt the positional numeral system (Van Egmond, 1976; Heeffer, 2011; Ambrosetti, 2008; Danna, 2021).

This was a major advancement over Roman numerals. Roman 'fractions' were based on the duodecimal subdivision of Roman units and only enabled the representation of a limited set of rational numbers, that is to say those based on the factors of twelve (Yeldham, 1927; Maher and Makowski, 2001). To address the problem of calculating multiplication and division, 'reckoners' usually relied on the counter abacus and on multiplication tables. Conversely, thanks to the concept of place value and to the symbol for zero, Hindu-Arabic numerals provide a numeral system that is at the same time a reckoning tool and a notation to record numbers. The principle of place value makes it possible to use an efficient set of symbols (i.e. only ten figures) to both represent any number and to make algorithmic calculations with them (Ifrah, 2000, p. 679). Moreover, the contemporaneous adoption from the Arabic world of the notation for fractions increased the potential for new uses of the positional numeral system, as it made it possible to represent and make calculations with virtually any rational number.

Given these characteristics, it is easy to see why merchant-bankers of the commercial revolution adopted Hindu-Arabic numerals for their calculations. Since using bills of exchange implies dealing with the slightest differences between exchange rates, one can wonder how (and to what extent) it would have been possible to develop such a financial instrument in the absence of a numerical notation that could handle any rational number. A similar argument can be made about the calculations related to insurance contracts and to the establishment of commercial *compagnie* (consider the distribution of profits and losses according to shares). Primary sources provide ample evidence that these merchants actually performed similar calculations. For example, exchange rates in bills of exchange and account books were recorded in terms of a certain amount and *fraction* thereof of currency x for currency y (Bolton and Guidi-Bruscoli, 2021). In commercial companies, partners usually received shares of profits according to quotas specified as *fractions*, or as a certain number of *soldi* and *denari* (and also *fractions*) thereof of a *lira*.[2] In the 16th century, these shares could be expressed in such sophisticated terms as 3 *soldi*, 5 *denari*, and 17/23 of *denaro* for a specific partner (Goldthwaite, 2015, p. 634; Goldthwaite, 2011, 309–10). It would have been particularly cumbersome to handle these calculations and record these numbers in the absence of Hindu-Arabic numerals and fractions.

2.2 Fibonacci's *Liber abaci*, practical arithmetic manuals and the 'abacus' schools

The first European economic agents to adopt Hindu-Arabic numerals were 13th-century Italian merchants. The *Liber abaci* by Leonardo Pisano (better known as Fibonacci), first completed in 1202, provides early evidence of the exchanges between Italian maritime republics and the western-Arabic world. According to the autobiographical preface of the work, the young Fibonacci first learned about Arabic arithmetic in 'Bugia' (Béjaïa, in today's Algeria), where his father worked as an official of the overseas customs of the republic of Pisa. Written in Latin, the *Liber abaci* is one of the most advanced mathematical texts of the century, as it provides a synthesis of the Arabic and the European mathematical traditions (Boncompagni, 1857; Giusti and D'Alessandro, 2020). As such, it cannot be considered as a text exclusively addressed to merchants. However, together with advanced mathematics, the *Liber abaci* also deals with the fundamentals of arithmetic, as it introduces the positional numeral system and the algorithms necessary to carry out calculations with the ten figures. Moreover, it explicitly applies these mathematical tools to the solution of practical and commercial problems. The *Liber abaci* provides at the same time an exceptional mathematical work and evidence that mathematical knowledge was being transferred between mercantile communities across the shores of the Mediterranean in the first half of the 13th century (Giusti, 2002; Giusti and Petti, 2002).

While there is no direct evidence of the first adoption of Hindu-Arabic numerals for economic purposes, they surely had been adopted by the end of the 13th century. In this period, Italian city states developed a system of vocational schools. Among these schools, there were the so-called 'abacus schools', which secured the intergenerational transmission of practical arithmetic

[2] A *lira* was the standard monetary unit of medieval Europe, where a *lira* was equivalent to 20 *soldi*, and a *soldo* to 12 *denari*.

skills (Black, 2007; Grendler, 1989). These were lay and vernacular schools where pupils between the age of ten and thirteen (the curriculum lasted on average two years) were trained to use Hindu-Arabic numerals, learned how to make calculations with them, and how to apply these mathematical skills to solve practical problems. Although there were some regional differences (especially between Tuscan and Venetian cities), the curricula of these schools did not significantly diverge in terms of mathematical subjects (Arrighi, 1966b; Grendler, 1989; Goldthwaite, 1972; Ulivi, 2008).

These schools were founded in most central-northern Italian cities, and there is evidence of their presence also in some centres in the south. Abacus schools were publicly funded in small centres, but in major cities, such as Florence and Venice, the demand for this kind of training was sufficiently strong for abacus schools only to rely on the fees of their students (Black, 2007, pp. 545–611; Ulivi, 2002b). These fees were low enough to make it possible also for children of small shopkeepers to afford a training in abacus mathematics (Van Egmond, 1977; Goldthwaite, 1972). Abacus masters belonged to a sort of urban middle class, and were also employed as expert reckoners, for example in purveying and in auditing accounts for both the city state and for private citizens (Ulivi, 2002a; Ulivi, 2004b; Ulivi, 2013; Van Egmond, 1977; Goldthwaite, 1972). Education in practical arithmetic was a marketised and competitive sector, with teachers competing to secure higher salaries and larger numbers of students (Arrighi, 1966a; Ulivi, 2008; Ulivi, 2004a). This competition was also active across cities, with masters challenging each other to solve increasingly complex mathematical problems, which in some cases stemmed directly from commercial practices (Ulivi, 2015).

The most important documentation we have from these schools are practical arithmetic manuals (the so-called 'abacus manuals'). These texts are mostly written in vernacular, i.e. the working language of merchants and of abacus schools. Despite what their name might suggest, these texts only present calculations with Hindu-Arabic numerals.[3] As the vast majority of the known authors of these manuals were abacus masters, it is reasonable to think that these texts were mainly used within abacus schools. The structure of these texts is also coherent with this hypothesis, as abacus manuals mainly consist of long lists of worked examples, organised into sections according to their domain of application (e.g. problems of conversion, of exchange, of division of profits and losses, etc.) (Bocchi, 2017, pp. 10–65; Franci, 2003; Franci, 2015; Franci, 2018). As there is evidence that they were passed down across generations of masters, these manuals were probably used by abacus masters while lecturing in their schools (Murano, 2015; Danna, 2019).

These texts had a wide social circulation. Estimates on the average enrolment of pupils in abacus schools range from 25 to 50 students (Grendler, 1989, p. 72; Van Egmond, 1976, 105–13).[4] Even by using the lower estimate, it is easy to see that in the span of a decade each abacus manual used in a school reached over a hundred pupils. As a consequence, abacus manuals used within schools often had a readership of a few generations of abacus masters, and an audience of several hundreds

[3] To the best of our knowledge, we are not aware of any manuscript abacus manual illustrating calculations with the counter abacus/reckoning board.

[4] While Grendler estimated an average enrolment of 25 to 40 pupils per abacus school, Van Egmond argued that this number ranged between 40 and 50 students per school.

of students. This makes abacus manuals distinctively public sources and, therefore, a particularly suitable source of evidence to study the diffusion of the mathematical skills they contain. Moreover, practical arithmetic was an essential step in the training of an accomplished merchant-banker, as (for the reasons given above) Hindu-Arabic numerals were fundamental tools to handle the advanced economic practices of the commercial revolution.

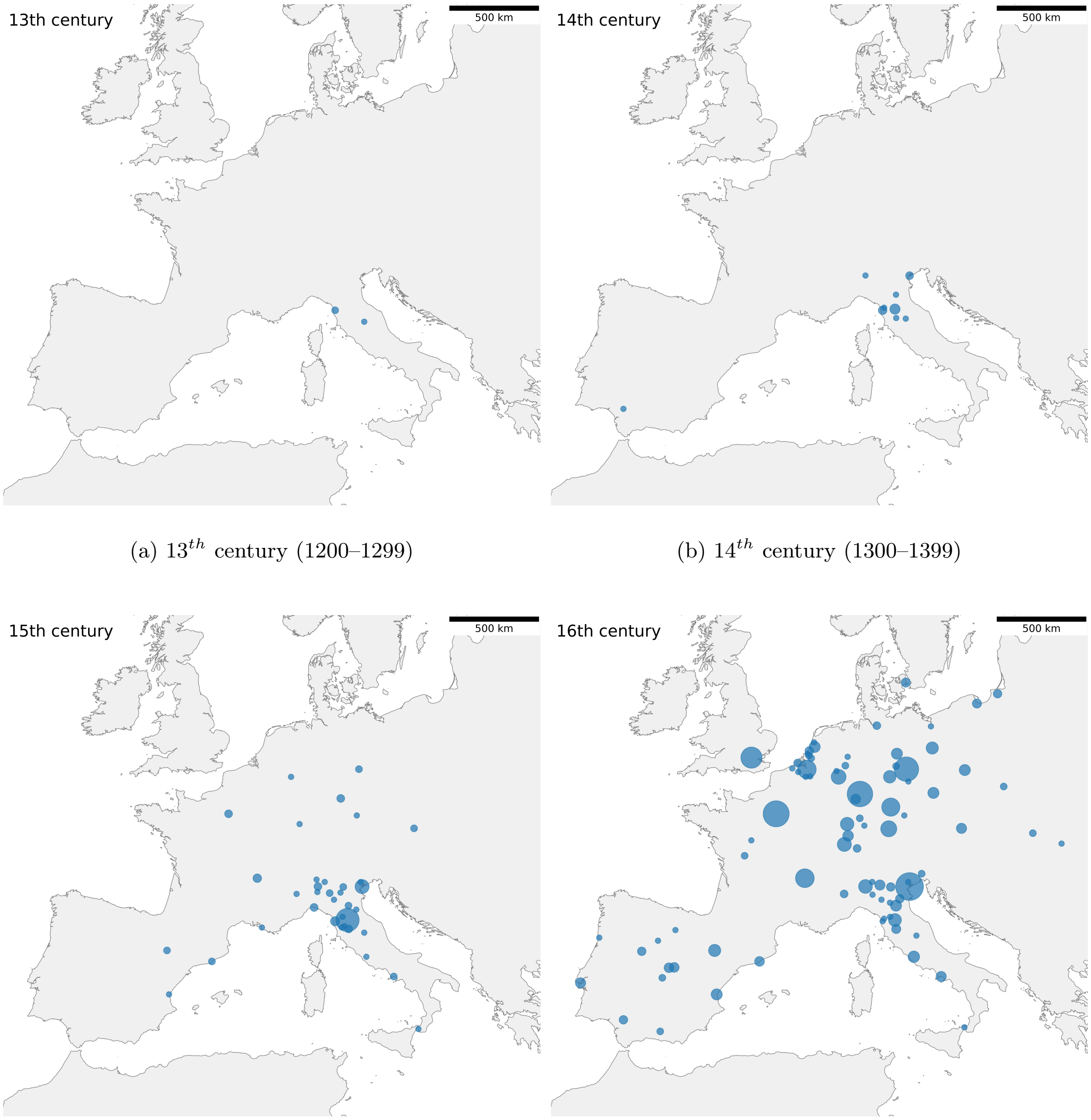
2.3 The diffusion of practical arithmetic in Europe

The historiographical evidence indicates that the manuals were mostly used within schools, and that their publication in a new city tended to be associated with the foundation of practical arithmetic schools. This is true not only for the Italian tradition of abacus mathematics, but for the entire European tradition (which we will call, more generally, the European tradition of practical arithmetic). The patterns of diffusion in Europe of practical arithmetic manuals are described in intertemporal maps in Figure 1.

