INSTITUTE
OF ECONOMICS

Scuola Superiore
Sant'Anna

LEM | Laboratory of Economics and Management

Institute of Economics
Scuola Superiore Sant'Anna

Piazza Martiri della Libertà, 33 - 56127 Pisa, Italy
ph. +39 050 88.33.43
institute.economics@sssup.it

LEM

WORKING PAPER SERIES

Increasing returns and labour markets in a predator-prey model

Giovanni Dosi [a]
Davide Usula [a]
Maria Enrica Virgillito [a]

[a] Institute of Economics, Scuola Superiore Sant'Anna, Pisa, Italy.

2023/21 May 2023
ISSN(ONLINE) 2284-0400

Increasing returns and labour markets in a predator-prey model

Giovanni Dosi[*] Davide Usula[†] Maria Enrica Virgillito[‡]

Abstract

The purpose of this work is to study the joint interaction of three founding elements of modern capitalism, namely endogenous technical change, income distribution and labour markets, within a low-dimensional nonlinear dynamic setup extending the Goodwin model. By going beyond the *conservative* structure typical of the predator-prey model, we insert an endogenous *source of energy*, namely a Kaldor-Verdoon increasing returns specification, that feeds the dynamics of the system over the long run and in that incorporates a transition to an (anti) dissipative framework. The qualitatively dynamics and ample array of topological structures reflect a wide range of Kaldorian stylised facts, as steady productivity growth and constant income distribution shares. The intensity of learning regimes and wage sensitivity to unemployment allow to mimic some typical traits of both Competitive and Fordist regimes of accumulation, showing the relevance of the demand-side engine, represented by the KV law, within an overall supply-side framework. High degrees of learning regimes stabilise the system and bring it out of an oscillatory trap. Even under regimes characterised by low degrees of learning, wage rigidity is able to stabilise the business cycle fluctuations and exert a positive effect on productivity growth.

Keywords: Capitalist system, Kaldor-Verdoon law, wage rigidity, dissipative complex systems.
JEL Codes: C61, C63, E11, E12, E32, E37

[*]Institute of Economics, Scuola Superiore Sant'Anna, Piazza Martiri della Libertà 33, 56127, Pisa, Italy.

[†]Institute of Economics, Scuola Superiore Sant'Anna, Piazza Martiri della Libertà 33, 56127, Pisa, Italy.

[‡]Corresponding author: Institute of Economics, Scuola Superiore Sant'Anna, Piazza Martiri della Libertà 33, 56127, Pisa, Italy.

1 Introduction

Is it possible to integrate a process of increasing returns of technical change with a conflicting class structure typical of a predator-prey model? If yes, how does productivity growth react to different elasticities to aggregate demand? How do unemployment and income distribution are affected by the interaction between labour markets and endogenous technical change? To address the above questions this work develops an extension of the Goodwin model (Goodwin, 1967).

The original model represents one of the most elegant, symbolic but also stylised representation of the recurrent cycles occurring in the capitalist system due to the conflicting class structure between capitalists and workers. Based on the Lotka-Volterra type predator-prey biological model (Lorenz, 1993), the Goodwin model presents a low dimensional nonlinear dynamical system, wherein business cycle fluctuations are due to functional income distribution showing opposing relations between profits and wages. The nature of such fluctuations is entirely *endogenous*, and in that the class struggle between predators and preys lies, providing a formalization of Marx's conflicting nature of capitalism (Dutt, 1992; Shaikh, 2016).

The model has been so seminal that an entire strand of literature has emerged with the attempt of modifying and extending the original framework, both from a purely modelling perspective and from an economic one. From the modelling perspective, extensions (Pohjola, 1981; Velupillai, 1979) have been proposed to overcome the topological structural instability of the system (Veneziani and Mohun, 2006). From an economic perspective, extensions focus on the dynamics of price formation through variations in the labour market equation (Desai, 1973), or to include new components such as the role of government expenditure (Wolfstetter, 1982). The majority of these extensions were in continuous time, while among discrete-time models, the paper by Canry (2005) advances beyond the original Classic *supply-side* scaffold and integrates a Keynesian endogenous source of demand. The contribution in Dosi et al. (2015) had the dual scope of overcoming structural instability and including endogenous sources of demand generation, by comparing alternative profit-led and demand-led investment drivers and labour market configurations.

So far, the majority of extensions has devoted few or no attention to the endogenization of technical progress, which in the model is constantly increasing at an exogenous rate. Exogeneity in productivity growth impedes any type of feedback mechanism from demand to supply, and, at the opposite, from productivity growth to labour markets and income distribution.

In the following, we intend to overcome the lack of a proper treatment of technical change and to study the ensuing feedback loops. In order to do that, in addition to the two original state variables – employment rate and wage share – we include a third element in the predator-prey scheme to allow the system to endogenously increase vis-à-vis its internal state. The specification we adopt encompasses a reduced form of the Kaldor-Verdoorn law in order to study the feedback from demand and income distribution to increasing returns and vice-versa (Verdoorn, 1949; Kaldor, 1966).

By including increasing returns, we make the original Goodwin model able to account for the three characteristic elements of a modern capitalist system, namely, income distribution, labour market and technical change, within a framework of both growth and business cycle. After introducing the new model specification, we start analyse a series of feedback dynamics by means of local sensitivity analysis upon alternative configurations of the parameter space. Our scenario analysis is performed upon two parameters, namely, the *learning coefficient*, that is the elasticity to economies of scale, and the *wage elasticity* to (un)employment as a proxy of labour market flexibility. By making varying the intensity of learning opportunities and labour market flexibility we intend to capture how the transition from informal economic systems, characterised by the absence of any type of increasing returns and protection in labour markets, to advanced capitalist systems, wherein technical change is affected by demand, and labour markets are more regulated and less volatile, affects the overall growth in labour productivity.

The paper is structured as follows. In section 2, we present the theoretical underpinnings behind the extension of the Goodwin model through a formalization à la Kaldor-Verdoorn. In section 3, we present the newly developed model including labor productivity as a state variable, and we discuss the ensuing stability properties. In section 4, we present a battery of simulations, comparing alternative degrees of formalization of economic systems, by means of local sensitivity analysis. Section 5 analyses the feedback mechanisms from income distribution to labour productivity growth. Finally, our conclusions are in section 6, together with a discussion on limitations and possible extensions.

2 Increasing returns and dissipative systems: beyond the Goodwin model

The lack of an explicit endogenous dynamics of technical change in the Goodwinian tradition implies the model being silent upon one of the main sources driving economic progress - typical of modern capitalism (Schumpeter, 1942). Indeed, a long research tradition has studied and modelled technical change with a *complex system* perspective (Arthur, 2009; Dosi and Virgillito, 2021; Dosi et al., 2022), emphasizing the role of evolution and increasing returns on knowledge accumulation to explain the very process of growth. In addition, the joint hypotheses of exogenous technical change and of Say's Law - according to which capitalists savings are immediately reinvested - make the Goodwinian model a conservative system, in line with the Lotka-Volterra framework (Lorenz, 1993).

At the opposite, socio-economic systems are rarely conservative and technical progress is exactly one of the main reason of violation of such dynamics. Indeed, endogenous technical progress allows the system to grow and generate more resources than those used. In this respect, the final output tends to be more than the used inputs, with a typical (anti) dissipative structure, whereby with (anti) dissipative systems we consider those endogenously generating activity, or energy from a physics perspective. Economies seen as complex evolving systems are therefore better characterised by such type of structures, wherein, beyond regular cycles, also out-of-equilibrium dynamics, as bifurcations and chaos, do emerge (Nicolis, 1977; Dosi et al., 2015).

By preserving the predator-prey setup, we insert an endogenous source of technical change into the unstable, cyclical and conflicting dynamics of the Goodwin model turning the system into a *dissipative* one. Endogenous technical change might however be both a *dis-equilibrating* (Arrow, 1996; Dosi and Nelson, 2010) and a *coordinating* force. From the Schumpeterian perspective of arrival of new technologies and paradigms, endogenous technical change creates destruction, more or less creative, but with an end result of reorganizing the system. From a Kaldor-Verdoorn (henceforth KV) perspective (Kaldor, 1966, 1972), technical change, rather than being a dis-equilibrium force, tends to create quite ordered dynamic increasing returns. Increasing returns occur in those sectors of activity more exposed to demand growth, therefore they are rather systematic, more than erratic. In fact, the KV law is based on a dynamic principle formalizing the role of aggregate demand as a driver of labour productivity growth. Therefore, beyond supply-side, investment formation and capital accumulation, productivity growth emerges out of demand.

In line with the KV law, we introduce an endogenous source of technical change making the dynamics of labour productivity dependent on the employment rate. Indeed, the employment rate can be considered a proxy of the general level of activity, being a procyclical and coincident variable with output. Therefore, an increase in the employment rate due to a phase of expansion of the cycle, and therefore driven by an increase in production (and so by the capital accumulation), stimulates the accumulation of knowledge and feeds the dynamics of continuous learning. These processes are indeed at the core of increasing returns that, according to Smith (Kaldor, 1972; Young, 1928), stimulate the rate of growth of labour productivity through the generation of economies of scale.

During an expansionary phase of the business cycle, firms will increase their demand for labour, with an increase in the employment rate. Compared to the basic Goodwin model, an increase in the employment rate will not only lead to an increase in workers' real wages but, at the same time, will have a positive impact on labour productivity growth. Such positive influence on the rate of productivity is expressed through the KV coefficient (learning coefficient) capturing the effects of increasing returns and economies of scale in the economy. The coefficient value – corresponding to a degree of dependence on increasing returns of the whole economy – modulates the impact on the growth of labour productivity.

