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The Excedance Set of a Permutation

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The *excedance set* of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is the set of indices i for which $\pi_i > i$. We give a formula for the number of permutations with a given excedance set and recursive formulas satisfied by these numbers. We prove log-concavity of certain sequences of these numbers and we show that the most common excedance set among permutations in the symmetric group $\mathcal{S}_n$ is $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. We also relate certain excedance set numbers to Stirling numbers of the second kind, and others to the Genocchi numbers.

1. INTRODUCTION

The theory of permutation statistics has a long history and has grown at a rapid pace in the last few decades. Two among the classical statistics are the number of descents and the number of excedances in a permutation. They were first studied by MacMahon [17] 100 years ago, and they still play an important role in the field. Of these, the number of descents has received the most attention, perhaps because the definition of descent generalizes to an arbitrary Coxeter group. Moreover, the descent set of a permutation has intriguing algebraic properties [1, 12, 22], as well as being a beautiful example of the theory of lattice path enumeration; see the work of Gessel and Viennot [13].

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Although the concepts of excedance and descent are closely related and can be considered as mirror images of each other, the story is quite different when it comes to descent sets versus excedance sets. The descent set of a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ is the set of indices i for which $\pi_i > \pi_{i+1}$, whereas the excedance set is the set of indices i for which $\pi_i > i$.

In this paper we study enumerative properties of the excedance set. Following the technique used in [8], we encode subsets of the set $\{1, \ldots, n-1\}$ as words in the letters a and b. This provides an effective notation to study the cardinalities of excedance sets. Thus, for example, the word *baaba* corresponds to the set $\{1, 4\}$, regarded as a subset of $\{1, 2, 3, 4, 5\}$. To denote the number of permutations in $\mathcal{S}_6$ with this excedance set we write $[baaba]$ or $[ba^2ba]$ for short. Using this notation we give recursive formulas for the number of permutations with a given excedance set and we also obtain an explicit inclusion–exclusion formula.

We also determine the most frequent excedance set among permutations in $\mathcal{S}_n$, that is, for which word w the bracket $[w]$ is maximized. The analogous problem for descent sets has raised a lot of interest [4, 18, 21]. The most recent method comes from relating the problem to the **cd**-index of the Boolean algebra, see [8, 20]. For recent developments of the **cd**-index of the Boolean algebra, see [2, 9, 19].

Furthermore, we determine the maximum for $[w]$ among all words w with a fixed number of *runs*, that is, a fixed number of maximal contiguous sequences of b's. The solution to the corresponding problem for descent sets was conjectured by Gessel and recently proved by Ehrenborg and Mahajan [7].

We hope that this paper will stimulate interest in the excedance set and that further research will be done in exploring the properties of this set-statistic. Especially, we would like to see an affirmative answer to the four inequalities in Conjecture 5.3.

The paper is organized as follows. In Section 2 we obtain the basic properties of the excedance statistic. In Sections 3 and 4 we discuss maximizing problems of the statistic over different sets. In Section 5 we prove that the sequence $[wa^k]$ is log-concave. We also state Conjecture 5.3, which can be viewed as a general log-concavity property. In Section 6 we prove the inclusion–exclusion formula. We use the fact that the bracket $[\cdot]$ can be viewed as a linear functional from the ring $\mathbb{Z}\langle a, b\rangle$ to the integers. In the last section we discuss some further research related to Genocchi numbers.

2. PRELIMINARIES

Let $\mathcal{S}_n$ denote the symmetric group on n elements, that is, all permutations of the elements $1, \ldots, n$. Let $\pi = \pi_1 \cdots \pi_n$ be a permutation in $\mathcal{S}_n$.

An *excedance* in π is an index i such that $\pi_i > i$. The *excedance set of* π is the set $E(\pi) = \{i : i \text{ is an excedance in } \pi\}$. Observe that $E(\pi)$ is a subset of $\{1, \ldots, n-1\}$ since the index n is never an excedance.