In France, an early vernacular manual (1470s) was written by one of the first practical arithmetic masters active in Lyon, Barthélemy de Romans (Spiesser, 2003; Spiesser, 2000). De Romans was soon followed by Nicholas Chuquet, the author of the important *Triparty en la science des nombres* (1484), who consolidated the tradition of Lyonnaise practical arithmetic schools (Flegg, 1988; Benoit, 1988; Flegg et al., 1985). These first manuscript texts were followed by a flourishing tradition of printed manuals, which grew steadily as Lyon became a key hub of European finance and trade. In Germany, the first manuscripts of practical arithmetic were written in the 1450s in Regensburg, under the influence of Venetian sources (Vogel, 1954). These were followed by the first printed vernacular manuals (the *Rechenbücher*), which were published in Bamberg by Ulrich Wagner in 1482-1483 (Vogel, 1980; Schröder, 1996). Wagner was a practical arithmetic master (*Rechenmeister*), and ran a school in Nuremberg in the second half of the 15th century. Following these first texts, the tradition of German *Rechenbücher* grew exponentially in subsequent decades together with the spread of the *Rechenschulen*, which provided the foundational training for generations of German merchants (Gebhardt and Albrecht, 1996; Denzel, 2002; Gebhardt, 2002).

Following the publication of *Die maniere om te leeren cyffren* in Brussels (1508) and in Antwerp (1510), the manuscript by the *rekenmeester* Christianus van Varenbraken (c. 1532) documents that also in the Low Countries the publication of these manuals was coupled with the foundation of practical arithmetic schools (Kool, 1988b; Kool, 1988a; Gärtner, 2000, 244–47). These schools flourished first in Antwerp and, subsequently, in Amsterdam (Meskens, 1996; Meskens, 2013). After the first manuals printed in the 1520s and 1530s under the influence of French and Dutch texts, the English tradition of practical arithmetic manuals took off in 1543 with the publication of Robert Recorde's *The Ground of Artes* (Williams, 2012; Bockstaele, 1960). The popularity of these texts grew together with the foundation of practical arithmetic schools in London by the first generation of English masters (Woodbridge, 2003, 3–5; Roberts, 2016).

These examples show that the European spread of practical arithmetic manuals can be considered as a proxy for the spread of a wider institutional setting (the practical arithmetic school),



(a) 13^{th} century (1200–1299)

(b) 14^{th} century (1300–1399)

(c) 15^{th} century (1400–1499)

(d) 16^{th} century (1500–1599)

Figure 1: Diffusion of Hindu-Arabic manuals across European cities. The circle size is proportional to the number of manuals published in the city in that century.

which was designed to ensure the diffusion among local economic agents of the skills necessary to handle the advanced commercial practices of the commercial revolution. The appearance of a practical arithmetic manual, in other words, is to be considered as a signal of wider phenomena: the institutional development of specialised schools catering to the training of generations of merchants, and the consequent diffusion of advanced commercial skills, which progressively spread throughout the European continent following a south-to-north axis.

Moreover, the chronology given above shows that data on practical arithmetic manuals does not provide a measure for economic activity in general, but rather a proxy for the spread of specific commercial skills. If this was not the case, we would observe practical arithmetic manuals with Hindu-Arabic numerals appear simultaneously in the main European trading and industrial centres of the 13th and 14th centuries – i.e. not only in Italy and southern Europe, but also in the Low Countries, Hansa cities, and England.

For example, both Florence and Bruges were key economic centres in 14th-century Europe, as both cities had a flourishing wool and cloth industry and relied on an extensive trade network (Goldthwaite, 2009; Murray, 2009). However, while the 14th century was the heyday of Florentine abacus mathematics – as marked by a booming production of manuscript manuals – in Bruges the first practical arithmetic manuals with Hindu-Arabic numerals were published only in the second half of the 16th century.[5] Significantly, this late appearance of vernacular practical arithmetic manuals corresponds with the fact that international finance in 14th-century Bruges was dominated by Florentines, and with the late adoption among Bruges merchants of the business practices of the commercial revolution (De Roover, 1948).

Similar considerations can be made about other important trade centres of northern Europe. Lübeck, for example, despite being among the leading Hanseatic cities in the 14th century – with its port providing a hub for maritime trade across the north sea – also had its first practical arithmetic manuals published only in the second half of the 16th century.[6] In other prominent Hanseatic commercial cities, such as Bremen and Lüneburg, no practical arithmetic manuals with Hindu-Arabic numerals were published before 1600. England was also a late adopter of Hindu-Arabic numerals in commercial practices, as the first manuals of practical arithmetic in England were published in London in the 1530s (Williams, 2012). It may not be by chance that until the late 15th century calculations of exchange proved 'of the greatest difficulty' for English merchants active in international trade (Hanham, 1985, pp. 165–66).

3 Data and variables

3.1 The practical arithmetic manuals

We study the impact of the diffusion of practical knowledge by observing the spread of practical arithmetic manuals. We rely on a new dataset of over 1290 practical arithmetic manuals which was compiled relying both on previous literature and on extensive archival research. The dataset gathers over 350 manuscripts and over 940 printed manuals written from the late 13th century to 1600. These manuals were written by more than 340 authors,[7] who were active in over 120 cities of western

[5]The first practical arithmetic manual published in Bruges was probably Adriaen van der Gucht's *Cijfer bouck, inhoudende vele nieuwe, fraye, ende gherievighe practijcken van arithmetica*, published in 1569.

[6]The first practical arithmetic manual with Hindu-Arabic numerals published in Lübeck was Kaspar Hützler's *Eyn behende und kuenstrike reekensbock up allerley koephandel*, printed in 1547.

[7]This estimate is due to the fact that the authors of a considerable share of texts (c. 20%) are anonymous.

Europe scattered between Austria, Bohemia, Denmark, England, France, Germany, Hungary, Italy, Low Countries, Poland, Portugal, Spain, and Switzerland. The database records texts written in the following languages: Castilian, Catalan, Czech, Danish, English, French, German, Greek, Hebrew, Hungarian, Italian, Latin, Netherlandish, Polish, Portuguese, and Provençal. A selection of these texts is used in the econometric analysis, as described in section 3.2.

This database provides the first reconstruction of the European tradition of practical arithmetic, and shows that the diffusion of these texts was a continuous and piecemeal process. The evidence provided in this database contributes to existing literature in two ways. First, as it covers a period of three centuries, it provides a long-run analysis which is based not on estimates of book production, but on clearly identifiable primary sources (Baten and Van Zanden, 2008). Second, as it is based on both manuscript and printed sources, this database makes it possible to identify a continuous transmission of useful knowledge from the commercial revolution of the 13th century to the onset of the European 'little divergence'. Moreover, for the reasons discussed above, this database covers a particularly significant subset of total European book production, providing a good proxy to study the spread of commercially-oriented knowledge.

As we cover the period from the late 13th century to 1600, the dataset includes both manuscript and printed sources. While manuscript sources are considered individually (with each manuscript being considered as an independent document), printed manuals have been recorded at the book-edition level. This implies that, while a reprint of a text is considered as a new entry (also in the case of a reprint in the same city), the database does not include information on print runs. This choice is due to three independent reasons: (1) the different characteristics of manuscript and printed sources, (2) the varying social circulation of European practical arithmetic manuals, both manuscript and printed, and (3) the nature of the evidence extant. With regards to reason (1), while there is a degree of variation between copies of the same edition of a printed text (and this is particularly true for early incunabula), these differences are not comparable to the diversity presented by manuscript copies of the same text. This is due to the malleability of the manuscript medium in comparison with the standardisation determined by movable-type printing. This is particularly true for practical sources such as practical arithmetic manuals, because manuscript practical texts were often not copied with the aim of reproducing the original text, but, rather, with the aim of meeting their users' individual needs (Bocchi, 2017; Long, 2009).

As it was common for manuscript manuals to be handed down across generations of users, manuscript manuals often show sections added by different hands at different moments in time. Just to mention a few examples, the earliest Venetian manual presents these characteristics, as it is a notebook written by several hands starting from the 1340s which covers arithmetical operations with fractions, a variety of commercial problems, a section on practical geometry, and one on commercial customs, weights, and measures.[8] With its 18 surviving manuscript copies, Maestro Benedetto's *Trattato d'abacho* (c. 1465) was probably the most influential manuscript abacus manual of the 15th century. These copies show a high degree of variation, as copyists selected the parts which were most

[8]Florence, Biblioteca Riccardiana, ms. Ricc. 2161.

interesting for them, and arranged the sections according to their needs (Franci, 2003; Ulivi, 2002a; Van Egmond, 1980). A late example of the highly distinctive characteristics of each manuscript manual is the so-called memorandum-book of Richard Hill, a grocer from London who in the first half of the 16th century compiled a manuscript which mixes Latin, French, and English, gathering transcriptions of popular carols, chronicles, tables of exchange for wool cloth across the Channel, personal notes, and a text on practical arithmetic.[9] This high specificity of each manuscript manual justifies the choice to consider copies of the same text as independent items in the database.

With regards to the social circulation of these texts – i.e. reason (2) – the 'social reach' of each practical arithmetic manual tended to decrease with time. As we have seen, early manuscript manuals were probably texts which circulated in the hands of practical arithmetic masters. This makes these texts distinctively public sources, which were transmitted across generations of masters, and whose mathematical contents 'reached' hundreds of students. This is not limited to the Italian tradition of abacus mathematics. One of the earliest documents attesting a French-vernacular tradition of practical arithmetic is a manuscript written by two early practical arithmetic masters active in the south of France in the 1470s, Barthélemy de Romans and Mathieu Préhoude. The latter was possibly one of the first practical arithmetic masters in Lyon, and the former's pupil (Spiesser, 2000; Spiesser, 2003; Hay, 1988).[10] One of the first texts to attest the knowledge of algebra in Germany is an anonymous manuscript written in 1481 which was used by several generations of German *Rechenmeister*) (Vogel, 1981; Folkerts, 2002).[11]

With time, there is an increasing number of manuscript manuals written or owned by scribes who were not masters. For example, we have evidence of 15th-century manuals written by a rag trader (1464)[12] and by a painter (c. 1480),[13] as well as of manuscripts owned by a carpenter (1479),[14] and by a medical practitioner (c. 1460).[15] This is again not limited to the Italian tradition, as there is evidence that manuscript manuals were circulating among Catalan merchants from the late 15th century (Rey, 2004).[16] As the first printed practical arithmetic manual was published in 1478,[17] and as the production of printed practical arithmetic manuals did not take off until the last decade of the 15th century, this increasing social circulation of practical arithmetic texts predated the effects on this tradition of the introduction of printing, which, as a consequence, acted as a rapid accelerator of a pre-existing trend. Thanks to much lower prices, printed manuals of practical arithmetic became affordable to a wider public, and therefore increasingly circulated in the hands of practitioners and of students. While the printing press dramatically expanded the number of copies in circulation, the 'social reach' of practical arithmetic manuals decreased, as they moved from being public texts (in the case of early manuscript manuals used to lecture in front of generations of students) to

[9] Oxford, Balliol College, ms. 354.
[10] Cesena, Biblioteca Malatestiana, ms S-XXVI-6.
[11] Dresden, Sächsische Landesbibliothek, Codex Dresdensis C 80.
[12] Bologna, Biblioteca Universitaria, ms. 1612.
[13] Florence, Biblioteca Mediceo-Laurenziana, ms Ash 359 (291).
[14] Florence, Biblioteca Marucelliana, ms. A. c. s. 47.
[15] Florence, Biblioteca Mediceo-Laurenziana, ms. Ash 1128.
[16] Arxiu Históric de Mallorca, ms Diversos 37B/2. Antigua signatura: C108.
[17] This is the anonymous *Aritmetica di Treviso*.

increasingly private documents (e.g. printed manuals used by a single individual).