Labour productivity dynamics becomes part of the class struggle: considering the conflicting nature of capitalism and the co-existence of increasing returns underlying the learning dynamics of the system, different degrees of labour market elasticity (wage elasticity) may affect labour productivity dynamics and ensuing income distribution. The conflict is therefore not anymore over an exogenous produce but over an endogenous one. In that, we link a (anti) dissipative learning dynamics with a perpetual class struggle between capitalists and workers, substantiated in the symbiotic relationship between income distribution, strength of the labour market and technical change.

3 The Model

Before introducing the new model specification, let us briefly recall the main assumptions and model specification of the original model.

3.1 The Goodwin model

The main assumptions of the model read as follows:

- Two economic forces: *employment effect* and *profits effect*.
- A constant productivity growth rate $\alpha > 0$
- A constant population growth rate $\beta > 0$.
- A constant capital/output ratio $\sigma > 0$.
- Wages are entirely spent, profits are entirely saved and reinvested.
- Output growth rate equals profit rate.
- The equilibrium growth rate will be equal to a *natural* growth rate given by the sum of population and productivity growth rates.
- Technical progress is assumed to be Harrod neutral.
- All quantities are real.

Productivity growth rate a, defined as output per capita, grows according to the following specification:

$$\dot{a}/a = (\dot{q}/\dot{l})/(q/l) = \dot{q}/q - \dot{l}/l = \alpha \tag{1}$$

Labour demand growth rate is defined as:

$$(\dot{l}/l) = (1-u)\sigma - \alpha \tag{2}$$

Employment growth rate is defined as:

$$(\dot{v}/v) = (1-u)\sigma - (\alpha + \beta) \tag{3}$$

The positive relation between real wages and employment is expressed by means of a linearized Phillips Curve[1]:

$$(\dot{w}/w) = -\gamma + \rho v \tag{4}$$

The final equations are expressed in terms of the employment rate v and the share of wages u:

$$\dot{v} = [(1/\sigma - (\alpha + \beta)) - 1/\sigma u]v \tag{5}$$

[1] It lies in between the Phillips Curve and the so called Wage Curve. The last one is a real relation between the *levels* of the wage rate and the unemployment rate

$$\dot{u} = [-(\alpha + \gamma) + \rho v]u \tag{6}$$

We can rewrite the system in the following way, obtaining the Lotka-Volterra type of formulation:

$$\dot{v} = (a - bu)v \tag{7}$$

$$\dot{u} = (-c + dv)u \tag{8}$$

3.2 Introducing increasing returns: a three dimensional predator-prey model

In the following, we present the basic structure of our three-dimensional predator-prey model. We start with the endogenous dynamics of technical change, according to the KV law, which reads as follows:

$$\frac{\dot{a}}{a} = \alpha' v, \quad \alpha' > 0 \tag{9}$$

Labour demand growth is equal to:

$$\frac{\dot{l}}{l} = \frac{1-u}{\sigma} - \alpha' v \tag{10}$$

Hence, the employment growth rate is given by the following dynamic equation:

$$\frac{\dot{v}}{v} = \frac{1-u}{\sigma} - \beta - \alpha' v, \quad v \in [0,1] \tag{11}$$

Differently from the original model, employment rate change now negatively depends on the level of employment rate in itself, implying that its growth trajectories over time are *anchored* to a level of employment reflecting the influence of an endogenous force of technical change. This is the first implication of the endogenous dynamics in technical change.

Conversely, the pure dynamics of wage growth is not affected by the introduction of an endogenous component in technical change:

$$\frac{\dot{w}}{w} = -\gamma + \rho v, \quad \gamma, \rho > 0 \tag{12}$$

However, the change in the wage share is affected by the new specification, in fact income distribution now depends on the difference between wage elasticity to unemployment, a proxy of the strength of the collective bargaining, and the rate of endogenous productivity $(\rho - \alpha')$:

$$\frac{\dot{u}}{u} = -\gamma + (\rho - \alpha')v, \quad u \in [0,1] \tag{13}$$

In our version, the overall stylised representation of the modern capitalism is described by a three-dimensional non linear dynamical system including an endogenous source of technical change, labour markets and income distribution. The system obtained by combining the three equations (9), (11) and (13), reads as:

$$\begin{cases} \frac{\dot{a}}{a} = \alpha' v \\ \frac{\dot{v}}{v} = \frac{1-u}{\sigma} - \beta - \alpha' v \\ \frac{\dot{u}}{u} = -\gamma + (\rho - \alpha')v \end{cases} \tag{14}$$

3.3 Stability analysis

In the following stability analysis we focus on the topological behaviour of equations (11) and (13), being equation (9) an ever-increasing process with no meaningful fixed point (Arrow, 1996).

System (14) presents three fixed points:

$$\begin{aligned} (\alpha_1'^*, v_1^*, u_1^*) &= (0,0,0) \\ (\alpha_2'^*, v_2^*, u_2^*) &= \left(0, \frac{1-\beta\sigma}{\alpha'\sigma}, 0\right) \end{aligned} \tag{15}$$

and

$$(\alpha_3'^*, v_3^*, u_3^*) = \left(0, \frac{\gamma}{\rho - \alpha'}, \frac{\rho - \alpha' - \alpha'\gamma\sigma - \beta\sigma\rho + \alpha'\sigma\beta}{\rho - \alpha'}\right) \tag{16}$$

Considering that two of the three state variables, employment rate and wage share, are limited within the square $[0,1]^2$, then their stationary points must also lay in the $[0,1]$ interval:

$$\gamma > 0 \wedge \rho \neq \alpha' \wedge \alpha' \leq \rho - \gamma \tag{17}$$

$$\alpha' > \frac{\beta\sigma\rho - \rho}{\sigma\beta - \gamma\sigma - 1} \wedge \alpha' \leq \frac{\beta\rho}{\beta - \gamma} \tag{18}$$

The study of the local dynamic properties of the fixed points is based on the Jacobian matrix of the dynamic system (14). In any generic point, the Jacobian matrix has the following specification:

$$\mathcal{J} = \begin{bmatrix} \frac{\alpha'\gamma}{\rho-\alpha'} & 0 & 0 \\ 0 & \frac{\alpha'-\rho-\alpha'\beta\sigma+\alpha'\gamma\sigma+\beta\rho\sigma}{\sigma(\rho-\alpha'} + \beta - \frac{2\alpha'\gamma}{\rho-\alpha'} - \frac{1}{\sigma} & -\frac{\gamma}{\sigma(\rho-\alpha'} \\ 0 & \alpha' - \alpha'\beta\sigma + \alpha'\gamma\sigma + \beta\rho\sigma - \rho & \gamma - \frac{\alpha'\gamma}{\rho-\alpha'} + \frac{\gamma\rho}{\rho-\alpha'} \end{bmatrix} \tag{19}$$

The eigenvalues λ_i are the roots of the *characteristic equation* $|\mathbf{A} - \lambda\mathbf{I}| = 0$. In a three dimensional continuous system, the eigenvalues associated to the Jacobian matrix (19) are three and can assume the following behaviour[2]:

1. **Monotonic convergence towards the fixed point**: if all eigenvalues λ_i are real and lower than zero. The fixed point is asymptotically stable.

2. **Dampening convergence towards the fixed point**: if there is at least a pair of complex conjugate eigenvalues $\lambda_k, \bar{\lambda}_{k+1}$ and the real parts of all $\lambda_k \in \mathbb{C}$, $\mathfrak{Re}(\lambda_k)$ are lower than zero. The stationary point is asymptotically stable.

3. **Monotonic divergence**: if all eigenvalues are real and strictly greater than zero, then the system diverges monotonically toward $+\infty$ or $-\infty$. The stationary point is unstable.

4. **Saddle point**: if $\lambda_{j,k} \in \mathbb{R}$ but some $\lambda_j > 0$ and some $\lambda_k < 0$.

5. **Unstable focus**: if there is a pair of eigenvalues complex conjugates, $\lambda_k, \bar{\lambda}_{k+1} \in \mathbb{C}$, whose real part is greater than zero, $\mathfrak{Re}(\lambda_k) > 0$, then the system produces diverging oscillations.

6. **Stable focus**: if the real part of complex conjugates eigenvalues is lower than zero, $\mathfrak{Re}(\lambda_k) < 0$, the system has converging oscillations.

7. **Center**: given a pair of complex conjugate eigenvalues, if $\mathfrak{Re}(\lambda_k) = 0$, the system exhibits constant oscillations.

According to these conditions, the fixed points (15) are saddle points, so they do not have a noteworthy economic significance. The only stationary point worthy of attention is the (16), which can be locally asymptotically stable, unstable or a center around which the trajectories of the system (14) infinitely oscillate.

[2] We restrict the properties in $\mathbb{R}^2$ considering the trivial nature of one fixed-point.

In addition, changes in critical parameters entering the fixed points closed form may also result in topological changes giving rise to the emergence of local bifurcations (Lorenz, 1993; Orlando et al., 2021). Differently from the original model, the technical change coefficient enters the closed form solutions and therefore also the eigenvalues associated to the Jacobian matrix (19). Hence, technical change has an explicit effect on the system stability (14).

4 Simulations

In order to study the possible changes in the topological structure of the model, we perform a battery of simulations on two critical parameters, namely the learning rate, α', and the wage elasticity to unemployment, ρ, on the macrodynamics of the system, considering the analysis of eigenvalues and the possible birth of local bifurcations. The variations of the intensity of these two coefficients might be considered alternative configurations of the macroeconomic system, with different degrees of formality in the economy, represented by the intensity of the learning coefficient, and degrees of labour market rigidity.