A convenient way to encode the subsets of a set is by words. Let a and b be non-commuting variables. For S a subset of $\{1, \ldots, n-1\}$ define u_S to be the word $u = u_1 \cdots u_{n-1}$ where $u_i = a$ if i does not belong to S and $u_i = b$ if i belongs to S. For $n = 1$ we set $u_\varnothing = 1$. For π a permutation in $\mathcal{S}_n$ let the *excedance word* $w(\pi)$ be the word $u_{E(\pi)} = u_1 \cdots u_{n-1}$. Observe that $u_i = b$ if and only if i is an excedance in π. Denote the number of permutations in $\mathcal{S}_n$ with excedance word w by the bracket $[w]$.

As an example, $w(3241) = bab$ and $[bab] = 3$ because there are exactly three permutations in $\mathcal{S}_4$ with excedance set $\{1, 3\}$, namely, 3241, 2143, and 3142. Also observe that $[1] = 1$ since there is exactly one permutation in $\mathcal{S}_1$ with excedance set $\varnothing$.

PROPOSITION 2.1. *Let v and w be ab-words. Then*

$$[vbaw] = [vabw] + [vbw] + [vaw].$$

Proof. Let S be the set $\{\pi \in \mathcal{S}_n : w(\pi) = vbaw\}$. We partition S into three sets and show that those sets are in a one-to-one correspondence with sets of cardinality $[vabw]$, $[vbw]$, and $[vaw]$, respectively.

Suppose the length of v is $k-2$. Let $\pi = \pi_1 \cdots \pi_{k-2}\ x\ y\ \pi_{k+1} \cdots \pi_n$ be a permutation in S with $w(\pi) = vbaw$ and thus $y \le k \le x$. Then π must satisfy exactly one of the following conditions:

(i) $x > k$ and $y = k$,

(ii) $x = k$ and $y < k$,

(iii) $x > k$ and $y < k$.

The permutations in S satisfying (i) are in a one-to-one correspondence with the permutations in $\mathcal{S}_{n-1}$ with excedance word vbw. Namely, removing the letter y from π and reducing by 1 each remaining letter in π that is larger than k gives a permutation in $\mathcal{S}_{n-1}$. It is straightforward to check that the excedances in the resulting permutation are the same as those in π. The permutations so obtained are thus counted by $[vbw]$.

By a similar argument, now removing x instead of y, the permutations in S satisfying (ii) are in one-to-one correspondence with the permutations in $\mathcal{S}_{n-1}$ with excedance word vaw.

Finally, suppose π satisfies (iii). Transposing x and y to get the permutation

$$\tau = \pi_1 \cdots \pi_{k-2}\ y\ x\ \pi_{k+1} \cdots \pi_n$$

defines a bijection to the set of permutations in $\mathcal{S}_n$ whose excedance word is $vabw$. $\square$

For an ab-word $u = u_1 u_2 \cdots u_n$, define the *dual word* u' by $u' = u'_n \cdots u'_2 u'_1$, where $u'_i = b$ if $u_i = a$ and $u'_i = a$ if $u_i = b$.

LEMMA 2.2. *For any ab-word w we have that $[w'] = [w]$.*

Proof. Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, define a permutation $\tau = \tau_1 \tau_2 \cdots \tau_n$ by setting $\tau_i = n + 1 - \pi_{n-i}$ for $i < n$ and $\tau_n = n + 1 - \pi_n$. This amounts to a bijective correspondence of $\mathcal{S}_n$ with itself. Then we have $\tau_i > i$ if and only if $\pi_{n-i} < (n - i + 1)$, which is equivalent to $\pi_{n-i} \leq (n - i)$. Thus, i is an excedance in τ if and only if $n - i$ is a non-excedance in π. $\square$

In what follows we will refer to the identity in Lemma 2.2 as *duality*.

LEMMA 2.3. *For all words w, we have $[aw] = [wb] = [w]$.*

Proof. A permutation $\pi \in \mathcal{S}_{n+1}$ with $w(\pi) = wb$ must have $\pi_n = (n + 1)$. Such permutations are in one-to-one correspondence with the permutations in $\mathcal{S}_n$ with excedance word $[w]$; simply remove the letter $(n + 1)$ from position n in π. By duality, $[aw] = [w'b] = [w'] = [w]$, which completes the proof. $\square$

The following curious fact now follows from Proposition 2.1.