The third reason to record printed manuals at the book-edition level relates to the evidence extant. While we have good evidence regarding the editions of early printed European texts, with over 27,000 editions known before 1501, we have details about their print runs in only 160 cases. Moreover, these are often editions with special characteristics, making this evidence exposed to selection bias (Nuovo, 2013, 104). As a consequence, the widespread scholarly practice is to set the focus of analysis to the book-edition level, and to consider the number of reprints as a proxy for the success of an early modern printed text. For example, scholarly catalogues, both printed and digital, typically include information at the book-edition level (Smith, 1908; Smith, 1939; Van Egmond, 1980; Hoock, 1991; Navarro Brotons, 2000). For all these reasons concerning the different characteristics of manuscript and printed texts, the social circulation of practical arithmetic texts, and the available evidence on printed editions, considering manuscript documents at the copy-level and printed documents at the edition-level seemed the preferable choice.

Furthermore, the cost of shipping printed books was often sufficiently high to make it more convenient to reprint a text in a new city, rather than to ship large numbers of volumes across cities. In other words, printed books spread through new editions rather than through inter-city trade (Dittmar, 2011, 1140–41). This is essential to corroborate the reliability of our measure of the spread of useful knowledge, and of its impact on European urban growth. As trade across cities of printed texts was limited, an analysis at the edition-level of printed manuals is a good proxy of their diffusion in space and time.

3.2 Variables and samples

We exploit this novel dataset to investigate the effect of the diffusion of practical mathematics on the growth of European cities.

Due to the lack of per capita income data for historic cities, we follow the extant literature (Acemoglu et al., 2005; Bairoch et al., 1988; Dittmar, 2011) and use population-growth data, which can be considered a good indicator of cities' well-being and technological advancement, as a proxy for economic growth. The population of European cities in pre-modern Europe is, indeed, well documented at the century-level. To perform our analysis, we combine the Bairoch et al. (1988) dataset on the population of European cities with at least 5,000 inhabitants in the 13th–19th centuries with more detailed and updated estimations about population in Italian cities by Malanima (1998) and Malanima (2011).[18] We then define population growth as:

$$\log\left(\text{City growth}_{c;\,t,t+1}\right) = \log\left(\frac{\text{Population}_{c;\,t+1}}{\text{Population}_{c;\,t}}\right), \tag{1}$$

where $\text{Population}_{c;\,t}$ is the population at the beginning of the century t (14th–17th) in the city c.

Since we collect practical arithmetic manuals in the 13th–16th century and we are interested in capturing their long-run effects on urban growth, we focus on population data in the 14th–17th cen-

[18] Dataset available at: Italian urban population 1300–1861, Paolo Malanima, 2005.

turies. Moreover, we consider only the thirteen countries (following the 20th-century classification) that are present in the manuals' dataset: Austria, Bohemia, Denmark, England, France, Germany, Hungary, Italy, Low Countries, Poland, Portugal, Spain, and Switzerland. The resulting dataset provides information about population growth in 815 European cities, with 1838 city-century total observations. We select from the entire database the practical arithmetic manuals published in these 815 cities, and obtain 1054 texts written in 97 cities. Table 1 provides more details about the distribution of European cities in our sample across countries and centuries. The number of observed cities grows considerably during the centuries: while we have information on population growth for only 298 cities in the 14th century, the same data are available for 728 cities in the 17th century.

Table 1: Historic cities and historic cities with at least one practical arithmetic manual across centuries and 20th-century countries.

(a) Total historic cities

Century Country	14th	15th	16th	17th
Austria	1	1	8	10
Bohemia	1	1	2	2
Denmark	1	1	2	7
England	16	18	34	54
France	41	40	49	60
Germany	40	52	83	100
Hungary	0	4	3	4
Italy	140	127	158	243
Low Countries	28	38	49	53
Poland	3	7	16	16
Portugal	2	2	7	6
Spain	20	24	73	166
Switzerland	5	8	6	7
Total	298	323	490	728

(b) Cities with manuals

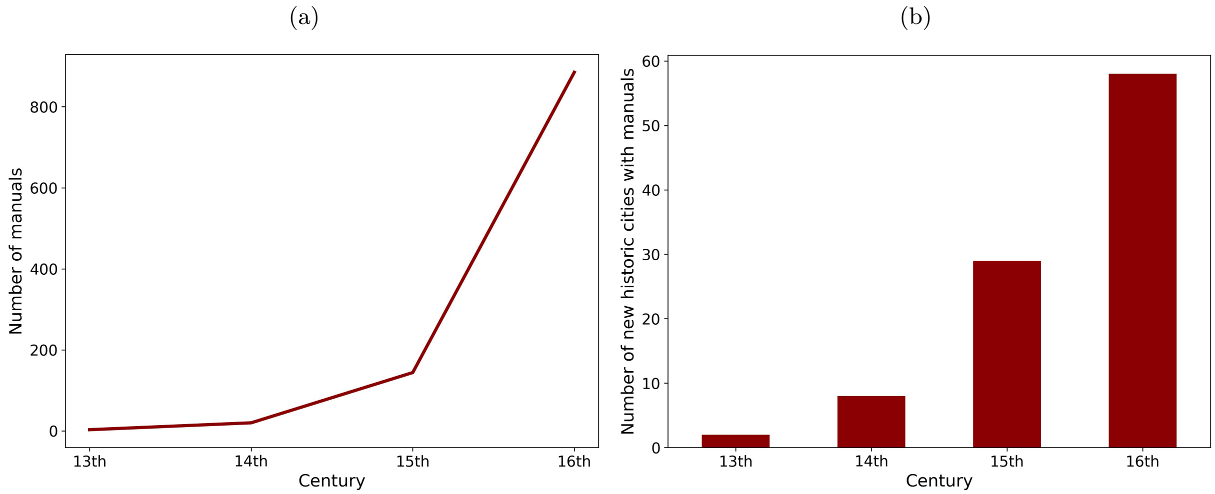
Century Country	13th	14th	15th	16th
Austria	0	0	1	1
Bohemia	0	0	0	1
Denmark	0	0	0	1
England	0	0	0	1
France	0	0	3	4
Germany	0	0	5	22
Hungary	0	0	0	2
Italy	2	8	24	22
Low Countries	0	0	0	12
Poland	0	0	0	3
Portugal	0	0	0	2
Spain	0	1	3	11
Switzerland	0	0	0	2
Total	2	9	36	84

Notes: The left-side table reports the number of historic cities in which we observe urban growth data across centuries and 20th-century countries. The bottom line summarizes the total number of cities per century. The right-side table reports the size of the subset of cities with at least one practical arithmetic manual across centuries and 20th-century countries.

Figure 2, instead, reports the diffusion, in terms of the number of manuals and number of new cities that adopt those manuals, of practical mathematics during the period of interest. Combined with the information provided in the previous table, this figure shows that for the period under investigation the spread of practical arithmetic manuals in Europe started in the 13th century with 3 manuals in 2 Italian cities and ended in the 16th century with 885 manuals in 84 cities across thirteen European countries. Additional details on the diffusion of practical arithmetic manuals across European cities in the centuries are included in Figure 1.

To better estimate the effect of the diffusion of practical mathematics on population growth, we also consider in the analysis a number of aspects that could affect the cities' economies in pre-modern

Figure 2: Number of practical arithmetic manuals (left-side panel) and number of new cities with manuals (right-side panel) in centuries.



Europe.

First of all, we control for the adoption of printing technology in the city. The first European movable-type printing press was introduced by Johannes Gutenberg in Mainz around 1450. In the following years, printing technology spread across the main European cities and substantially contributed to the decrease of book prices and to the dissemination of merchants' manuals and intellectual works.[19] The adoption of this technology has, therefore, a potentially relevant effect on cities' economies, as proven by Dittmar (2011). Specifically, Dittmar's investigation estimates a positive impact of the adoption of printing technology in 1450–1500 on city growth during the 16th century. In our analysis, we also consider the presence of printing technology in European cities during the 16th century so as to cover our entire period of interest. To create an indicator of the presence of the printing press in a city, we rely on the *Incunabula Short Title Catalogue*, for what concerns the books published up to 1500, and on the *Universal Short Title Catalogue*, for data after 1500.[20]

In addition to the control for the presence of printing technology, we follow previous literature and include in the analysis a series of dummy variables that capture cities' cultural, political, and geographical statusFirst of all, we include variable indicating the presence of a university in the city, as a proxy for its intellectual activities and cultural development. This variable may vary across centuries. We extract this information from De Ridder-Symoens and Rüegg (1992), and obtain observations on 106 different universities in our sample of European cities during the 14th–17th centuries.

[19]The classic historical study on the subject is Eisenstein (1980).

[20]The *Incunabula Short Title Catalogue* is available at: ISCT, maintained by the British Library (2016). In February 2022, the database collected 26297 printed books published before 1500 in 283 European cities. The *Universal Short Title Catalogue*, instead, is available at: USTC, hosted by the University of St Andrews (2022). In February 2022, we identified 311665 printed books published between 1500 and 1600 in 448 cities.

To capture the political status of cities, we define a dummy variable indicating whether the city was a state capital. We integrate the data on state capitals in pre-modern Europe by Bosker et al. (2013) with historical sources to check whether cities were state capitals throughout the century and not only at its beginning. As expected, this variable changes over time, and we count 46 capitals in our period of observation.

Concerning geographical location, we control for features – such as the presence of roads and ports – that may favour trade with other citiesWe obtain information about the presence of Roman roads in the proximity of each city by integrating data by Bosker et al. (2013) with the geolocalisation of Roman roads.[21] We also define a variable accounting for the presence of a navigable river close to the city. Once again, we rely on the information collected in Bosker et al. (2013). This data has been compared with the information about European cities in Buringh (2021) and checked manually for the remaining cities. Finally, we identify cities with seaports by distinguishing between Mediterranean, Atlantic, Baltic, and North Sea ports. We use data by Acemoglu et al. (2005) and Buringh (2021) as main sources of information on these city characteristics.

Table A1 in Appendix A summarises descriptive statistics concerning dependent, independent, and control variables in our panel dataset. Each observation refers to city-century pairs.

4 Econometric analysis and results

4.1 Descriptive evidence

Pooled OLS estimations In this section, we estimate the impact of the diffusion of practical mathematics on pre-modern European economies. We focus on the analysis of long-run effects by studying population growth, as a proxy for economic growth, with respect to the diffusion of practical arithmetic manuals in the previous century. In this exercise, we consider city growth in the 14th – 17th centuries, while the diffusion of manuals across European cities occurs in the 13th – 16th centuries. We, therefore, estimate the following model by an ordinary least square (OLS) regression, pooling together all centuries of interest:

$$\log(\text{City growth}_{c;\,t,t+1}) = \beta_0 + \beta_1\,\text{Manuals}_{c;\,t-1} + \gamma_{c;\,t} + \delta_t + \phi_a + \epsilon_{c;\,t}, \tag{2}$$

where c is a European city, t is the century (from the 14th to the 17th century), a is the country the city belongs to, $\gamma_{c;\,t}$ are a set of controls at the city level, δ_t and ϕ_a capture, respectively, century and country (following the 20th century classification) fixed effects, and $\epsilon_{c;\,t}$ is the error term. In all estimates, we report clustered standard error by country.