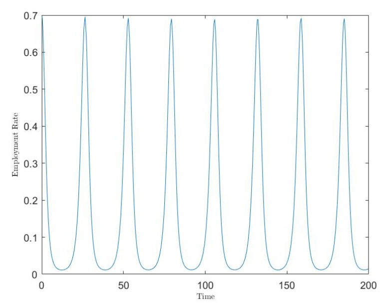
Table 1 presents the baseline coefficients of the model, and the range of variations we are going to consider, also given the restrictions in conditions (17) and (18).

Parameters of the Baseline Model		
Parameters	Baseline Value	Variation Range
α'	0.001	0.001-0.2
σ	0.4	-
β	1	-
γ	0.05	-
ρ	0.3	0.1-0.5

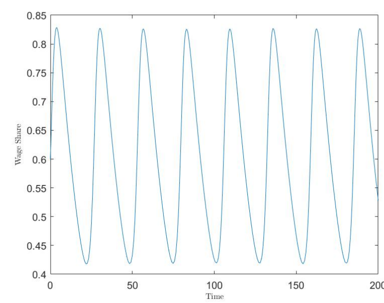
Table 1: Baseline Parameter Values and Relative Range of Variation

4.1 Stages of development: variation of α'

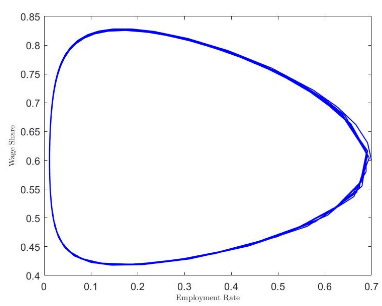
We start by asking what is the effect of the increase in the KV α' coefficient on the dynamics of modern capitalism, hereby represented by the three-dimensional predator-prey model. We interpret the learning coefficient of the KV law as a process that differently manifests along different stages of capitalist development, and therefore associated with distinct phases of modern capitalism. Our interpretation derives from the fact that learning regimes and economies of scale are typical of the manufacturing sector, a sector that characterises industrialising capitalist systems exiting from informal, putting-out, agricultural systems.



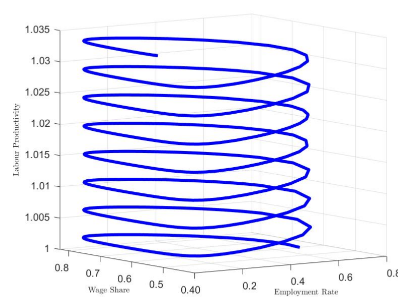
(a) Employment Rate Time Series



(b) Wage Share Time Series



(c) Limit Cycle



(d) Oscillatory Loop in a 3D Space

Figure 1: Macrodynamics with α'=0.001

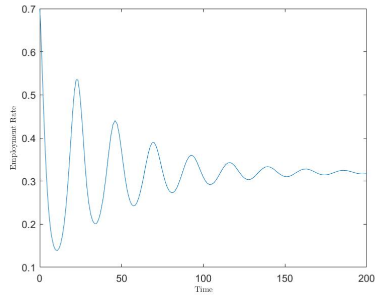
In our baseline setup, shown in figure 1, the typical nature of centre of the equilibrium point a' la Goodwin is presented, with a perpetual oscillatory dynamics in both wage share and employment rate. Being the value of the learning coefficient dramatically low, vis-à-vis our range of parameter variations, simulation results show that economies characterised by a poor dependence on increasing returns exhibit strong macroeconomic instability and business cycle volatility, manifested through the oscillatory dynamics in income distribution. The low, almost zero, elasticity of productivity growth to employment dynamics, proxy of the level of economic activity, means that the economic system is not able to fully exploit economies of scale.

Therefore, learning by doing processes are such irrelevant that they do not allow the generation of increasing returns for labor productivity growth in the long run. Indeed, the model maintains the typical conservative structure of the Lotka-Volterra model.

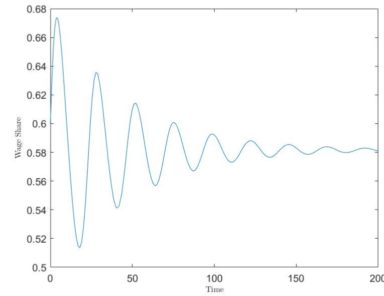
Employment rate and wage share, shown in figures 1(a) and 1(b) respectively, manifest a structural weakness: indeed, low elasticity of productivity to employment dynamics exacerbates the unstable and conflicting nature of capitalism, because the latter is not able to sufficiently stimulate the endogenous engine of growth. Therefore, within the predator-prey skeleton, a slow dynamism in labour productivity fuels class struggle and makes the system even more unstable. Notably, the amplitude of oscillations are quite remarkable, with strong upswing and downswing phases. High amplitude of cycles imply large perpetual instability, with a large range of potential values that both variables can reach. In fact, the lack of any transfer of levels of output (employment) toward productivity makes wages compressed by profits, and the ensuing conflicting dynamics strongly exacerbated.

The absence of an endogenous engine capable of fuelling the whole system in the long run not only contributes to making the economy unstable, but at the same time does not allow a proper *coordination* between the components of the system. Low learning regimes coexist with a persistent unstable and cyclic loop in labour productivity, as shown in figure (1(d)). Due to learning processes not able to generate sufficient increasing returns, the macroeconomic dynamic stops and locks in an *oscillatory trap* induced by a very low learning coefficient. Adopting the metaphor of the 'bicycle postulate' (Dosi and Virgillito, 2021), it is as if the slow learning dynamics, typical of a certain development stage of modern capitalism, are not able to generate enough *kinetic energy* to progress and move forward. This implies floating to infinity around a dynamic center, as in figure (1(c)), by exactly replicating goodwinian patterns.

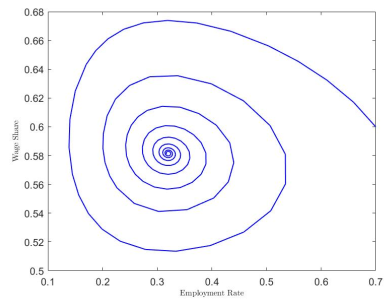
Let us now show the effects of changes in the learning coefficient. The following figures depict the impact on the temporal evolution of the employment rate, wage share and the two-dimensional phase portraits, respectively.



(a) Employment Rate Time Series



(b) Wage Share Time Series



(c) Unstable Focus

Figure 2: Macrodynamics with α'=0.2

Firstly, a qualitative change in the nature of the stationary point (16) from a center to an unstable focus, as shown in figure (2(c)), occurs. From a topological point of view (Arnold, 2012), the learning coefficient, α', is the bifurcation parameter of the system (14). Our analysis is concerned on the category of local bifurcations, that is topological changes of the system around the fixed points that can be analysed by linearisation (Kuznetsov, 1998). The phase transition from a dynamic center to an unstable focus is exactly due to the intensity of the learning parameter. The change in the topological structure is the opposite path with respect to the *Hopf Bifurcation*.[3]

[3] See appendix A for a formal exposition.

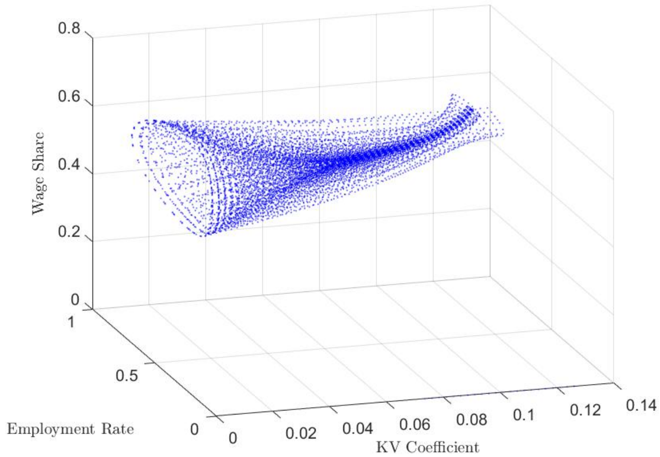


Figure 3: Hopf Bifurcation Diagram with respect to α'
Source: Our Simulation

Hopf bifurcation arises in presence of non-conservative dynamical systems (Lorenz, 1993). Consequently, the transition from a center to a focus for employment rate and wage share coupled dynamics is an evidence that, compared to the original Goodwin model, we have introduced an (anti) dissipative component for the final created and transmitted energy within the system, given by an endogenous technical change source.

The (anti) dissipative structure is however a source of stabilisation for the system. In fact, contrary to the conservative setup characterised by low levels of α', as the parameter increases, the sensitivity of the system to economies of scale induces a progressive change in the trajectories that tend to *converge* towards an unstable focus. Due to the loss of fragility and oscillatory properties as the bifurcation parameter increases, the introduction of an endogenous source of technical change helps to remove the trajectories of the system from a *corridor of stability* (Lorenz, 1993). Until the elasticity of the capitalist system with respect to economies of scale is very low, the conflicting and symbiotic dynamic of the two predator-prey components is persistently attracted by the limit cycles, this phase being marked by a less dark color at the beginning of the figure 3. Higher coefficients allow the fixed point to exit from the oscillatory trap.

A high elasticity of the economic system to scale economies is an attribute of a given, quite advanced, stage of development of modern capitalism, in line with the North-South literature. High elasticity to scale economies allows the entire system to benefit from increasing returns, which stimulate labour productivity, in line with the KV law (Kaldor, 1960; Fingleton and McCombie, 1998; McCombie et al., 2002; Deleidi et al., 2021). In such stage, increasing returns generated by learning by doing processes grant exponentially and persistently growing labour productivity.