COROLLARY 2.4. *For any word w, $[w]$ is odd.*

Proof. By Proposition 2.1, $[w]$ can be written in terms of three words, two of which are shorter than w and one of which has the same letters as w but where one of the a's has been moved closer to the beginning of the word. Since $[a^k b^m] = [1] = 1$, this implies by induction that for any word w the quantity $[w]$ can be written as the sum of an odd number of ones. $\square$

PROPOSITION 2.5. *We have*

$$[b^n aw] = \sum_{i=0}^{n} \binom{n+1}{i} [b^i w] \qquad \text{and} \qquad [wba^n] = \sum_{i=0}^{n} \binom{n+1}{i} [wa^i].$$

Proof. The proof of the first identity is by induction on n. By Lemma 2.3 we have $[aw] = [w]$. This proves the induction case $n = 0$. Assume now that the statement is true for n. Using Proposition 2.1 and applying the inductive hypothesis to $[b^n abw]$ and to $[b^n aw]$ we have

$$\begin{aligned}
[b^{n+1}aw] = [b^n baw] &= [b^n abw] + [b^n bw] + [b^n aw] \\
&= \sum_{i=0}^{n} \binom{n+1}{i} [b^i bw] + [b^{n+1}w] + \sum_{i=0}^{n} \binom{n+1}{i} [b^i w] \\
&= \sum_{i=1}^{n+1} \binom{n+1}{i-1} [b^i w] + \sum_{i=0}^{n+1} \binom{n+1}{i} [b^i w] \\
&= \sum_{i=0}^{n+1} \left[\binom{n+1}{i-1} + \binom{n+1}{i} \right] [b^i w] \\
&= \sum_{i=0}^{n+1} \binom{n+2}{i} [b^i w],
\end{aligned}$$

as desired. The second identity follows by duality. □

3. UNIMODALITY OF $[b^k a^{n-k}]$

A sequence of positive real numbers $a_0, a_1, a_2, \ldots$ is said to be *unimodal* if, for some integer k with $0 \le k \le n$, we have $a_0 \le a_1 \le \cdots \le a_k \ge a_{k+1} \ge \cdots \ge a_n$. We say that the sequence has a *peak* at the integer k. Observe that a unimodal sequence can have several peaks. In this section we are interested in the unimodality of the sequence $[b^k a^{n-k}]$, $k = 0, \ldots, n$. This sequence is symmetric, that is, by duality we have $[b^k a^{n-k}] = [b^{n-k} a^k]$.

THEOREM 3.1. *The sequence* $[b^k a^{n-k}]$, $k = 0, \ldots, n$, *is unimodal with peak(s) at* $k = \lfloor n/2 \rfloor$ *and* $k = \lceil n/2 \rceil$.

Proof. Let $m = n - k$. By symmetry it is enough to prove that if $1 \le k \le m \le \lfloor n/2 \rfloor$ then $[b^{k-1}a^{m+1}] < [b^k a^m]$. We prove this by induction on n. The base case is straightforward.

The induction step is as follows. We wish to show that $[b^k a^m] > [b^{k-1}a^{m+1}]$ or, equivalently, that $[b^k a^m] - [b^{k-1}a^{m+1}] > 0$. By Proposition 2.5 we have

$$\left[b^k a^m\right] = 1 + \sum_{i=1}^{k} \binom{k+1}{i} \left[b^i a^{m-1}\right]$$

$$= 1 + \sum_{i=0}^{k-1} \binom{k+1}{i+1} \left[b^{i+1} a^{m-1} \right],$$

$$\left[b^{k-1} a^{m+1} \right] = \sum_{i=0}^{k-1} \binom{k}{i} \left[b^i a^m \right].$$