We lag the variable Manuals because, as practical arithmetic schools began training a growing number of local merchants, some time had to pass between the publication of the first practical arithmetic manuals and the achievement of a social circulation of advanced commercial skills

[21] We measure the distance between each city and Roman roads and consider cities with a distance of up to 2.5 km from those roads. Geolocalised data are from Harvard Center for Geographic Analysis - Roman World.

Table 2: Effect of the presence of practical arithmetic manuals on the city growth of the following century - Pooled OLS estimations with country and time fixed effects.

	log(City growth)$_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of manuals$_{t-1}$)	0.112***		
	(0.032)		
log(Nb. of manuals 2^{nd} half-century$_{t-1}$)		0.156***	
		(0.023)	
Manuals dummy$_{t-1}$			0.142**
			(0.056)
Printed books$_{t-1}$	0.122*	0.119	0.123*
	(0.068)	(0.068)	(0.068)
University	0.091**	0.093**	0.092***
	(0.033)	(0.035)	(0.030)
Capital	0.424***	0.421***	0.434***
	(0.054)	(0.054)	(0.058)
Roman road	0.031	0.032	0.029
	(0.034)	(0.033)	(0.033)
Navigable river	0.089***	0.090***	0.088***
	(0.018)	(0.016)	(0.018)
Mediterranean port	0.095	0.095	0.097
	(0.066)	(0.065)	(0.065)
Atlantic port	0.291**	0.293**	0.289**
	(0.104)	(0.106)	(0.106)
Baltic port	0.455**	0.451**	0.452**
	(0.155)	(0.153)	(0.158)
North Sea port	0.189***	0.186***	0.194***
	(0.043)	(0.042)	(0.041)
log(Population)	-0.302***	-0.305***	-0.297***
	(0.016)	(0.014)	(0.016)
Fixed-effects			
Century	Yes	Yes	Yes
Country	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	0.309	0.312	0.306
Within R^2	0.182	0.185	0.179

Notes: This table presents pooled OLS estimates of equation 2. The dependent variable is the logarithm of the population growth during the century t (equation 1), with t ranging between 14^{th} and 17^{th} century. The *Manuals* variables are indicators of the presence of practical arithmetic manuals in the city in the century $t-1$. Control variables include the presence of printing presses in the century $t-1$, university, state capital, roman road, navigable river, and (Mediterranean, Atlantic, Baltic or North Sea) seaport. We also control for the logarithm of the city population at the beginning of the century t, and we add country and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the country level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

sufficiently wide to generate the measurable impact we observe in the empirical analyses. The lag observed between the first appearance of practical arithmetic manuals and subsequent urban growth corresponds to the time necessary for the new knowledge to be sufficiently widespread in the local commercial community to produce tangible changes.

To better understand the economic effects of the diffusion of practical arithmetic manuals in a given city, we provide a set of alternative variables to measure the presence of manuals in that city: the *number of manuals* in the previous century, the number of manuals *in the second-half* of the previous century, and, finally, a dummy variable that captures *the presence of at least one manual* in the previous century. Due to the skewed nature of distributions of the number of manuals, these variables are all log-transformed.

Following previous studies (Acemoglu et al., 2005; Dittmar, 2011), controls at the city level include the presence, during century t, of a university or the state capital, as well as access to navigable rivers or seaports. They also include a dummy variable that signals the presence of the printing technology during the same century $(t-1)$ in which we observe practical arithmetic manuals. Finally, we control for the logarithm of population at the beginning of the century to account for the initial level of economic development.

Table 2 reports the result. For all the indicators capturing practical arithmetic manuals, we observe a positive and significant association between the diffusion of texts and city growth in the subsequent century. More in detail, specification (1) estimates that a 10% increase in the number of manuals in the century $t-1$ leads to a 1.12% increment in city growth in century t. This effect is greater if we consider texts in the second-half of the century (specification (2)): a 10% increase in the number of manuals in the second-half of the century is associated with a 1.56% increase in city growth. Finally, specification (3) shows that the presence of at least one manual in the city during the previous century increases city growth by 15.3%.

Results are qualitatively unchanged when we replace country fixed effects with variables controlling for the *longitude of the city*, its *latitude*, and *their interaction* (see table A3).

Panel fixed-effect estimations To address possible sources of endogeneity such as the omitted variable bias, and because poolability tests for fixed effects suggest the presence of city effects, we exploit the panel nature of our database, and we control for unobserved cities' characteristics by estimating the following model with city and century fixed effects:

$$\log(\text{City growth}_{c;\,t,t+1}) = \beta_0 + \beta_1\,\text{Manuals}_{c;\,t-1} + \gamma_{c;\,t} + \delta_t + \phi_c\epsilon_{c;\,t}, \tag{3}$$

where c is a city, t is the century, $\gamma_{c;\,t}$ are a set of controls at the city level that vary over time, δ_t and ϕ_c capture, respectively, time and city fixed effect, and $\epsilon_{c;\,t}$ is the error term. All estimations report clustered standard errors at the city level.

Control variables at city level include the presence of a university or state capital during the century of interest, and the presence of print technology in the previous century. For what concerns cities' geographical features, we include only the dummy variable signaling the presence of a navigable river, since the other variables do not vary over time. As in the previous section, we consider three different indicators of the presence of practical arithmetic manuals in the city.

Table 3: Effect of the presence of practical arithmetic manuals on the city growth of the following century - Panel fixed-effect estimations.

	log(City growth)$_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of manuals$_{t-1}$)	0.061*		
	(0.033)		
log(Nb. of manuals 2^{nd} half-century$_{t-1}$)		0.083**	
		(0.039)	
Manuals dummy$_{t-1}$			0.080
			(0.058)
Printed books$_{t-1}$	0.027	0.024	0.029
	(0.042)	(0.043)	(0.043)
University	0.024	0.027	0.023
	(0.064)	(0.064)	(0.064)
Capital	0.293**	0.291**	0.293**
	(0.114)	(0.113)	(0.115)
Navigable river	0.512	0.519	0.527
	(0.346)	(0.346)	(0.353)
log(Population)	-0.776***	-0.778***	-0.774***
	(0.038)	(0.038)	(0.039)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	0.726	0.727	0.726
Within R^2	0.414	0.415	0.413

Notes: This table presents panel fixed-effect OLS estimates of equation 3. The dependent variable is the logarithm of the population growth during the century t (equation 1), with t ranging between 14^{th} and 17^{th} century. The *Manuals* variables are indicators of the presence of practical arithmetic manuals in the city in the century $t-1$. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

Overall, panel estimates confirm the results, and even when we control for unobserved urban characteristics, practical arithmetic manuals have a positive and significant effect on city growth in the subsequent century, as reported in table 3. Indeed, an increase of 10% in the number of texts in circulation in a given century produces an increment of 0.61% in city growth in the following century (specification (1)). As in the pooled OLS estimations, the manuals published in the second-half of the century are particularly relevant: an increment of 10% in the number of manuals in circulation in the second part of the previous century leads to a 0.83% increase in urban growth. However, in this case the effect estimated through the dummy variable is positive but not significant.

4.2 Causal evidence

4.2.1 The mechanism

The publication of a practical arithmetic manual is a powerful proxy for deeper institutional change at the city level. It can be documented that these developments occurred as a result of deliberate urban policies to invest in local useful knowledge, and that these investments were designed to promote growth. Several Italian city states chose to devote part of their budget to fund abacus schools. This possibly dates back to the very origin of the tradition of abacus mathematics, as in 1241 the republic of Pisa provided Fibonacci with an annual salary in compensation for his knowledge ('merito dilectionis et gratie, atcque scientie sue prerogativa'), identifying him as a 'magister' (Ulivi, 2011, 256–57). The documentation from other Italian city states shows that the foundation of abacus schools was understood as an intentional economic policy. In the 14th century, the city of Pistoia justified the foundation of its abacus school stating that without practical arithmetic its merchants and artisans could not have run their businesses in appropriate ways ('Sine scientia abaci mercatores et artifices utiliter et bene se exercere non possunt') (Grendler, 1989, p. 22).

Following the establishment of state-supported fairs in the 1460s, which granted particularly advantageous conditions to foreign merchants, Lyon quickly became one of the main European centres for international trade and foreign exchange, with a growing number of Italian merchant-banking companies establishing their branches in the city. The rapidly expanding number of practical arithmetic schools in Lyon in the second half of the 15th century can be understood as a policy from local economic agents to catch up with this rapid inflow of foreign merchant-bankers. The Lyonnaise merchant François Garin, for example, showed the awareness of this gap in competences among the local mercantile community in the 1460s, when he urged his son to master the new mathematical techniques (Dubuis, 1978). Similar developments can be seen later on in England. In the preface to the 1574 edition of his *The Well spring of Sciences*, dedicated to the company of Merchant Adventurers, Humphrey Baker was sure he could appeal to a widely held opinion among the London commercial *élite* with regards to the usefulness of practical arithmetic for business purposes.[22]

This choice to invest in practical arithmetic skills can be understood as a result of the first exposure of a local mercantile community to advanced commercial practices used abroad. In some cases, the authors of the first manuals were themselves the very first adopters of advanced commercial techniques. After learning these techniques abroad, these early adopters wrote texts to facilitate their dissemination in their area of origin. Fibonacci's preface to his *Liber abaci* can be read as a deliberate choice to introduce Arabic practical arithmetic into the western-Latin world (Giusti and D'Alessandro, 2020, p. 4).[23] Jan Ympyn Christoffels, who published the first treatise on bookkeeping

[22]'And heerein I am sure yee are good witnesses with mee, howe foolishe and vayne is their opinion whiche beside youre most commendable affaires, suppose and affirme that Arithmeticke is of small use unto any other men, seynge that the lawes of sundry Realmes well instituted and guyded have deservedly accompted for fooles, and unfit members (to rule or deale in a Common welthe) all suche as wanted the skill of natural Arithmeticke, deprived them bothe of Landes and livinge, whiche as it tendeth unto no small prayse and credit of Arithmeticke.'

[23]'Summam huius libri quam intelligibilius potui in quindecim capitulis distinctam componere laboravi, fere omnia que inserui certa probatione ostendens, ut ex tam perfecto pre ceteris modo hanc scientiam appetentes instruantur,

published in northern Europe, had spent a decade in Venice, where he had been sent by his merchant father to learn business practices and accounting. Ympyn's work was published in 1543 in Antwerp in both Netherlandish and French, was quickly translated into English, and printed in London in 1547 (De Roover, 1974; Yamey and Bywater, 1982).

In other cases, it can be documented that these developments were sparked when a generation of local merchants moved to an area of previous adoption not just to trade, but to get an education in advanced commercial techniques. These early adopters then backed the foundation of practical arithmetic schools in their original cities. The first *Rechenschulen* were founded in Upper Germany after the first generation of German international merchant-bankers had moved to Venice and other Italian cities to gain a cutting-edge business education (Braunstein, 2016, 342–44; Weissen, 2000). Jacob Fugger, Lucas Rem and Matthäus Schwarz are a few examples of a number of German merchants who moved to Italy in their teen-ages to master advanced financial and accounting practices in the early 16th century (Braunstein, 1992; Häberlein, 2012; Greiff, 1861). The first *Rechenschulen* served as the institutional arrangement to secure the diffusion across Upper German mercantile communities of the skills first acquired abroad by these early adopters.