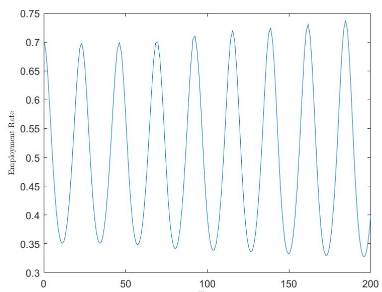
Sustained productivity growth allows employers to compensate for the increase in real wages following the expansion of the business cycle. The wage share tends to stabilise, as shown figure (2(b)). Consequently, the profit share will stabilise as well. In that, the class struggle dynamics is tamed and distributive shares will no longer be subject to wide fluctuations thanks to the effect of the endogenous source of technical progress, which indeed stabilises the whole economic cycle and allows to escape from the purely oscillatory and unstable dynamics. At the same time, the employment rate (2(a)) also tends to stabilise and looses persistent fluctuations, endemic of an economy marked by low learning coefficients.

In terms of empirical counterpart, our results are in line with *Kaldorian* stylised facts (Kaldor, 1961), according to which distributive shares of wages and profits are constant over the long run, and not perpetually oscillating as predicted by the predator-prey model. The Kaldorian stylised facts emerged in a historical phase wherein the stimulus to growth and coordination originated from the manufacturing sector, the *engine of growth* (Kaldor, 1960). Indeed, the way-out from the classical Goodwin phase occurs by means of a phase transition due to an endogenous source of technical change such as the KV law. Increasing returns are able to smooth the cycle, promoting a coordination between its elements, although imperfect due to the unstable nature of the node (Dosi and Orsenigo, 1988; Dosi and Virgillito, 2021).

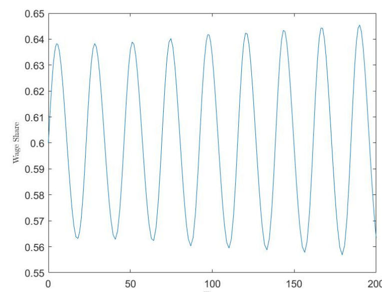
Beyond constant distributive shares over the long run, the introduction of increasing returns allows to regulate and depict different stages of economic development, just varying the intensity of the learning coefficient. Indeed, the change of the topological nature of the fixed point with respect to higher levels of α' allows to distinguish different phases of capitalist development, characterised by different degrees of stability/instability and macroeconomic fragility (Aglietta, 1976; Boyer and Saillard, 2005). Moreover, it emerges a stabilisation of income distribution dynamics thanks to an energy-pushing technical change, the latter allowing to maintain coupled the joint dynamics between real wages and labour productivity. Technical change has therefore a beneficial stabilising effect on functional income distribution, meaning that gains from productivity growth are not appropriated by capitalists, but are rather transferred into wages, due to the fact that employment level stabilises and anchors employment growth.

4.2 Flexibility in labour markets: variation of ρ

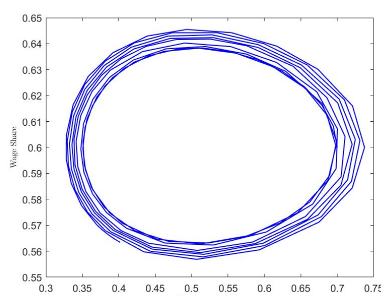
Let us now analyse the impact on the macroeconomic dynamics of a change in the coefficient capturing the elasticity of real wages relative to the employment rate, that is the coefficient of the quasi Phillips Curve (from here on, PC). We consider the intensity of the elasticity of wage changes to employment level a sort of *thermometer* of strength (weakness) of class conflict, in terms of the exposition of wages to variations in employment levels. In the following, we present a transition dynamics from a *quasi*-inelastic labour market to a setup characterised by high levels of elasticity. Clearly, low levels of ρ stand for high labour power, while at the opposite, high levels of ρ stand for low labour power.



(a) Employment Rate Time Series

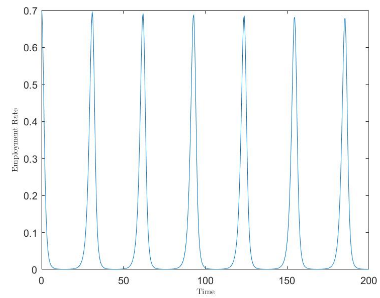


(b) Wage Share Time Series

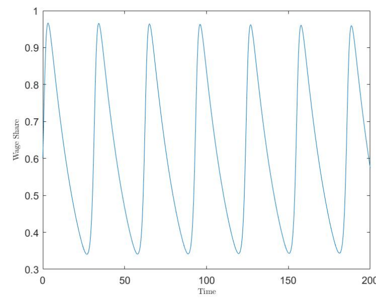


(c) Limit Cycle

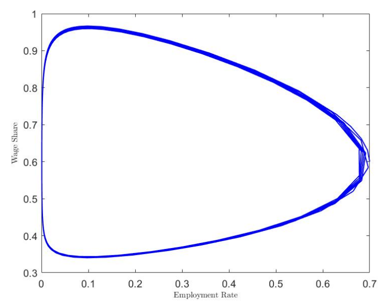
Figure 4: Macrodynamics with ρ=0.1



(a) Employment Rate Time Series



(b) Wage Share Time Series



(c) Limit Cycle

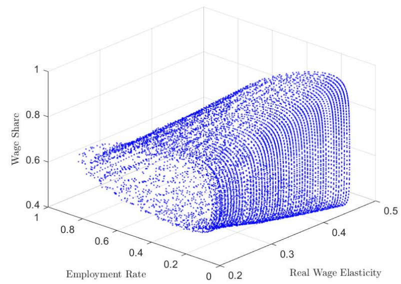
Figure 5: Macrodynamics with ρ=0.5

Figures (4(a)) and (5(a)) show the effect on the employment rate dynamics of the transition from a non-elastic to a more sensitive and elastic labour market. In the latter configuration, the labor market is very sensitive to business cycle fluctuations. Even in the absence of strong learning regimes, a lower wage elasticity reduces the amplitude of oscillations of the cycle, dampening volatility in employment rate, as figure (4(a)) shows. From the simulation results, it emerges that a low wage elasticity has a beneficial effect on the unstable cyclicality of the employment rate because it does not expose real wages to cyclical fluctuations. This setup ensures a higher employment rate, since it fluctuates in ranges of values overall more restricted compared to the opposite case, depicted in figure (5(a)). On the contrary, under higher wage elasticities, the amplitude of oscillations is so high that employment rate can reach the boundary values. This is due to a decline in real wage growth rate and to a further weakening of the share of income allocated to workers. Indeed, a very *elastic* labour market further discourages dynamism in recessionary periods and tends to overheat the dynamic path of the system exacerbating its macroeconomic fragility and cycle volatility.

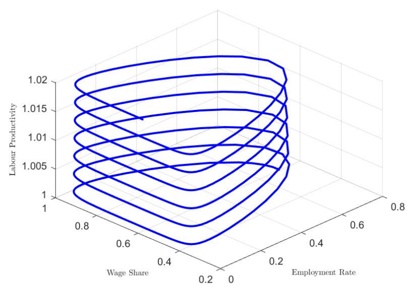
Figure (4(b)) shows that stronger wage rigidity guarantees a dynamic variation of the wage share within a small range. This lower volatility on the side of wage earners is consequently also reflected for capitalists. On the contrary, as shown in figure (5(b)), volatility and fluctuations of the wage share are much sharpened under high wage elasticity making the all system more unstable, including profits and their investment.

The transition from a low to a highly flexible labour market has an effect on the coupled dynamics of employment rate and wage share, whose trajectories in the two-dimensional space are represented in figures (4(c)) and (5(c)), in a rigid and a flexible setup respectively. The increase in flexibility causes a higher weakness of the system with the consequent increase of fluctuations and above all of the amplitude of the limit cycles, as depicted in figure (6), with a three-dimensional representation.

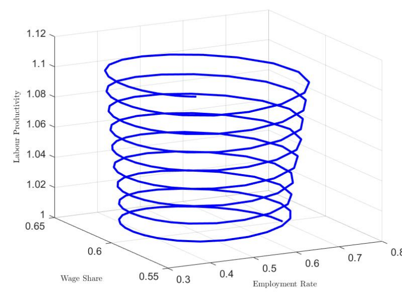
Wage flexibility does not necessarily guarantee a good coordination of the entire capitalist system which, instead, benefits from a certain degree of wage rigidity. In addition to being qualitatively consistent with some stylised facts of the post-war phase of modern capitalism (Boyer and Saillard, 2005; Dosi and Virgillito, 2019), our simulations are also in line with agent-based modelling results that have highlighted the detriment effects of labour market flexibilization on micro, meso and macroeconomic dynamics (Dosi et al., 2017, 2018a,b). In general, high degrees of labour power on the class struggle conflict appear to act as a crisis stabilisers.



(a) Amplitude of Oscillations



(b) Wage Flexibility



(c) Wage Rigidity

Figure 6: Amplitude of Oscillations in 3D Space

5 Feedback mechanisms to productivity growth

In this section we analyse the effects of alternative setups of both stages of development and degrees of labour market flexibility on the dynamics of labor productivity growth, in order to detect the feedback mechanisms from KV law and income distribution, in that fuelling economic growth.