The difference is given by

$$\begin{aligned} \left[b^k a^m \right] - \left[b^{k-1} a^{m+1} \right] &= 1 + \sum_{i=0}^{k-1} \left(\binom{k+1}{i+1} \left[b^{i+1} a^{m-1} \right] - \binom{k}{i} \left[b^i a^m \right] \right) \\ &\geq 1 + \sum_{i=0}^{k-1} \left(\binom{k}{i} \left[b^{i+1} a^{m-1} \right] - \binom{k}{i} \left[b^i a^m \right] \right) \\ &= 1 + \sum_{i=0}^{k-1} \binom{k}{i} \cdot \left(\left[b^{i+1} a^{m-1} \right] - \left[b^i a^m \right] \right). \end{aligned}$$

By the induction hypothesis we know that $[b^{i+1} a^{m-1}] - [b^i a^m] \geq 0$, except for the case when $k = m$ and $i = k - 1$. But in this case we have that $[b^{i+1} a^{m-1}] = [b^k a^{k-1}] = [b^{k-1} a^k] = [b^i a^m]$. Hence we conclude that $[b^k a^m] - [b^{k-1} a^{m+1}] \geq 1$ and the induction step is proved. □

COROLLARY 3.2. *Among all words w of length n, the maximum of $[w]$ is attained for $w = b^k a^{n-k}$, where $k = \lfloor n/2 \rfloor$.*

Proof. By Proposition 2.1 we have that $[ubav] > [uabv]$. That is, transposing ab in $uabv$ to get $ubav$ increases the bracket. Thus, among all words of length n and with exactly k b's, the maximum for the bracket is reached by $b^k a^{n-k}$. By Theorem 3.1, $[b^k a^{n-k}]$ is maximized when $k = \lfloor n/2 \rfloor$. □

4. WORDS WITH EXACTLY k RUNS

A *descent-run* in a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ is a maximal set $\{k, k+1, \ldots, k+m\}$, where $m > 0$, such that $\pi_k > \pi_{k+1} > \cdots > \pi_{k+m}$. An *ascent-run* in π is defined similarly, with $>$ replaced by $<$. Gessel conjectured that the most common descent set among permutations in $\mathcal{S}_n$ with exactly r descent-runs and r ascent-runs is, roughly speaking, the set corresponding to the word $a^{n/k} b^{n/k} a^{n/k} b^{n/k} \cdots$, where $k = 2r$. The exact rounding of the exponents is formulated in the conjecture. This conjecture was recently proved by Ehrenborg and Mahajan [7].

We will in this section consider the analogous question for the excedance set.

DEFINITION 4.1. *A run in an ab-word is a maximal contiguous subword consisting solely of b's.*

As an example, the word *bbabaabbba* has three runs, of lengths 2, 1, and 3, respectively.

PROPOSITION 4.2. *Among the ab-words of length n with exactly $r+1$ runs, the bracket $[\cdot]$ is maximized by $b^m(ab)^r a^p$, where $\lfloor p/2 \rfloor = \lfloor m/2 \rfloor$ and $m + p = n - 2r$. In particular, if n is even then $m = p$.*

Proof. By duality, we may restrict our attention to words with at least as many b's as a's. Moreover, by Lemma 2.3 it suffices to consider words that begin with b and end with a. We first show that among the words of length n with exactly $r+1$ runs and exactly s b's, the bracket is maximized by $b^{s-r}(ab)^r a^p$ (where $p = n - s - r$).

Let $w = vab^t ua^s$, where we assume $t > 1$. Then $[w] < [vbab^{t-1}ua^s]$, by Proposition 2.1, and thus we can "move" the first b in a run of length $t > 1$ successively leftwards until it is "absorbed" by the preceding run, always increasing the bracket. As an example,

$$[babbaabbbaa] < [babbababbaa] < [babbbaabbaa].$$

Repeating this process will move all b's but one from each run all the way left to the first run of the word, increasing the bracket and preserving the number of runs. Similarly, moving all a's but one from each contiguous string of a's to the end of the word will increase the bracket and preserve the number of runs. This proves the claim.