As a consequence, it is common to find that the first practical arithmetic masters active in a city moved (or were called) to that city from an area of previous adoption. The first master documented in the north of Italy – Maestro Lotto, who was appointed as public abacus master in Verona in 1284 – moved to that area from Tuscany, as he was originally from Florence (Black, 2007, p. 227). German *Rechenmeister* played a key role in the diffusion of the teaching of practical arithmetic in the Low Countries. For example, lists of practical arithmetic masters in 16th-century Antwerp report German names in around 50% of cases (Meskens, 2013, 91–94). The diffusion of practical arithmetic to the northern Low Countries, and especially to Amsterdam, was triggered by the migration of Antwerp's Protestant citizens in the aftermath of the Dutch revolt, that included a number of *rekenmeesters* (Meskens, 2013, 52–55). Humphrey Baker's *The welspring of sciences* (1574) attests that, in the third quarter of the 16th century, practical arithmetic skills were in growing demand in London among mercantile classes, and that such demand had so far been satisfied mostly by masters coming from the continent (Otis, 2017). As the exchange of ideas and the movement of people were the key channels through which this useful knowledge spread, the European diffusion of practical arithmetic did not follow the maritime routes of international trade, but rather inland channels based on proximity, as can be observed from the maps reported in Figure 1.

The appearance of a practical arithmetic manual, therefore, occurred after the first early adopters of new commercial techniques decided to invest in local useful knowledge, in order to secure the intergenerational transmission of advanced commercial skills. The outcome of this investment was the appointment of practical arithmetic masters, who often came from areas of previous adoption, and the foundation of the first local practical arithmetic schools. Once a school was established, a market for practical arithmetic texts opened up, which led to the publication of the first practical arithmetic manuals in that city. This is the framework in which we explain the publication of

et gens latina de cetero, sicut hactenus absque illa minime inveniatur' practical arithmetic manuals in a new city and its effect on growth.

4.2.2 Quasi-experimental designs

Table 4: Centres of diffusions of practical arithmetic manuals in different countries.

City	Country	Century
Pisa	Italy	13^{th}
Sevilla	Spain	14^{th}
Wien	Austria	15^{th}
Lyon	France	15^{th}
Regensburg	Germany	15^{th}
Praha	Bohemia	16^{th}
Kobenhavn	Denmark	16^{th}
London	England	16^{th}
Debrecen	Hungary	16^{th}
Bruxelles	Low Countries	16^{th}
Kraków	Poland	16^{th}
Lisboa	Portugal	16^{th}
Basel	Switzerland	16^{th}

Notes: The table reports the city and century of first occurrence of practical arithmetic manuals in each country.

Following Dittmar (2011), we propose a quasi-experimental design based on the introduction of an instrumental variable defined as the distance of the city under investigation from the location of first occurrence of a practical arithmetic manual. Differently from Dittmar (2011), we do not observe a concentric spread of these texts, but rather a progressive diffusion of practical arithmetic manuals from the south to the north of the European continent. Across centuries, new centres of diffusion emerged, as we can see from Table 4, which reports the cities and centuries of first occurrence of texts in each country.[24] Centres of new diffusion are identified at the country level because the first appearance of a manual in a new country tended to coincide with its translation into that country's language. For example, the first manuals belonging to the Italian tradition of abacus mathematics were also the first practical arithmetic manuals written in Italian vernaculars; the anonymous *Die maniere om te leeren cyffren*, printed in Bruxelles in 1508, was both the first practical arithmetic manual published in the Low Countries and the first text of this tradition to be written in Netherlandish, and the English tradition of practical arithmetic started off with the first translations of French and Netherlandish texts into English (Williams, 2012). As the translation of this mathematical knowledge into a new vernacular language boosted its circulation in the new country of adoption, we design an instrument that exploits this feature of diffusion.

Building on the multicentric diffusion of manuals across centuries, we define the instrumental variable as the distance between the city under investigation and the city of the same country where a practical arithmetic manual appeared first. If no manual had yet appeared in that country, the

[24]Table 4 reports cities for which population data are available.

Table 5: Instrumental variable estimates: Effect of the presence of practical arithmetic manuals on the city growth of the following century.

	log(City growth)$_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of manuals$_{t-1}$)	0.375***		
	(0.143)		
log(Nb. of manuals 2^{nd} half-century$_{t-1}$)		0.447**	
		(0.187)	
Manuals dummy$_{t-1}$			0.729***
			(0.271)
Printed books$_{t-1}$	-0.059	-0.062	-0.086
	(0.057)	(0.060)	(0.063)
University	0.042	0.056	0.042
	(0.072)	(0.070)	(0.071)
Capital	0.335***	0.322***	0.359***
	(0.110)	(0.109)	(0.125)
Navigable river	0.602**	0.627**	0.787**
	(0.293)	(0.303)	(0.319)
log(Population)	-0.808***	-0.814***	-0.804***
	(0.039)	(0.039)	(0.038)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	0.696	0.696	0.689
Within R^2	0.349	0.349	0.333
F-test (1^{st} stage)	219.6***	199.4***	106.9***

Notes: This table presents panel fixed-effect 2SLS estimates of Equation 3. The dependent variable is the logarithm of the population growth during the century t (equation 1). The $Manuals$ variables are indicators of the presence of practical arithmetic manuals in the city in the century $t-1$. We instrument these variables with the distance of the city from the location of first occurrence of practical arithmetic manuals in the country. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

instrumental variable is equal to the minimum distance from the already existing centres of diffusion.

The distance from the first centre of diffusion in each country is a means to measure a pattern of diffusion of practical arithmetic that is exogenous from the determinants of urban growth in other cities of the same country. For example, while Regensburg was a centre of exchanges between Venice and the German world in the 15th century, the observed patterns of growth of German cities in the same period are scattered independently of the distance from Regensburg. Moreover, the most important urban centres of the period – i.e. the Hansa cities – were particularly distant from the southern Bavarian city. Similar arguments apply to the distance from other centres of first adoption.

We therefore estimate a panel fixed-effect two-stage least square (2SLS) regressions for the three different indicators of the presence of manuals in the city. The first-stage results (available in table A4) show that the distance from the first manual in the area is significantly and negatively correlated to the presence and the number of manuals in the city, confirming the importance of regional centres for the diffusion of these texts across Europe. Moreover, the high values of F statistics for the IV signal that this distance is a strong instrument.

Table 5 reports the second-stage results. Overall, these estimations provide clear causal evidence of the positive and significant effects of practical arithmetic on city growth. More specifically, a 10% increase in the number of manuals leads to a 3.8% increase city's growth in the following century. The effect is stronger for manuals published in the second half of the century: the same increment in the number of manuals corresponds to a 4.45% increase in city growth. The effect is positive and significant also when we measure the presence of those texts with a dummy variable, since cities with at least one manual increase their urban growth by 107% in the subsequent century.

Compared with the OLS panel fixed-effect estimates, the results obtained by introducing the instrumental variable show a stronger effect of practical arithmetic on growth. A very plausible explanation of this difference lies in the ability of this instrument to detect fine-grained differences in the adoption and spread of Hindu-Arabic numerals in commercial practices. First, the distance from the first centre of adoption reduces the attenuation bias resulting from the possible presence of missing practical arithmetic manuals in nearby cities. Second, the number of practical arithmetic manuals published in the city is arguably an underestimation of the actual degree of integration of Hindu-Arabic numerals into local commercial practice. By capturing, instead, patterns of diffusion of practical arithmetic across European cities on the basis of distance, the instrument allows us to distinguish between cities with the same number of manuals, but with differences in local knowledge assimilation, thus generating more accurate estimates of the economic effect of this numerical revolution.

We also apply an alternative quasi-experimental design based on the matching of cities with and without practical arithmetic manuals in the period of observation. We compute a propensity score, estimated with a logistic regression, to identify pairs of cities in terms of the presence of university and state capital, country, list of centuries in which we have information on city growth, average population in the observed period, and geographical features (roman roads, seaports, navigable rivers).

The resulting sample includes 97 cities with at least one practical arithmetic manual in 13^{th}-16^{th} centuries and 97 cities without this kind of texts in the observed period. Cities are matched through a nearest neighbour matching without replacement based on the propensity score difference.

Results are reported in table 6 and, once again, confirm the importance of practical arithmetic manuals for city growth. More in detail, an increment of a 10% in the number of manuals leads to an increase of 0.76% in the city growth in the following century. When considering only manuals in the second part of the century, the increase is 0.98%. Finally, the presence of at least one practical arithmetic manual in the previous century entails an increment of 11% in the city growth. All in all, the estimated effect of the presence of practical arithmetic manuals on the city growth of the following century is similar to the one obtained in Table 3. However, by reducing the sample of cities and balancing places with and without texts, in this set of estimates the coefficient associated with the dummy variable is positive and significant, and is a useful complement to the results obtained with the number of variables in the interpretation of the economic magnitude of the effect.

Results are confirmed when considering alternative matching strategies such as nearest neighbour matching based on Mahalanobis distance and optimal pair matching. These additional results are presented in tables A5 and A6.[25]

Table 6: Matching estimates: Effect of the presence of practical arithmetic manuals on the city growth of the following century.

	$\log(\text{City growth})_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of manuals$_{t-1}$)	0.076**		
	(0.033)		
log(Nb. of manuals 2^{nd} half-century$_{t-1}$)		0.098**	
		(0.038)	
Manuals dummy$_{t-1}$			0.107*
			(0.058)
Printed books$_{t-1}$	0.023	0.018	0.019
	(0.064)	(0.065)	(0.065)
University	0.037	0.042	0.033
	(0.068)	(0.068)	(0.069)
Capital	0.372***	0.371***	0.374***
	(0.109)	(0.109)	(0.111)
Navigable river	0.489	0.498	0.506
	(0.312)	(0.313)	(0.321)
log(Population)	-0.745***	-0.749***	-0.740***
	(0.064)	(0.063)	(0.065)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	662	662	662
R^2	0.625	0.626	0.623
Within R^2	0.403	0.405	0.399

Notes: This table presents panel fixed-effect OLS estimates of Equation 3 on matched cities. Matching is performed with a nearest neighbour matching based on propensity score difference among cities. The dependent variable is the logarithm of the population growth during the century t (equation 1). The *Manuals* variables are indicators of the presence of practical arithmetic manuals in the city in the century $t-1$. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

[25]It is interesting to notice that in the quasi-experimental designs there is no effect associated with the presence of the printing press.

4.3 Placebo test: the manuscript tradition of Ptolemaic astronomy

It can be argued that the effects attributed to practical arithmetic manuals are part of a more general effect of mathematical knowledge that could have diffused via alternative channels. To show that this is not the case, and therefore rule out this alternative explanation, we design a placebo test in which we analyse the effect on urban growth of arithmetical knowledge diffused in universities and monasteries. The rationale of this test is to explore the importance for economic performance not just of the diffusion of knowledge in general, but also, and more specifically, of its social circulation and its application to economic problems. To do so, we focus on the circulation of Hindu-Arabic numerals among academic sources. While the main evidence of this paper is provided by data on the diffusion of vernacular manuals which circulated among practitioners, in this section we focus on mathematical texts that circulated among scholars.