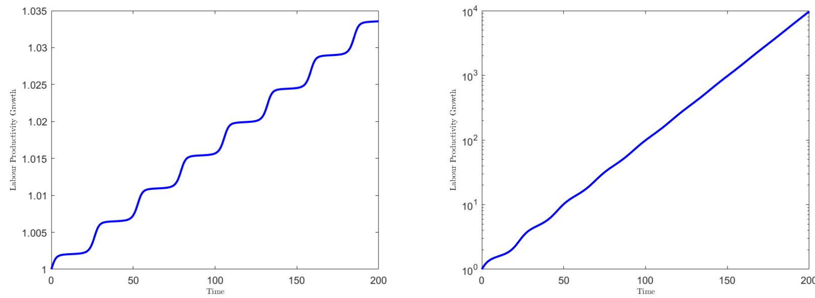
5.1 Learning regimes and productivity dynamics

In the following, we show the retroactive effects upon labour productivity of two different parametrizations of the learning coefficient.

Starting with the baseline parametrization, in figure (7(a)), under a low learning coefficient, productivity growth is not sufficiently stimulated by the KV effect. Low learning coefficients are not able to spur economic growth, due to low opportunities for increasing returns in the economic activity. In line with the analysis presented to so far, not only employment and income distribution but also productivity growth is stacked in the very initial stage of economic development. Notably, under low learning regimes the system is trapped into a form of looping hysteresis, that is, there is no escape from the oscillatory trap of the Goodwinian cycle. This latter topological structure represents bad lock-in and path-dependent patterns (David, 2000; Castaldi and Dosi, 2006; Setterfield, 2009). The positive retroactive effects are instead quite visible in figure (7(b)), where the dynamics of labour productivity is now presented in the case of a high learning coefficient. In this case we do observe a setup of stages of development in which employment rate reflects into high learning regimes and opportunity to growth. Therefore, the escape from the oscillatory trap, that allows to stabilise employment and income share, also induces higher opportunities for economic growth.

Which is the speed of learning in the system? Or better, how much time does the system take in order to benefit from learning regimes? We zoom in into the case of $\alpha' = 0.2$, by focusing on the first 200 steps. We first analyse the first 50 time steps in figure 8(a) and we then move to the whole range in figure 8(b). By comparing the two figures it emerges a *time-to-learn* effect in the system, which maps into a threshold behaviour reached at $t = 100$. Indeed, the step-wise dynamic that we detect in the baseline (figure (7(a))) and in the high learning regime specification until $t = 50$ are very similar among them. The system requires a given period of time to absorb the gains from such learning opportunities, and the dynamic becomes exponential after a given time-to-learn threshold is overcome.

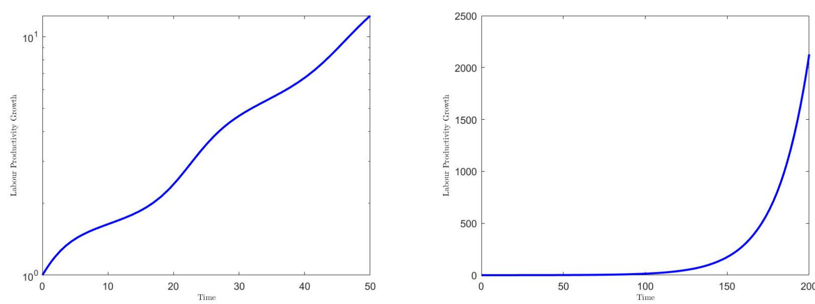
This time-to-learn effect mimics the time necessary to build increasing returns, and with a micro-level reference, the time required by workers to accumulate knowledge and know-how (Hartley, 1965; Dosi and Nelson, 2010). The time-to-learn is likewise the propagation speed to feed effects into the system (Young, 1928).



(a) $\alpha' = 0.001$

(b) $\alpha' = 0.2$

Figure 7: Labour Productivity Dynamics



(a) Step-wise

(b) Exponential

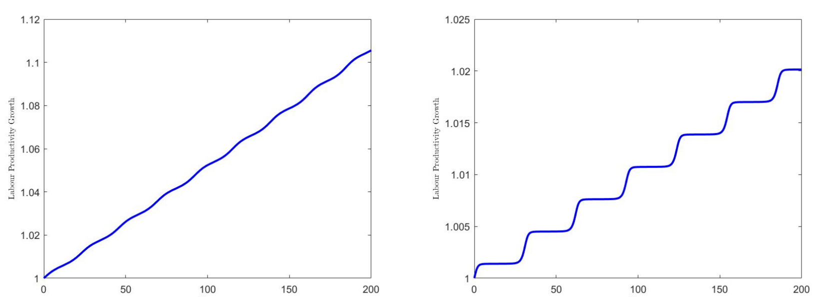
Figure 8: Time-to-learn threshold point in Labour Productivity, $\alpha' = 0.2$

5.2 Labour power and productivity dynamics

Let us now discuss the feedback effects from a rigid to a flexible labour market on productivity growth, whose parametrizations represent two different degrees of labour power in the economy. We present in figure (15) the dynamics of labour productivity under a rigid (figure 9(a)) and a flexible (figure (9(b))) labour market regime, keeping the other parameters at the baseline configuration. Under a more rigid wage growth dynamics vis-à-vis employment ($\rho = 0.1$), even under a baseline parametrization of low learning regimes ($\alpha' = 0.001$), wage rigidity prevents the labour market from being exposed to large cycle fluctuations. Therefore, although the chances to reap the benefits of increasing returns are quite low, wage rigidity, via dampening of oscillations, is more conducive to (relatively low) labour productivity growth, as shown by a rather smooth linear trend dynamics. The increase in productivity originates from the positive feedback mechanisms of small amplitudes in the cycle oscillations, e.g., from the pattern of income distribution pushing economic growth.

At the opposite, the case of high wage elasticity to employment dynamics is a volatility-fuelling setting, with the labour market more prone to business cycle fluctuations. Large oscillations in income distribution translate into large fluctuations in the employment rate, which tends to be less stable. The weak and oscillating employment dynamics causes a slowdown in productivity growth, because of low output accumulation and volatility in learning opportunities. Indeed, instability in income distribution maps into unstable productivity dynamics, which keeps growing but with a distinct, slow-moving step-wise dynamics.

Notably, positive and negative feedbacks from income distribution to productivity growth do not only manifest in the amplitude of oscillations but also in the final level of cumulated productivity reached during the same time period ($t = 200$), which under high labour power and low wage sensitivity to employment increases more than 10 p.p. compared to a 2 p.p. increase under the low labour power setting.



(a) $\rho = 0.1$ (b) $\rho = 0.5$

Figure 9: Labour Productivity Dynamics

5.3 Coordination setups and bifurcation regions

So far, we have performed a local, one at-the-time, parameter analysis. We now move to a two-dimensional bifurcation diagram in order to detect regions of the constellation parameters allowing to identify the nature of the system with respect to a two-dimensional parameter setup. The underlying question we address is what are the parameter constellations, and ensuing economic configurations, which allow to reach coordination patterns, defined as phases of cumulative and increasing growth, coupled with low degrees of fluctuations in income distribution.

Figure (10) shows the bifurcation regions where different configurations of the two parameters determine alternative dynamic trajectories of the system, as marked by colours. We detect a greater complexity than the original Goodwin model, with a division of the phase portrait into three areas, depending on the values of the elasticity of the labor market ρ (ordinate axis) and the learning coefficient α' (abscissa axis).

To briefly summarise, a higher elasticity to scale economies and increasing returns contributes to make convergent the entire dynamics towards an unstable focus, with ensuing effects upon the dampening of oscillations in employment rate and wage share. This implies an exponential productivity growth after a period of adaptation of the system to the impulse originated by increasing returns: the time to learn effect. At the same time, wage rigidity has a positive (and therefore dampening) effect on the oscillatory dynamics of employment rate and wage share, with a propagation feedback effect on labour productivity growth, through the long run KV relationship.

The two-parameter diagram allows to have a synthetic and joint look on both dynamics.

- Red area: for low values of α' (low dependence of the system on scale economies) and for values of ρ relatively low (wage rigidity), the macrodynamics of the system assumes a *Goodwinian pattern*, that is, presents persistent limit cycles around the dynamic centre. The emergence of such Goodwinian trajectories is due to low energy input from the endogenous source of technical change. This oscillatory dimension is found both in two- (employment rate vs wage share, figure (1(c))) and in three-dimensional space (oscillatory loop, (1(d))). Even with a low wage elasticity able to guarantee low fluctuations in the limit cycles, very low values of α' nail the system into a stage of capitalism characterised by fragility and cyclical instability. This proves that the parameter that induces a phase transition is α', the bifurcation parameter.

- Blue area: for constantly increasing values of α' and for low values of ρ (in some intervals of the diagram even lower than those of the red area), there is a topological change of the macrodynamics with the emergence of a bifurcation region. This region represents the combinations of wage elasticities and learning coefficients that give birth to Hopf Bifurcations: from an oscillatory dynamics (Goodwinian setup) to a more stable configuration that sees the phase transition from limit cycles to an unstable focus. As a result, positive feedback chains emerge from the interaction of low labour market elasticity (or high labour power), greater macroeconomic stability and sustained labour productivity growth, thanks to higher α' values that give the necessary stimulus to the system to grow and self-coordinate, albeit imperfectly, over the long run, out of the oscillatory trap. Hence, a form of coordination among labour market, income distribution and endogenous technical change emerges as a necessary condition for this macrodynamic regime.

- Black area: value combinations of the two parameters leading to explosive trajectories and divergence from any *quasi* stationary point, such as the dynamic center or the unstable focus. This area represents one of the drawback of the model which, being based on a Lotka-Volterra skeleton, remains characterised by an inherent structural instability.