It remains to be shown that among all words of the form $b^m(ab)^r a^p$, where $m + p = n - 2r$, the bracket reaches its maximum when $\lfloor p/2 \rfloor = \lfloor m/2 \rfloor$. By duality, $[b^m(ab)^r a^p] = [b^p(ab)^r a^m]$, so we may take $m \leq p$ and then it suffices to show that

$$[b^m(ab)^r a^p] - [b^{m-1}(ab)^r a^{p+1}] \geq 0.$$

Note that this will prove that the sequence $[b^m(ab)^r a^{k-m}]$, indexed by m, is unimodal with peak "in the middle." We now proceed by induction on r, assuming this to be true for all $r \leq k$. The basis step is $r = 0$, which follows from Theorem 3.1. Now, by Proposition 2.5, we have

$$\begin{aligned}
&\left[b^m(ab)^{k+1}a^p\right] - \left[b^{m-1}(ab)^{k+1}a^{p+1}\right] \\
&= \left[b^m ab(ab)^k a^p\right] - \left[b^{m-1}ab(ab)^k a^{p+1}\right] \\
&= \left[b(ab)^k a^p\right] + \sum_{i=1}^{m} \binom{m+1}{i} \left[b^{i+1}(ab)^k a^p\right] - \sum_{i=0}^{m-1} \binom{m}{i} \left[b^{i+1}(ab)^k a^{p+1}\right].
\end{aligned}$$

Since $\binom{m+1}{i} \geq \binom{m}{i-1}$ we need only show that

$$\left[b^{i+1}(ab)^k a^p\right] \geq \left[b^i(ab)^k a^{p+1}\right].$$

But this is covered by the inductive hypothesis, except in the case when $i = m = p$. In that case we have, by duality,

$$\left[b^{i+1}(ab)^k a^p\right] = \left[b^p(ab)^k a^{i+1}\right] = \left[b^i(ab)^k a^{p+1}\right],$$

as desired. $\square$

5. LOG-CONCAVITY RESULTS

A sequence of (real) numbers $a_0, a_1, a_2, \ldots$ is said to be *log-concave* if, for any $k > 0$, we have $a_{k-1} \cdot a_{k+1} \leq a_k^2$. It is straightforward to verify that this is equivalent to $a_k \cdot a_m \leq a_{k+i} \cdot a_{m-i}$, for $0 \leq i \leq m-k$. Moreover, for finite positive sequences log-concavity implies unimodality.

In this section we prove the following result.

PROPOSITION 5.1. *For any word w, the sequence $\{[wa^k]\}_{k\geq 0}$ is log-concave.*

In order to prove this proposition we need the following lemma.

LEMMA 5.2. *Let $a_0, a_1, a_2, \ldots$ be a log-concave sequence of non-negative real numbers. Then the sequence*

$$A_n = \sum_{i=0}^{n-1} \binom{n}{i} a_i$$

for $n = 1, 2, \ldots$ is log-concave.

Proof. We need to show that $D_n = A_n^2 - A_{n-1}A_{n+1} \geq 0$. For $i \leq j$, let $C_{i,j}$ be the coefficient of $a_i a_j$ in D_n. For convenience of notation, set $C_{i,j} = 0$ if $i > j$. Let $S_k = \sum_i C_{i,k-i} \cdot a_i a_{k-i}$. Then we have

$$D_n = \sum_{i,j} C_{i,j} \cdot a_i a_j \;=\; \sum_k S_k,$$

so it suffices to show that $S_k \geq 0$ for all k.

We prove this in two steps. First, we show that $\sum_i C_{i,k-i} = \binom{n-1}{k-n+1}$. We then demonstrate that for each k and for $m \leq k/2$, the sequence $C_{0,k}, C_{1,k-1}, \ldots, C_{m,k-m}$, has the property that all of its terms are non-negative after a certain point, say $C_{j,k-j}$, and non-positive before that.

But, by log-concavity of the sequence $a_0, a_1, a_2, \ldots$, we have that $a_0 a_k \leq a_1 a_{k-1} \leq a_2 a_{k-2} \leq \cdots$, so

$$S_k = \sum_i C_{i,k-i} \cdot a_i a_{k-i} \ \geq \ a_j a_{k-j} \sum_i C_{i,k-i}.$$

Since $a_j a_{k-j} \geq 0$, it suffices to know that $\sum_i C_{i,k-i}$ is non-negative.