In fact, the vernacular tradition of practical arithmetic was not the only tradition spreading Hindu-Arabic numerals in Europe. The initial circulation of the positional numeral system in Europe started in 12th-century *al-Andalus* with the first Latin translations of Arabic mathematical works (Folkerts and Kunitzsch, 1997; Folkerts, 2003). These translations were carried out by European scholars proficient in both Arabic and Latin, and their subsequent copies constitute the tradition of the so-called Latin *algorismi*. These texts are brief mathematical primers which were used to show how to use Hindu-Arabic numerals among people who could access a Latin education. While they include fundamentally the same mathematical knowledge as vernacular manuals, Latin *algorismi* are theoretical texts, and generally do not deal with the practical applications of mathematics. As such, they were quickly adopted in the main centres of European learning – i.e. monasteries and universities – as they were relevant for scholars interested in advanced calculations. These were needed primarily to solve problems stemming in astronomy, such as accurate calendrical calculations. As a consequence, Latin *algorismi* are often found in manuscripts which also include texts on astronomy (Allard, 1991; Ambrosetti, 2008; Nothaft, 2014). As they were primarily circulating in monasteries and universities, the geographical spread of Latin *algorismi* was different from that of practical arithmetic texts.

While these texts include the same mathematical theory as practical arithmetic manuals – the same 'technology' – they circulated in different social and institutional contexts, and their mathematical knowledge was applied to solve different problems. As observed in the incipit of a Florentine practical arithmetic manual written around 1457, the learned mathematics spread by Latin *algorismi* was of little use to economic agents active on the market.[26] We therefore use evidence on the diffusion of this Latin tradition to investigate the impact of the same factor as that of practical arithmetic (i.e. Hindu-Arabic numerals) applied in different social and institutional

[26] This is Francesco di Carlo de Macigni's *Libretto alla praticha della merchatantia*, preserved in Florence, Biblioteca Mediceo-Laurenziana, ms. Ash. 352. In the incipit of the text, Francesco remarked: 'uno che voglia essere marchatante non ha bisogno d'inparare a misurare le stelle e pianeti e 'l chorso de' pianeti e movimenti de' cieli, ma solo ha bisogno d'intendere la marchatantia e lle ragioni ch'elli adoperano in essa merchatantia', i.e. 'One who wants to become a merchant does not need to learn how to measure stars and planets and their course or that of the skies, but only needs to know well the trade of merchants and the problems which are needed in their business'.

contexts and to the solution of different problems. Specifically, we replicate the previous analyses by considering the diffusion in European cities in the 13th–16th centuries of Latin astronomical texts that use advanced arithmetic, relying on data on the Latin tradition of Ptolemaic texts.[27]

Table 7: Effect of the presence of Ptolemy's manuscripts on the city growth of the following century.

	$\log(\text{City growth})_{t,t+1}$		
	(1)	(2)	(3)
$\log(\text{Nb. of Ptolemy's manuscripts}_{t-1})$	-0.061		
	(0.051)		
$\log(\text{Nb. of Ptolemy's manuscripts}_{t-1}\ 2^{nd}\ \text{half-century}_{t-1})$		-0.060	
		(0.045)	
Ptolemy's manuscripts dummy_{t-1}			-0.104*
			(0.063)
Printed books_{t-1}	0.048	0.045	0.049
	(0.043)	(0.043)	(0.043)
University	0.017	0.020	0.015
	(0.063)	(0.063)	(0.063)
Capital	0.284**	0.288**	0.280**
	(0.116)	(0.116)	(0.116)
Navigable river	0.489	0.483	0.496
	(0.353)	(0.349)	(0.358)
log(Population)	-0.768***	-0.769***	-0.767***
	(0.039)	(0.039)	(0.039)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	0.725	0.725	0.726
Within R^2	0.412	0.412	0.413

Notes: This table presents panel fixed-effect OLS estimates on the placebo dataset. The dependent variable is the logarithm of the population growth during the century t (equation 1), with t ranging between 14^{th} and 17^{th} century. The *Ptolemy's manuscripts* variables are indicators of the presence of Ptolemy's manuscripts in the city in the century $t-1$. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

As in the case of Latin *algorismi*, Europe came in contact with Ptolemy's works thanks to the first translations of his texts from Arabic into Latin. For example, one of Ptolemy's main works – the *Almagest* – was rediscovered by western Europe thanks to the translations from Arabic to Latin carried out by Gerard of Cremona in late 12th-century *al-Andalus* (Kunitzsch, 1990). Ptolemy's works were among the most advanced astronomical texts of the time, and required advanced mathematical skills to be understood.

We retrieve information on Ptolemy's manuscripts written in Latin from the data collected by

[27] There is no complete survey of all surviving Latin *algorismi*, but evidence of the Latin Ptolemaic tradition is a reliable proxy of the circulation of Arabic arithmetic in a given city.

the *Ptolemaeus Arabus et Latinus* (PAL) project, which provides a reconstruction of the entire Latin *corpus Ptolemaicus*, i.e. of all Latin manuscripts which include works by or attributed to Ptolemy (Folkerts and Lorch, 2000; Kunitzsch, 2004).[28] The database provides information about the estimated date, origin, and provenance of 681 manuscripts belonging to the Ptolemaic tradition. By exploiting the origin information provided by the PAL project, we can locate Ptolemaic manuscripts in 61 cities (10 countries) of our database and replicate the analysis of the previous sections by replacing practical arithmetic manuals with Ptolemaic manuscripts.

The results are reported in table 7 and show no significant effects of the presence of Ptolemaic manuscripts on urban growth in the following century. Results are confirmed by quasi-experimental designs that replicate the ones we ran for the practical arithmetic manuals. We first introduce an instrumental variable equal to the distance of the city from the location of the first occurrence of Ptolemaic manuscripts in the country. If these manuscripts are not yet present in the area, the instrument is equal to the minimum distance of the city from the other centres of diffusion. [29] As a second quasi-experimental design, we match cities via nearest neighbour matching based on propensity score difference, and replicate the second test contained in Section 4.4.2. Tables A7 and A9 report the results. These rule out the academic channel as a possible alternative mechanism of growth.

5 Conclusions

In this contribution we have explored the idea that growth depends on socially distributed applications of knowledge to problems, whose solution generates economic value by means of new tools and techniques that open up unprecedented avenues for economic development. We have provided both qualitative evidence and quantitative analyses of the effects of the diffusion of Hindu-Arabic numerals on city-level growth: the detailed historiographical methods shed light on the underlying economic mechanism, and the econometric analysis provide robust descriptive and quasi-experimental evidence.

With respect to science and technology, mathematics provides less immediate illustrations of discontinuities associated, for example, with industrial revolutions. Yet, mathematical tools share some of the characteristics identified in general-purpose technologies (Bresnahan and Trajtenberg, 1995; Jovanovic and Rousseau, 2005; Bresnahan, 2010) and can indeed produce paradigmatic change (Dosi, 1988) in the structure and evolution of economic systems. The statistical and data-science foundations of a variety of advanced digital technologies may come to mind. There are of course other examples. In his path-breaking paper on the economics of scientific research, Nelson (1959) spoke eloquently about the importance of Maxwell's equation for technical progress. The particular type of innovation we have studied in this paper – the introduction of Hindu-Arabic numerals – was

[28] Data are available here: PAL: Latin Manuscripts.

[29] Since the patterns of diffusion of Ptolemy's manuscripts are different from those of practical arithmetic manuals, in this case the instrument is significantly correlated to the number of manuscripts only when we consider texts in the second half of the century, as shown in the first stage estimates (table A8).

an instrument that provided a range of very practical uses for the institutions of modern markets. The same innovation circulating in a narrower subset of the European population and applied to the solution of more theoretical problems – in our analysis exemplified by manuscript texts of Ptolemaic astronomy circulating among European scholars – generated no visible productivity gains. We argue that the socially distributed diffusion of this kind of mathematics in Europe was indeed a numerical revolution, without which the prospects of economic growth would have been very different.

The diffusion of this particular body of knowledge was organized in society through purposeful learning activities in institutions, the practical arithmetic schools, which formed a vernacular curriculum parallel to that of the universities, at a time when universities were also coming into existence. It is important to stress that, as we have shown in sections 4.2.1 and 3, the practical arithmetic texts reached, at different times for different places, a wide social circulation through a series of bottom-up decisions that did not follow academic hierarchies. While the tradition of the Latin *algorismi* was dominated by the names of a few authors,[30] the European tradition of practical arithmetic comprises hundreds of 'minor' authors who incrementally contributed to the development of a mathematical/commercial culture which eventually brought about measurable economic effects.

The social, temporal, and spatial diffusion of Hindu-Arabic numerals in European vernacular societies unleashed the economic potential of the new techniques of calculation. These became a sort of 'social technology' (Nelson and Sampat, 2001) for devising new ways to manage trade and make commercial decisions. One is left to wonder what type of market economy would have developed without this particular application of human intellect. Certainly, modern economies (and economics) owe a great deal to this mathematical innovation that came to Europe late relatively to the place of their invention (5th-century India).

In the end, this combination of mathematics, commercial practices, and their social circulation proved a formidable engine of epistemic, technical, and social changes which may help us reflect on the complex dynamics of economic development, and on the roots and boundaries of our modern economic categories and techniques.

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[30] Such as Alexander of Villedieu and Johannes de Sacrobosco.

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A Data appendix

A.1 Description of database sources

Both secondary and primary sources were used in the construction of the database. Catalogues are the most important secondary source. With respect to the Italian abacus tradition, the database includes all the evidence from the catalogue edited by Van Egmond (1980), which remains the most comprehensive source of its kind documenting this tradition. Even though this catalogue is relatively more thorough on manuscript sources than printed books (it does not provide summaries of content for the latter), whenever possible, these sources were consulted directly, either through direct inspection or through digitised copies, and their content was manually recorded. Moreover, several findings of successive research in the field were integrated in the evidence provided by Van Egmond (Bocchi, 2017; Franci, 2015; Franci, 2003; Ulivi, 2011; Ulivi, 2002a; Long, 2009)

With respect to the European tradition, the database includes a variety of sources. Their foundation is the study by Smith (1908), a monumental work based on extensive archival research efforts. This source needed to be treated with care, and required the definition of precise criteria since it also includes texts that are arguably no part of a European tradition of practical arithmetic. Firstly, our database excludes all texts that do not present Hindu-Arabic numerals. Secondly, among the texts that do use Hindu-Arabic numerals, we retained only those that are part of the practical arithmetic tradition. This implies the exclusion of the early modern reprints of classical, as well as early medieval, sources (e.g. all the reprints of Boethius' Arithmetica and all the early modern texts in this tradition, jointly with works related to numerology and the computi for the calculation of the calendar).[31] Thirdly, the database includes only texts that were relevant for practice in that they 1) contained practical applications of mathematics; 2) were explicitly addressed to practitioners, or 3) could have been used in activities of commercial training. The implication is that purely theoretical works developed in universities (e.g. most of the algorismi tradition) did not enter the database. Theoretically-oriented algebraic texts were only included if they contained at least some practical applications.[32]

Relevant online repositories, depending on the kind of text under investigation, were used to check Smith's records. These included: the *Incunabula Short Title Catalogue* (ISTC), the *Universal Short Title Catalogue* (USTC), the *English Short Title Catalogue* (ESTC), the online catalogue of the Bibliothèque Nationale de France and their digitalisation project (Gallica), the Münchener

[31] The computi were texts that taught how to calculate the calendar and were widely used, particularly in ecclesiastical contexts, for the calculation of important dates, such as Easter. They constituted a relevant channel of diffusion for Hindu-Arabic numerals and in a number of cases they preceded by a considerable margin the spread of practical arithmetic Nothaft, 2014.