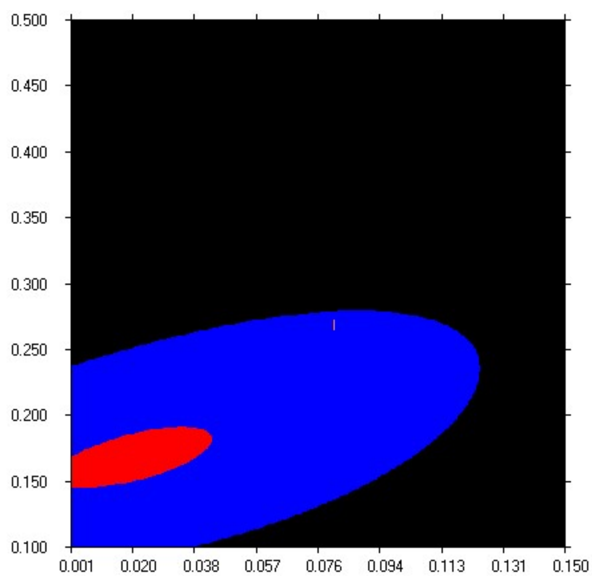


Figure 10: Bifurcation Regions

6 Conclusions

The scope of this work has been to study the joint interaction of three founding elements of modern capitalism, namely endogenous technical change, income distribution and labour markets, within a low-dimensional nonlinear dynamic setup. By going beyond the conservative structure typical of the predator-prey model, we inserted an endogenous source of energy, a Kaldor-Verdoon increasing returns specification, that fed the dynamics of the system over the long run and in that incorporated a transition to an (anti) dissipative framework.

Our extension allows to enrich the dynamics of the Goodwin model and to include a new element in the dynamics of modern capitalism, that is endogenous productivity growth. Firstly, our model is able to reproduce some typical dynamics of the Goodwin original model. For low values of the endogenous response coefficient of labor productivity, the limit cycles are kept. This reflects the unstable and conflicting nature of capitalism. At the same time, by increasing the learning coefficient, there is a gradual but constant transition towards a less oscillatory and fragile dynamics, attracted by an unstable focus. With respect to the original framework, productivity growth acts as a *kinetic force* that in the long run is able to keep together the unstable and conflicting elements of modern capitalism.

The qualitatively dynamics and ample array of topological structures reflect a wide range of Kaldorian stylised facts, as constant productivity growth and constant income distribution shares (Kaldor, 1961; Boyer and Saillard, 2005; Dosi and Virgillito, 2019). Indeed, the intensity of learning regimes and wage sensitivity to unemployment allow to mimic some typical traits of both Competitive and Fordist regimes of accumulation, and the eventual transition from one configuration to another. We show the relevance of the demand-side engine represented by the KV law, within an overall supply-side framework typical of the Goodwin model. High degrees of learning regimes stabilise the system and bring it out of an oscillatory trap. Even under a low degree of learning regimes, wage rigidity is able to stabilise the fluctuations of business cycle and to exert a positive effect on productivity growth.

Limitations of our model include, first, the lack of one-to-one realism, in terms of modeling dimensionality and counterpart empirical evidence, and, second, the continuous time setting, poorly appropriate to model discrete time decisions. Therefore, future advancements entail a discrete time version of the model (Goodwin, 1990; Dosi et al., 2015), together with the possible inclusion of a North-South gap and international trade structure, in order increase the dimensionality of the system and account for other relevant patterns of modern capitalism, such as specialisation and persistent structural asymmetries between countries/regions. Considering that some of the Kaldor stylised facts, as the constant functional income distribution, are loosing empirical relevance, it would be interesting to move beyond a predator-prey setting, and possibly allowing in the model for a declining wage share and an increasing profit share, as a result of the neoliberal turn, modelling therefore the structural weakening of labour power over the long run. Alternatively, the study of a counterbalancing dissipative effect acting against the KV law, like financialization and retained profits not invested, would be a relevant extension to include forms of rentified capitalism (Dosi and Virgillito, 2019). Finally, the modelling of sources of inflation, extending the model via a price equation and studying formation of wage-spiral vs profit-spiral configurations is a further avenue of research.

References

Aglietta, M. (1976). Régulation et crises du capitalisme: l'expérience des États-Unis. Paris : Calmann-Lévy.

Arnold, V. I. (2012). Geometrical methods in the theory of ordinary differential equations, volume 250. Springer Science & Business Media.

Arrow, K. J. (1996). Technical information and industrial structure. Industrial and corporate change, 5(2):645–652.

Arthur, W. B. (2009). Complexity and the economy. In Handbook of Research on Complexity. Edward Elgar Publishing.

Boyer, R. and Saillard, Y. (2005). Régulation theory: the state of the art.

Canry, N. (2005). Wage-led regime, profit-led regime and cycles: a model (french title for the publication: Régime wage-led, régime profit-led et cycles: un modèle). Économie appliquée: archives de l'Institut de science économique appliquée, 58(1):143–163.

Castaldi, C. and Dosi, G. (2006). The grip of history and the scope for novelty: some results and open questions on path dependence in economic processes. In Understanding change, pages 99–128. Springer.

David, P. A. (2000). Path dependence, its critics and the quest for 'historical economics'. Market Failure or Success, page 79.

Deleidi, M., Paternesi Meloni, W., Salvati, L., and Tosi, F. (2021). Output, investment and productivity: the italian north–south regional divide from a kaldor–verdoorn approach. Regional Studies, pages 1–12.

Desai, M. (1973). Growth cycles and inflation in a model of the class struggle. Journal of economic theory, 6(6):527–545.

Dosi, G. and Nelson, R. R. (2010). Technical change and industrial dynamics as evolutionary processes. Handbook of the Economics of Innovation, 1:51–127.

Dosi, G. and Orsenigo, L. (1988). Coordination and transformation: an overview of structures, behaviours and change in evolutionary environments. Technical change and economic theory, pages 13–37.

Dosi, G., Pereira, M. C., Roventini, A., and Virgillito, M. E. (2017). When more flexibility yields more fragility: The microfoundations of keynesian aggregate unemployment. Journal of Economic Dynamics and Control, 81:162–186.

Dosi, G., Pereira, M. C., Roventini, A., and Virgillito, M. E. (2018a). Causes and consequences of hysteresis: aggregate demand, productivity, and employment. Industrial and Corporate Change, 27(6):1015–1044.

Dosi, G., Pereira, M. C., Roventini, A., and Virgillito, M. E. (2018b). The effects of labour market reforms upon unemployment and income inequalities: an agent-based model. Socio-Economic Review, 16(4):687–720.

Dosi, G., Pereira, M. C., Roventini, A., and Virgillito, M. E. (2022). A complexity view on the future of work. Meta-modelling exploration of the multi-sector K+S agent based model. Technical report, Laboratory of Economics and Management (LEM), Sant'Anna School of Advanced Studies.

Dosi, G., Sodini, M., and Virgillito, M. E. (2015). Profit-driven and demand-driven investment growth and fluctuations in different accumulation regimes. Journal of Evolutionary Economics, 25(4):707–728.

Dosi, G. and Virgillito, M. E. (2019). Whither the evolution of the contemporary social fabric? new technologies and old socio-economic trends. International Labour Review, 158(4):593–625.

Dosi, G. and Virgillito, M. E. (2021). In order to stand up you must keep cycling: Change and coordination in complex evolving economies. Structural Change and Economic Dynamics, 56:353–364.

Dutt, A. K. (1992). Conflict inflation, distribution, cyclical accumulation and crises. European journal of political economy, 8(4):579–597.

Fingleton, B. and McCombie, J. S. (1998). Increasing returns and economic growth: some evidence for manufacturing from the european union regions. Oxford Economic Papers, 50(1):89–105.

Goodwin, R. M. (1967). A growth cycle. In Feinstein, C.H. (Ed.), Socialism, Capitalism and Economic Growth. Cambridge University Press, Cambridge.

Goodwin, R. M. (1990). Chaotic economic dynamics.

Hartley, K. (1965). The learning curve and its application to the aircraft industry. The Journal of Industrial Economics, pages 122–128.

Kaldor, N. (1960). Causes of growth and stagnation in the world economy. Cambridge University Press, Cambridge.

Kaldor, N. (1961). Capital accumulation and economic growth. In The theory of capital, pages 177–222. Springer.

Kaldor, N. (1966). Causes of the slow rate of economic growth of the United Kingdom: an inaugural lecture. Cambridge University Press.

Kaldor, N. (1972). The irrelevance of equilibrium economics. The Economic Journal, 82(328):1237–1255.

Kuznetsov, Y. (1998). Elements of applied bifurcation theory, volume 112. Springer.

Lorenz, H.-W. (1993). Nonlinear dynamical economics and chaotic motion, volume 334. Springer.

McCombie, J., Pugno, M., and Soro, B. (2002). Productivity growth and economic performance: essays on Verdoorn's law. Springer.

Nicolis, G. (1977). Self-organization in nonequilibrium systems. Dissipative Structures to Order through Fluctuations, pages 339–426.

Orlando, G., Pisarchik, A. N., and Stoop, R. (2021). Nonlinearities in Economics. Springer.

Pohjola, M. T. (1981). Stable, cyclic and chaotic growth: The dynamics of a discrete-time version of goodwin's growth cycle model. Zeitschrift für Nationalökonomie/Journal of Economics, 41(1/2):27–38.

Schumpeter, J. A. (1942). Capitalism, socialism and democracy. New York and London: Harper & Brothers Publishers.

Setterfield, M. (2009). Path dependency, hysteresis and macrodynamics. In Path dependency and macroeconomics, pages 37–79. Springer.