[32] Among the relevant examples, the De arte supputandi libri quattuor by Cuthbert Tonstall (London, 1522) is included in the database despite its predominant theoretical focus because it is the first text published in England that was entirely dedicated to arithmetic. Similarly, Petrus Ramus' Arithmeticae libri tres (first ed. Paris, 1555) is included because given the importance attributed by the author to applied mathematics Angelini, 2008. Among the texts that are excluded even though they were of demonstrable practical relevance are the manuals concerning finger reckoning. Some of the late manuals on finger reckoning present Hindu-Arabic numerals, but belong to a different tradition than that of practical arithmetic.

DigitalisierungsZentrum of the Bayerische Staatsbibliothek, and the Biblioteca Virtual Miguel de Cervantes (Cervantes Virtual). These research tools, which were unavailable to both Smith and Van Egmond, allowed for a significant expansion of the information provided on every text, including identification of new texts by the same author, identification of new authors, and discovery of 1) editions not included in older catalogues, 2) holding institutions and 3) classmarks of original copies. Moreover, it was possible to identify and report every digital reproduction. When referring to these research tools, the risk of circularity was avoided by including their additional evidence only if it was based on information richer or independent from the sources that were already being consulted.

The research protocol we have just described (i.e. the criteria for inclusion in the database and the use of online repositories) was applied to the consultation of all catalogues. The texts cited in Smith (1939) were included only if it is clear that they had been consulted by Smith himself or if the USTC provided expanded the information on them, therefore corroborating Smith's information. Navarro Brotons (2000) contained valuable information on Iberian sources, whereas Hoock (1991) helped to consolidate the evidence available for central and northern Europe. The information provided by these sources was integrated by means of an extensive list of specialistic papers and local studies.

Moreover, one of the authors of this paper personally visited Italian archives in Florence and in Bologna, as well as archives in the British Library in London, the Cambridge University Library in Cambridge (UK) and the Bibliothèque nationale de France in Paris. The visits generated first-hand evidence and new documentation on early primary sources – of great interest – of Italian and European practical arithmetic traditions. Whenever possible, during the visits, the historian inspected the content of the texts. Through a combination of direct inspection of original or digitised copies, and through secondary studies, this work has generated data on the contents of 1051 texts (82% of all recorded texts).

How representative is this sample of texts? We cannot provide a definitive answer to this question because the size of the population of reference is unknown. It is extremely difficult to estimate the sample's representativeness considering that we do not know how many manuals of practical arithmetic were written in Europe between the 13th and the end of the 16th century. We do know, however, that the Italian abacus tradition is reasonably well documented during the first part of the observation period (14th and early 15th centuries). The same cannot be said for other parts of Europe, e.g. Spain and the south of France, where there is evidence of far fewer manuscripts than the case of Italy. As we have already mentioned, we do not have a good picture of the mathematical exchanges that may have taken place in the early phase in the Mediterranean region because several sources did not survive to the present day. It is possible that the manuscript material of the very early stages of the diffusion of European practical mathematics may be lost, except for relevant Italian sources. Of the known surviving manuscripts from the pre-printing age, our database offers very good coverage.

With the advent of printing, the quantity and quality of available sources increases considerably as we enter the age of the printing press. An overwhelming amount of the texts was produced in this period. Although it is conceivable that some materials are missing, the most reliable repositories currently available were used to compile detailed information on these stages of diffusion, and whenever possible each and every record has been manually double-checked.[33] The sources concerning the printing-press-era manuscripts from outside of Italy have not been studied as comprehensively as would be desirable. Nevertheless, it is important to emphasise that 1) known sources of European printed texts can probably be considered as a representative sample of their respective traditions,[34] and 2) the database provides a highly satisfactory coverage of them. The quality of the data is even better for Italy, for which the data can be considered a reliable sample of both the manuscript and printed traditions.

To summarise, in the case of Italy, both the manuscript and printed traditions have been studied and are relatively well preserved. The database offers good coverage of both. As far as the broader European region is concerned, few early manuscripts have been preserved, but the known sources have been studied in detail and recorded in the database. Conversely, manuscript material from the printing-press-era has not been comprehensively studied and, as a consequence, is probably not recorded in full. Printed European material is instead abundant: it can be considered qualitatively and quantitatively representative of its tradition, and has been thoroughly recorded in the database. Given the type of historical sources, the database contains as good a sample as possible of the known sources, and one that is arguably representative for the purposes of our study.

[33]The database relies on sources that cover most western-European areas. The data collected is arguably stronger for Italian, German, English printed texts, given the amount of studies that have addressed the areas (especially Italy) and the quality of the work that has been done on cataloguing and digitising primary sources (especially for Germany and England).

[34]Information on editions of early European printed texts is abundant, as we know of over 27,000 editions of printed texts between 1450 and 1500 (Nuovo, 2013, p. 104).

A.2 Additional descriptive statistics

Tables A1 and A2, instead, reports the descriptive statistics and correlation matrix of variables used in the regression analysis, respectively.

Table A1: Descriptive statistics

(a) Continuous variables

Variable	Mean	Std	Min	Max	Count
log(City growth)	0.06	0.59	-2.71	2.56	1838
log(Population)	2.03	0.94	0.00	5.70	1838
Nb. of manuals	0.57	4.99	0.00	101.00	1838
Nb. of manuals 2^{nd} half century	0.38	3.46	0.00	67.00	1838

(b) Binary variables

Variable	N	%	Count
Manuals dummy	131	7.13	1838
Printed books	456	24.80	1838
University	234	12.73	1838
Capital	118	6.42	1838
Roman road	973	52.94	1838
Navigable river	953	51.85	1838
Mediterranean port	184	10.01	1838
Atlantic port	35	1.90	1838
Baltic port	27	1.47	1838
North Sea port	105	5.71	1838

Table A2: Correlation matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1: log(City growth)	1.00	-0.30	0.02	0.03	0.00	0.03	0.02	0.07	-0.03	0.07	-0.00	0.04	0.05	0.09
2: log(Population)	-0.30	1.00	0.23	0.22	0.35	0.34	0.31	0.38	0.26	0.18	0.08	0.06	0.07	-0.02
3: Nb. manuals	0.02	0.23	1.00	0.97	0.41	0.19	0.10	0.17	0.02	0.07	0.02	-0.00	-0.00	0.02
4: Nb. manuals 2^{nd} half	0.03	0.22	0.97	1.00	0.40	0.19	0.09	0.16	0.02	0.07	0.01	-0.00	0.00	0.04
5: Manuals dummy	0.00	0.35	0.41	0.40	1.00	0.40	0.26	0.23	0.09	0.14	-0.00	0.04	0.02	0.00
6: Printed books	0.03	0.34	0.19	0.19	0.40	1.00	0.32	0.15	0.12	0.21	-0.04	0.04	0.02	-0.01
7: University	0.02	0.31	0.10	0.09	0.26	0.32	1.00	0.22	0.18	0.15	0.04	-0.01	0.05	-0.07
8: Capital	0.07	0.38	0.17	0.16	0.23	0.15	0.22	1.00	0.11	0.08	0.12	0.03	0.02	-0.02
9: Roman road	-0.03	0.26	0.02	0.02	0.09	0.12	0.18	0.11	1.00	0.10	0.19	0.04	-0.13	-0.14
10: Navigable river	0.07	0.18	0.07	0.07	0.14	0.21	0.15	0.08	0.10	1.00	-0.29	0.02	0.03	0.04
11: Mediterranean port	-0.00	0.08	0.02	0.01	-0.00	-0.04	0.04	0.12	0.19	-0.29	1.00	-0.05	-0.04	-0.08
12: Atlantic port	0.04	0.06	-0.00	-0.00	0.04	0.04	-0.01	0.03	0.04	0.02	-0.05	1.00	-0.02	0.03
13: Baltic port	0.05	0.07	-0.00	0.00	0.02	0.02	0.05	0.02	-0.13	0.03	-0.04	-0.02	1.00	-0.03
14: North Sea port	0.09	-0.02	0.02	0.04	0.00	-0.01	-0.07	-0.02	-0.14	0.04	-0.08	0.03	-0.03	1.00

A.3 Robustness checks

Table A3 reports results of pooled OLS estimates with the city longitude, its latitude and their interaction as control variables.

Table A4 reports the first stage of the 2SLS regression with the distance from the city of first occurrence of practical arithmetic manual as instrument. Figure A1 instead shows the difference between the characteristics of cities with and without manuals before and after the nearest neighbour matching based on propensity score difference.

Figure A1: Distance between cities with and without practical arithmetic manuals before and after matching.

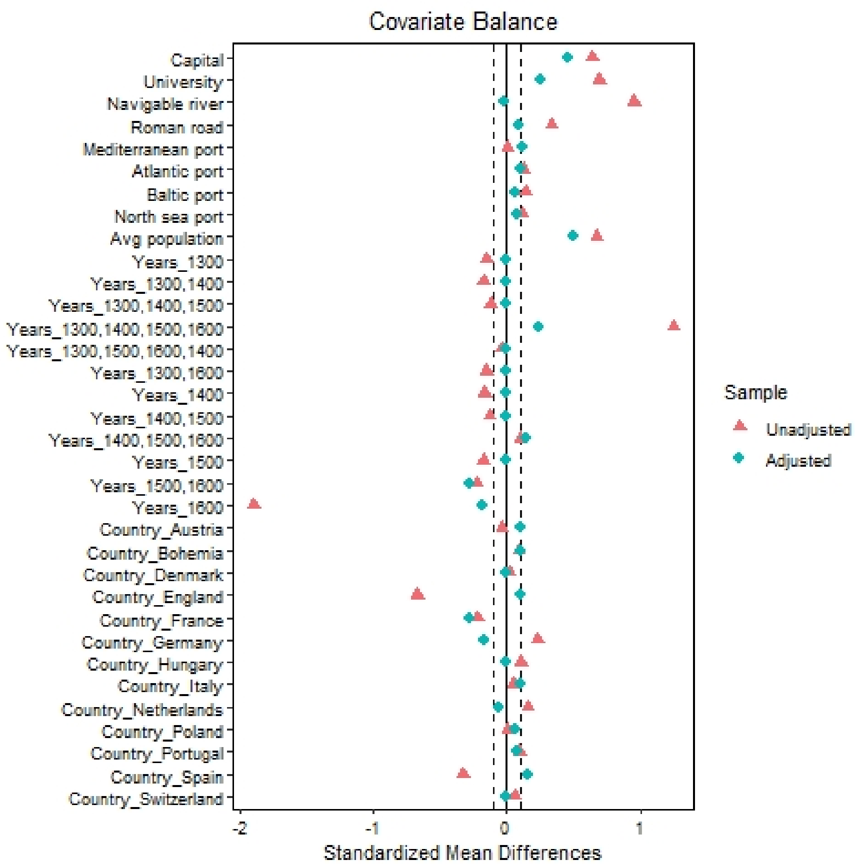

Table A3: Effect of the presence of practical arithmetic manuals on the city growth of the following century - Pooled OLS estimations with controls for city's longitude and latitude and time fixed effects.