Shaikh, A. (2016). Capitalism: Competition, conflict, crises. Oxford University Press.

Velupillai, K. (1979). Some stability properties of goodwin's growth cycle. Zeitschrift für Nationalökonomie, 39(3-4):245–257.

Veneziani, R. and Mohun, S. (2006). Structural stability and goodwin's growth cycle. Structural Change and Economic Dynamics, 17(4):437–451.

Verdoorn, P. J. (1949). Fattori che regolano lo sviluppo della produttivita del lavoro. Ed. L'industria.

Wolfstetter, E. (1982). Fiscal policy and the classical growth cycle. Zeitschrift für Nationalökonomie/Journal of Economics, 42(4):375–393.

Young, A. A. (1928). Increasing returns and economic progress. The economic journal, 38(152):527–542.

A Eigenvalues Analysis

In the following appendix, we insert the formal treatment from a mathematical point of view regarding the extension of the model in section 3.

Firstly, let us present the formal treatment of Hopf Bifurcation when there is a three dimensional space. To deal with this specific topological change, let us provide the following Hopf theorem for a three-dimensional nonlinear dynamical system[4].

Theorem 1 (Hopf Bifurcation) *Consider the system of ordinary differential equations on an open set* $U \subseteq \mathbb{R}^n$*,*

$$\dot{x} = f(x, \mu)$$

where $x \in U$ *and* μ *is a real parameter varying in some open interval* $I \subseteq \mathbb{R}$*. Suppose that for each* μ *there exists an equilibrium point* $x^* = x^*(\mu)$*. Assume that the Jacobian matrix of* f *with respect to* x*, evaluated at* $x(\mu)$*, has a pair of complex conjugate eigenvalues,* $\lambda(\mu)$ *and* $\bar{\lambda}(\mu)$*, which satisfy the following (transversality conditions of the Hopf bifurcation):*

$$\mathfrak{Re}[\lambda(\mu_H)] = 0, \quad \mathfrak{Im}[\lambda(\mu_H)] \neq 0$$

$$\left. \frac{d\,\mathfrak{Re}[\lambda(\mu)]}{d\mu} \right|_{\mu=\mu_H} \neq 0$$

while $\mathfrak{Re}[\gamma(\mu_H)] \neq 0$ *for any other eigenvalues* γ*. Then, the system has a family of nonconstant, periodic solutions.*

Let us now consider the above theorem applied to the nonlinear system (14). Therefore, let us take into account the associated Jacobian matrix (19). Given the three-dimensional reference system, the characteristic polynomial associated with the Jacobian matrix has the following functional form:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where $\lambda \in \mathbb{R}$ denotes the characteristic roots and it is a scalar. Indeed, the characteristic roots are the eigenvalues of the Jacobian matrix. Each coefficient of the characteristic polynomial is given by:

$$a_1 = -tr(J) = -(J_{11} + J_{22} + J_{33})$$

[4] This version is adopted from (Orlando et al., 2021).

$$a_2 = \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & 0 \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{vmatrix} = J_{22} \cdot J_{33} - J_{23} \cdot J_{32} + J_{11} \cdot J_{33} + J_{11} \cdot J_{22}$$

$$a_3 = -det(J) = -\begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix}$$

The necessary and sufficient condition for the local stability of the fixed point (16) is that all characteristic roots of the polynomial (i.e. the eigenvalues of the Jacobian matrix) have negative real parts. From the Routh-Hurwitz condition, it is equivalent to say that:

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_1 \cdot a_2 - a_3 > 0$$

Consistent with the Hopf theorem and the results we obtain with respect to the specific fixed points nature presented in section 3, let us now report each individual eigenvalue analysis for each specific setup according to its parameterization.

A.1 Low learning coefficient, $\alpha = 0.001$

We start with the baseline case to which we have associated, in our interpretative analysis, a low level of learning. With a low α' value and all other parameter values equal to the baseline of table (1), the Jacobian matrix (19) assumes the following algebraic form:

$$J = \begin{bmatrix} 1/5980 & 0 & 0 \\ 0 & -1/5980 & -125/299 \\ 0 & 8969/50000 & 0 \end{bmatrix}$$

Since the associated Jacobian matrix has dimension 3x3, the relative associated eigenvalues are three.

In matrix (A.1) it is possible to notice that the trace is equal to zero. This in $\mathbb{R}^2$ would be sufficient to affirm the presence of limit cycles around the stationary point; however, in $\mathbb{R}^3$ there might be a couple of complex conjugated eigenvalues, of the type $\lambda_{k,k+1} = \alpha \pm \beta i$.

The eigenvalues associated with the matrix are the following three:

$$\lambda_1 = 1/5980; \lambda_2 = -(\sqrt{10726923} \cdot 1i)/11960 - 1/11960; \lambda_3 = (\sqrt{10726923} \cdot 1i)/11960 - 1/11960$$

The real part of the pair of conjugated complex eigenvalues is almost null and numerically equal to:

$$\Re\mathfrak{e}(\lambda_{2,3}) = -1/11960 = 0,000083 \Longrightarrow \Re\mathfrak{e}(\lambda_{2,3}) \simeq 0$$

Being the real part of the conjugated complex eigenvalues almost zero implies that the oscillations of the trajectory of the system are stationary over the long run and have a regular and constant amplitude. In other words, from an economic point of view, it means that fluctuations in state variables, such as employment rates and wage shares, are persistent and do not appear to be decreasing. This explains the dynamics of the employment rate and wage share in figures (1(a)) and (1(b)), just as it is possible to show the presence of limit cycles concerning precisely the joint dynamics of these two state variables (1(c)) to which are associated the complex and conjugated eigenvalues with real part equal to zero.

A.2 High learning coefficient, $\alpha = 0.2$

In this case we show the calculations relating to the variation of the coefficient of KV.

First we show the Jacobian matrix related to this specific setup.

$$J = \begin{bmatrix} 1/50 & 0 & 0 \\ 0 & -1/50 & -1/4 \\ 0 & 111/625 & 0 \end{bmatrix}$$

We now calculate the eigenvalues associated with the matrix.

$$\lambda_1 = 1/50; \lambda_2 = 1/100 + (\sqrt{443} \cdot 1i)/100; \lambda_3 = 1/100 - (\sqrt{443} \cdot 1i)/100$$

The amplitude of the oscillations, as we have also in the previous case, depends on the real part of the complex and conjugated eigenvalues. In this case, the fact that the real part of the eigenvalues is positive tells us that we are faced with an unstable focus, as we have shown in the figure (2(c)) regarding the joint dynamics of the employment rate and the wage share.

A.3 Low Wage Elasticity, $\rho = 0.1$

In this circumstance we analyse a variation of the elasticity of real wages. The Jacobian matrix with reference to the baseline values of the table (1) is:

$$J = \begin{bmatrix} 1/1980 & 0 & 0 \\ 0 & -1/1980 & -125/99 \\ 0 & 2969/50000 & 0 \end{bmatrix}$$

The associated eigenvalues are the following:

$$\lambda_1 = 1/1980; \lambda_2 = -(\sqrt{1175723} \cdot 1i)/3960 - 1/3960; \lambda_3 = (\sqrt{1175723} \cdot 1i)/3960 - 1/3960$$

If we take into account the real part of the conjugated complex eigenvalue pair, it is close to zero. Consequently this means that the oscillations of the variables, and more generally the fluctuations of the trajectory of the system, are persistent and constant, as it is possible to show in the figure 6(b).

$$\Re\mathfrak{e}(\lambda_{2,3}) = 1/3960 = 0,000252 \Longrightarrow \Re\mathfrak{e}(\lambda_{2,3}) \simeq 0$$

A.4 High Wage Elasticity, $\rho = 0.5$

$$J = \begin{bmatrix} 1/9980 & 0 & 0 \\ 0 & -1/9980 & -125/499 \\ 0 & 14969/50000 & 0 \end{bmatrix}$$

The associated eigenvalues are as follows:

$$\lambda_1 = 1/9980; \lambda_2 = -(\sqrt{29878123} \cdot 1i)/19960 - 1/19960; \lambda_3 = (\sqrt{29878123} \cdot 1i)/19960 - 1/19960$$

Also in this case the real part of the couple of the conjugated complex eigenvalues is null and this means that the fluctuations of the trajectory of the system are persistent and constant over time. Indeed, as in the case of the rigid labour market, we show the persistent fluctuations in the figure 6(c) where it can be noted that compared to the previous case the size of the closed orbits is much greater indicating a greater fragility of the economy as a result of the process of flexibility of the labour market.

$$\mathfrak{Re}(\lambda_{2,3}) = 1/19960 = 0,000050 \Longrightarrow \mathfrak{Re}(\lambda_{2,3}) \simeq 0$$

B Robustness Checks

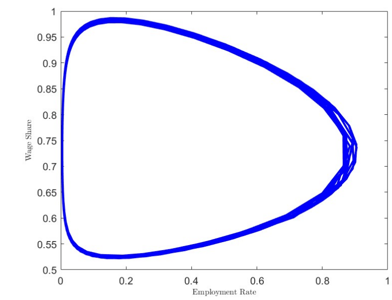
We report some robustness checks on the other parameters of the model. Although we have proven that the parameters of relevance are α' and ρ, these robustness checks complete the whole parameter analysis of the system. Table 2 shows the range of variation from the baseline setting. Below, we report the simulations. The values chosen are such that the boundary conditions are always satisfied, so that the state variables, especially employment rate and wage share, do not lose economic significance. For each of the parameters of interest we perform a simulation exercise at high and low levels of the range of variation.