	log(City growth)$_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of manuals$_{t-1}$)	0.096**		
	(0.031)		
log(Nb. of manuals 2^{nd} half-century$_{t-1}$)		0.140***	
		(0.022)	
Manuals dummy$_{t-1}$			0.110
			(0.063)
Printed books$_{t-1}$	0.157**	0.153**	0.160**
	(0.058)	(0.059)	(0.059)
University	0.085*	0.086*	0.087*
	(0.043)	(0.044)	(0.041)
Capital	0.393***	0.389***	0.403***
	(0.040)	(0.040)	(0.041)
Roman road	0.042	0.043	0.040
	(0.028)	(0.028)	(0.029)
Navigable river	0.134***	0.135***	0.135***
	(0.022)	(0.022)	(0.022)
Mediterranean port	0.068	0.069	0.070
	(0.055)	(0.054)	(0.055)
Atlantic port	0.390***	0.391***	0.389***
	(0.075)	(0.076)	(0.076)
Baltic port	0.411***	0.408***	0.409**
	(0.131)	(0.129)	(0.134)
North sea port	0.234***	0.230***	0.238***
	(0.053)	(0.051)	(0.052)
Longitude	0.074**	0.074**	0.074**
	(0.025)	(0.025)	(0.025)
Latitude	0.009	0.009	0.009
	(0.007)	(0.007)	(0.007)
Longitude × Latitude	-0.002**	-0.002**	-0.002**
	(0.0006)	(0.0006)	(0.0006)
log(Population)	-0.284***	-0.287***	-0.279***
	(0.017)	(0.016)	(0.018)
Fixed-effects			
Century	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	0.292	0.295	0.290
Within R^2	0.204	0.207	0.201

Notes: This table presents pooled OLS estimates of equation 2. The dependent variable is the log population growth during the century t (equation 1). The $Manuals$ variables are indicators of the presence of practical arithmetic manuals in the century $t-1$. Control variables include the presence of printing presses in the century $t-1$, university, state capital, roman road, navigable river, and seaport. We control for the log city population, longitude, and latitude. We add century fixed effects. Heteroskedasticity-robust standard errors are clustered at the country level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

Table A4: Instrumental variable estimates: First stage.

	log(Nb. manuals$_{t-1}$)	log(Nb. of manuals 2^{nd} half-century$_{t-1}$)	Manuals dummy$_{t-1}$
	(1)	(2)	(3)
log(Distance from first manual$_{t-1}$)	-0.150***	-0.126***	-0.077***
	(0.037)	(0.038)	(0.012)
Printed books$_{t-1}$	0.235***	0.203***	0.159***
	(0.043)	(0.038)	(0.025)
University	-0.035	-0.061	-0.018
	(0.097)	(0.077)	(0.047)
Capital	-0.140	-0.090	-0.106
	(0.119)	(0.111)	(0.096)
Navigable river	-0.170	-0.198	-0.340***
	(0.186)	(0.141)	(0.079)
log(Population)	0.085***	0.083***	0.037**
	(0.029)	(0.026)	(0.016)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	0.537	0.517	0.564
Within R^2	0.104	0.102	0.117

Notes: This table presents panel fixed-effect OLS estimates on European cities in the 14^{th}–17^{th} centuries. Dependent variables are the logarithm of the number of practical arithmetic manuals in the city in the century $t-1$, the logarithm of the number of manuals in the second half of the century, and a dummy variable which signals the presence of manuals in the century $t-1$. *Distance from first manual* refers to the distance of the city from the location of first occurrence of practical arithmetic manuals in the country. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

In table A5, we report the estimated effect of the presence of practical arithmetic manuals on city growth on a sample of cities matched minimising the Mahalanobis distance among cities. The matching is performed considering geographical characteristics, the presence of universities or state capitals in the observed period, the list of centuries for which we have information on city growth, the country, and the average population of the city at the beginning of 14^{th}-17^{th} centuries.

Table A5: Matching based on Mahalanobis distance: effect of the presence of practical arithmetic manuals on the city growth of the following century.

	log(City growth)$_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of manuals$_{t-1}$)	0.077**		
	(0.033)		
log(Nb. of manuals 2^{nd} half-century$_{t-1}$)		0.100***	
		(0.038)	
Manuals dummy$_{t-1}$			0.103*
			(0.059)
Printed books$_{t-1}$	0.004	-0.001	0.004
	(0.067)	(0.068)	(0.068)
University	0.044	0.050	0.040
	(0.076)	(0.076)	(0.077)
Capital	0.338***	0.337***	0.338***
	(0.106)	(0.106)	(0.108)
Navigable river	0.492	0.500	0.507
	(0.305)	(0.306)	(0.317)
log(Population)	-0.725***	-0.729***	-0.719***
	(0.067)	(0.066)	(0.068)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	637	637	637
R^2	0.641	0.642	0.639
Within R^2	0.389	0.391	0.385

Notes: This table presents panel fixed-effect OLS estimates of Equation 3 on matched cities. Matching is performed with a nearest neighbour matching based on Mahalanobis distance among cities. The dependent variable is the logarithm of the population growth during the century t (equation 1). The *Manuals* variables are indicators of the presence of practical arithmetic manuals in the city in the century $t-1$. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

Table A6: Optimal pair matching: effect of the presence of practical arithmetic manuals on the city growth of the following century.

	$\log(\text{City growth})_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of manuals$_{t-1}$)	0.076**		
	(0.033)		
log(Nb. of manuals 2^{nd} half-century$_{t-1}$)		0.098**	
		(0.037)	
Manuals dummy$_{t-1}$			0.109*
			(0.058)
Printed books$_{t-1}$	0.060	0.056	0.056
	(0.062)	(0.062)	(0.063)
University	0.046	0.051	0.042
	(0.068)	(0.068)	(0.069)
Capital	0.370***	0.369***	0.371***
	(0.108)	(0.108)	(0.110)
Navigable river	0.482	0.490	0.499
	(0.303)	(0.304)	(0.313)
log(Population)	-0.734***	-0.738***	-0.728***
	(0.065)	(0.064)	(0.066)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	666	666	666
R^2	0.629	0.630	0.627
Within R^2	0.403	0.405	0.400

Notes: This table presents panel fixed-effect OLS estimates of Equation 3 on matched cities. Matching is performed with an optimal pair matching based on propensity score difference among cities. The dependent variable is the logarithm of the population growth during the century t (equation 1). The *Manuals* variables are indicators of the presence of practical arithmetic manuals in the city in the century $t-1$. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

Tables A7 and A9 report, respectively, the results of two quasi-experiments (IV and matching) on the placebo dataset. In both cases, the presence of Ptolemaic manuscripts does not affect the city growth in the following century. Table A8 shows the results of first-stage regressions.

Table A7: Instrumental variable: effect of the presence of Ptolemy's manuscripts on the city growth of the following century.

	$\log(\text{City growth})_{t,t+1}$		
	(1)	(2)	(3)
log(Nb. of Ptolemy's manuscripts$_{t-1}$)	21.0		
	(387.7)		
log(Nb. of Ptolemy's manuscripts$_{t-1}$ 2^{nd} half-century$_{t-1}$)		0.567	
		(0.385)	
Ptolemy's manuscripts dummy$_{t-1}$			2.83
			(4.75)
Printed books$_{t-1}$	-1.49	0.024	-0.123
	(28.5)	(0.045)	(0.297)
University	1.35	0.029	0.172
	(24.5)	(0.066)	(0.291)
Capital	0.534	0.250**	0.400
	(5.02)	(0.119)	(0.274)
Navigable river	2.49	0.599	0.447
	(36.7)	(0.430)	(0.328)
log(Population)	-1.38	-0.776***	-0.835***
	(11.2)	(0.040)	(0.129)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	-38.7	0.701	0.122
Within R^2	-84.0	0.359	-0.880

Notes: This table presents panel fixed-effect 2SLS estimates on the placebo dataset. The dependent variable is the logarithm of the population growth during the century t (equation 1), with t ranging between 14^{th} and 17^{th} century. The *Ptolemy's manuscripts* variables are indicators of the presence of Ptolemy's manuscripts in the city in the century $t-1$. We instrument these variables with the distance of the city from the location of first occurrence of Ptolemaic manuscripts in the country. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

Table A8: Instrumental variable on Ptolemy's manuscripts: First stage.

	log(Nb. of Ptolemy's man_{t-1})	log(Nb. of Ptolemy's man_{t-1} 2^{nd} half-century_{t-1})	Ptolemy's man. dummy_{t-1}
	(1)	(2)	(3)
log(Distance from 1^{st} Ptolemy's man)	0.002	0.078^{***}	0.015
	(0.039)	(0.019)	(0.028)
Printed books_{t-1}	0.073^{***}	0.044^{*}	0.061^{***}
	(0.028)	(0.025)	(0.022)
University	-0.063	-0.018	-0.054
	(0.041)	(0.027)	(0.045)
Capital	-0.012	0.040	-0.045
	(0.080)	(0.056)	(0.065)
Navigable river	-0.095^{*}	-0.196^{*}	0.014
	(0.056)	(0.113)	(0.020)
log(Population)	0.029^{**}	0.022^{*}	0.025^{*}
	(0.013)	(0.011)	(0.014)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	1,838	1,838	1,838
R^2	0.489	0.386	0.455
Within R^2	0.014	0.051	0.013

Notes: This table presents panel fixed-effect 2SLS estimates on the placebo dataset. The dependent variable is the logarithm of the population growth during the century t (equation 1), with t ranging between 14^{th} and 17^{th} century. The *Ptolemy's manuscripts* variables are indicators of the presence of Ptolemy's manuscripts in the city in the century $t-1$. We instrument these variables with the distance of the city from the location of first occurrence of Ptolemaic manuscripts in the country. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: *p<0.1; **p<0.05; ***p<0.01

Table A9: Matching: effect of the presence of Ptolemy's manuscripts on the city growth of the following century.

	$\log(\text{City growth})_{t,t+1}$		
	(1)	(2)	(3)
$\log(\text{Nb. of Ptolemy's manuscripts}_{t-1})$	-0.044		
	(0.052)		
$\log(\text{Nb. of Ptolemy's manuscripts}_{t-1}\ 2^{nd}\ \text{half-century}_{t-1})$		-0.050	
		(0.046)	
Ptolemy's manuscripts dummy$_{t-1}$			-0.089
			(0.063)
Printed books$_{t-1}$	0.066	0.062	0.071
	(0.074)	(0.075)	(0.074)
University	0.080	0.085	0.076
	(0.096)	(0.096)	(0.095)
Capital	0.312***	0.317***	0.307***
	(0.116)	(0.115)	(0.116)
Navigable river	0.964***	0.955***	0.974***
	(0.090)	(0.090)	(0.089)
log(Population)	-0.837***	-0.840***	-0.834***
	(0.078)	(0.078)	(0.077)
Fixed-effects			
City	Yes	Yes	Yes
Century	Yes	Yes	Yes
Observations	427	427	427
R^2	0.64432	0.64424	0.64625
Within R^2	0.45352	0.45340	0.45649

Notes: This table presents panel fixed-effect OLS estimates on matched cities in the placebo dataset. Matching is performed with a nearest neighbour matching based on propensity score difference among cities. The dependent variable is the logarithm of the population growth during the century t (equation 1), with t ranging between 14^{th} and 17^{th} century. The *Ptolemy's manuscripts* variables are indicators of the presence of Ptolemy's manuscripts in the city in the century $t-1$. We instrument these variables with the distance of the city from the location of first occurrence of Ptolemaic manuscripts in the country. Control variables include the presence of printing presses in the century $t-1$, university, state capital, and navigable river. We also control for the logarithm of the city population at the beginning of the century t, and we add city and century fixed effects. Heteroskedasticity-robust standard errors are clustered at the city level and reported in parentheses. Significance level: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$