Regarding the capital-output ratio σ, although analysed at its boundary conditions, the topological nature of the system keeps assuming the original Goodwinian patterns as shown in figure 11 in which the system is trapped into an oscillatory loop. Regarding the rate of exogenous population growth (see figures 12), the oscillatory and unstable à la Goodwin dynamic continues to persist, with no effect on productivity dynamics and in general on possible phase transitions. Finally, for the parameter γ, shown in figure 13, the intercept of the *quasi* PC, starting from a floor value in the baseline scenario, is stretched until 0.25. Even at this juncture, the oscillatory and highly unstable nature of the macrodynamics of the system persists, with a slight difference in the amplitude of the oscillations, now less marked than the baseline scenario.

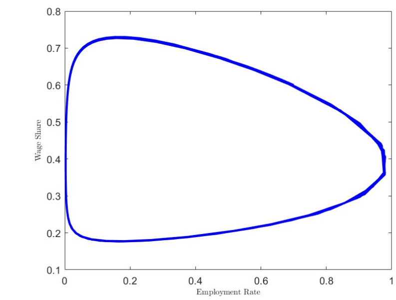
After carrying out this robustness analysis, we confirm that the macrodynamic behavior à la Goodwin persists even in the above scenarios. The topological nature of the system is not unaltered in a structural way (Veneziani and Mohun, 2006), regardless of the values assigned to the other parameters. Therefore, the learning coefficient $\alpha^{'}$, actually the bifurcation parameter of the system, is the only one governing the phase transition due to the insertion of a non-conservative force, which causes the emergence of Hopf bifurcations. Notably, the three parameters, rather than affecting the topological structure, affect the path of productivity growth, which can exert both step-wise versus smoother linear trends, vis-à-vis the alternative parameter settings.

Parameters of the baseline model		
Parameters	Baseline Value	Variation Range
σ	0.4	0.27-0.61
β	1	0.75-1.5
γ	0.05	0.05-0.25

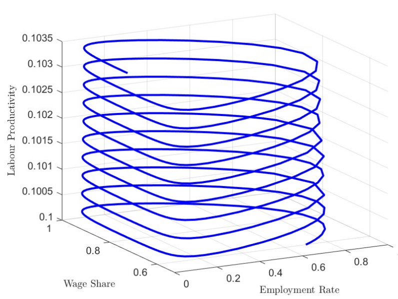
Table 2: Range of parameters variation



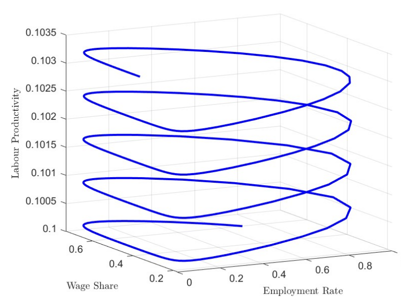
(a) Limit Cycle



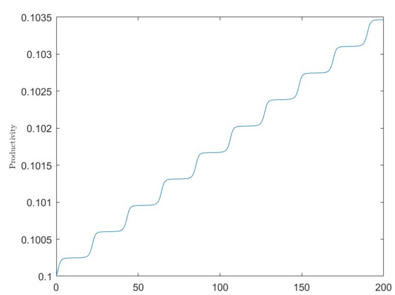
(b) Limit Cycle



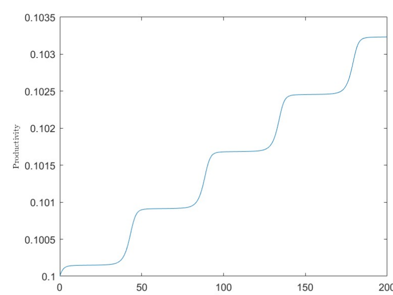
(c) Dynamics in a 3D Space



(d) Dynamics in a 3D Space

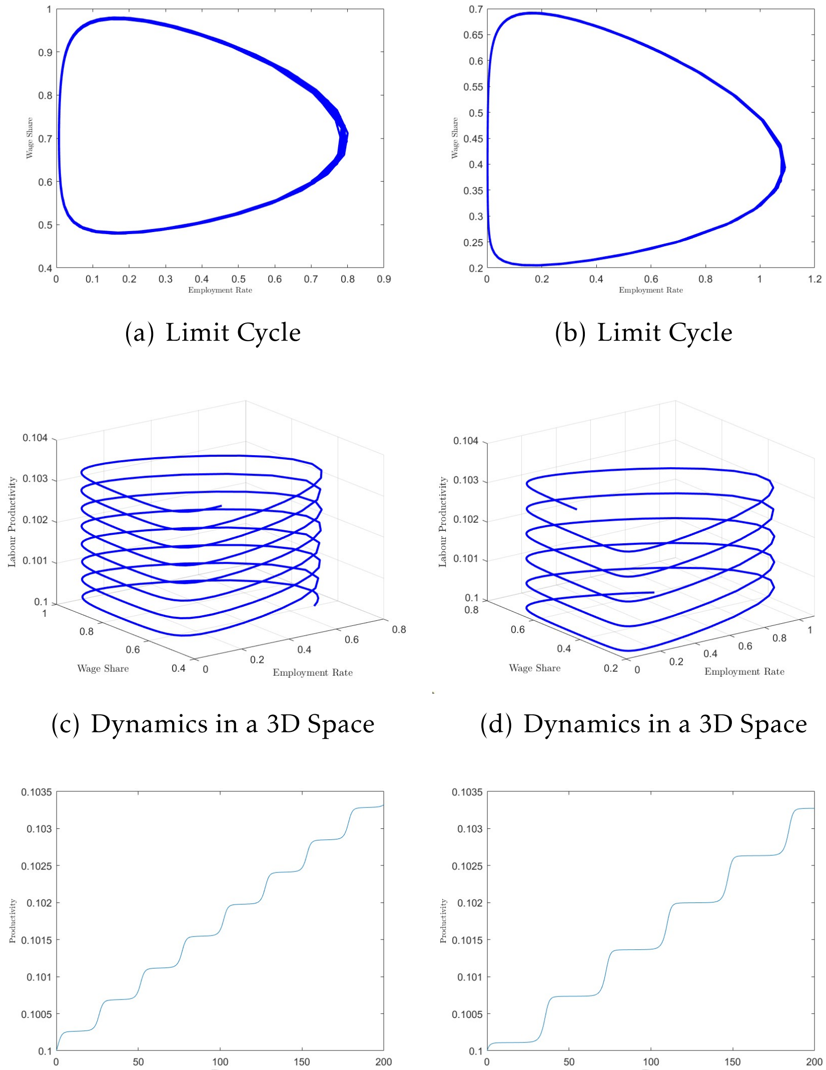


(e) Productivity Dynamic



(f) Productivity Dynamic

Figure 11: Macrodynamics with low (0.27) and high (0.61) values of σ

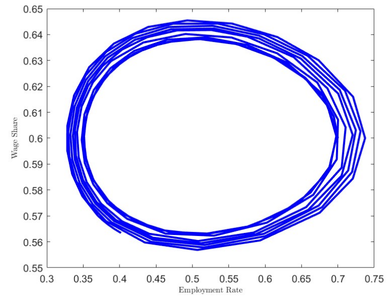


(a) Limit Cycle (b) Limit Cycle

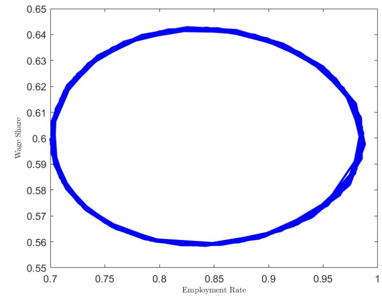
(c) Dynamics in a 3D Space (d) Dynamics in a 3D Space

(e) Productivity Dynamic (f) Productivity Dynamic

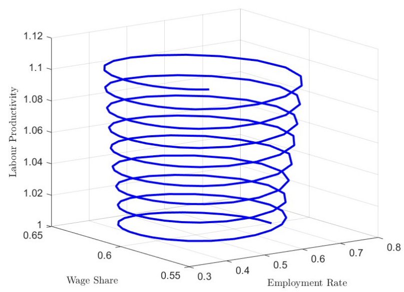
Figure 12: Macrodynamics with low (0.75) and high (1.5) values of β



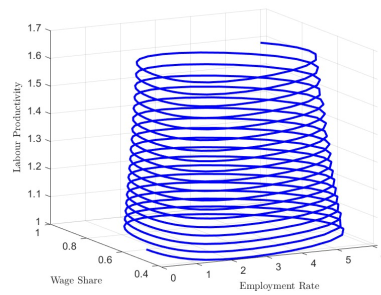
(a) Limit Cycle



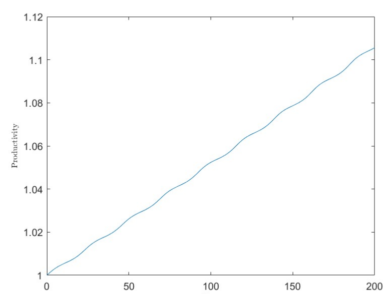
(b) Limit Cycle



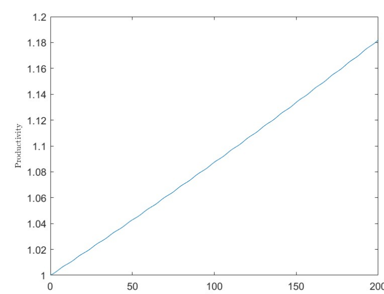
(c) Dynamics in a 3D Space



(d) Dynamics in a 3D Space



(e) Productivity Dynamic



(f) Productivity Dynamic

Figure 13: Macrodynamics with low (0.05) and high (0.25) values of γ