Observe that $A_n = -a_n + \sum_{i=0}^{n} \binom{n}{i} a_i$. Now, $C_{i,k-i}$ is the coefficient to $a_i a_{k-i}$ in

$$\begin{aligned} D_n &= A_n^2 - A_{n-1} A_{n+1} \\ &= \left[-a_n + \sum_{i=0}^{n} \binom{n}{i} a_i \right]^2 \\ &\quad - \left[-a_{n-1} + \sum_{i=0}^{n-1} \binom{n-1}{i} a_i \right] \cdot \left[-a_{n+1} + \sum_{i=0}^{n+1} \binom{n+1}{i} a_i \right], \end{aligned}$$

so we have

$$\begin{aligned} \sum_i C_{i,k-i} &= \sum_i \left[\binom{n}{i} \binom{n}{k-i} - \binom{n+1}{i} \binom{n-1}{k-i} \right] \\ &\quad - \binom{n}{n} \binom{n}{k-n} - \binom{n}{k-n} \binom{n}{n} \binom{n+1}{n+1} \binom{n-1}{k-n-1} \\ &\quad + \binom{n+1}{k-n+1} \binom{n-1}{n-1} \\ &= \sum_i \left[\binom{n}{i} \binom{n}{k-i} - \binom{n+1}{i} \binom{n-1}{k-i} \right] \\ &\quad - 2 \binom{n}{k-n} + \binom{n-1}{k-n-1} + \binom{n+1}{k-n+1}. \end{aligned}$$

The following identity is a special case of the *Vandermonde convolution*; see for instance [15, p. 174].

$$\sum_i \binom{n+1}{i} \binom{n-1}{k-i} \ = \sum_i \binom{n}{i} \binom{n}{k-i} = \binom{2n}{k}.$$

Thus we have

$$\sum_i C_{i,k-i} = \ - 2 \binom{n}{k-n} + \binom{n-1}{k-n-1} + \binom{n+1}{k-n+1},$$

which, by a straightforward manipulation, equals $\binom{n-1}{k-n+1}$ as claimed.

It remains to be shown that for some m the sequence $C_{i,k-i}$ is non-positive for $i \le m$ and non-negative for $i > m$.

We have

$$C(i, k-i) = 2\binom{n}{i}\binom{n}{k-i} - \binom{n+1}{i}\binom{n-1}{k-i} - \binom{n+1}{k-i}\binom{n-1}{i},$$

except when $i \ge n-1$ or $k-i \ge n-1$. These two latter cases are easily treated.

When $k \le n$, we have

$$\begin{aligned} C(0, k) &= \left[\binom{n}{k} - \binom{n-1}{k}\right] - \left[\binom{n+1}{k} - \binom{n}{k}\right] \\ &= \binom{n-1}{k-1} - \binom{n}{k-1} = -\binom{n-1}{k-2} < 0. \end{aligned}$$

When $k > n$ we have, for $i = k-n$, that

$$C(i, k-i) = C(k-n, n) = -\binom{n+1}{k-i}\binom{n-1}{i} < 0.$$

Now, apart from the above mentioned exceptions, we have

$$C(i, k-i) = \frac{(n-1)!^2}{i! \cdot (k-1)! \cdot (n-i+1)! \cdot (n-k+i+1)!} A(i, k),$$

where

$$\begin{aligned} A(i, k) &= 2n^2(n-i+1)(n-k+i+1) \\ &\qquad -(n+1)(n-k+i)(n-k+i+1) - (n+1)(n-i+1)(n-i). \end{aligned}$$

Clearly the sign of $C(i, k-i)$ is the same as that of $A(i, k)$. The derivative of $A(i, k)$ with respect to i is $2(k-2i)(1+n+n^2)$ so, as a function of i, $A(i, k)$ has its only critical point at $i = 2/k$. The cases where $k \le 2$ are easily checked. When $k > 2$, this means that the only critical point of $A(i, k)$ lies in the interval $(0, 1)$, so $A(i, k)$ changes sign at most once in the interval of interest to us. As it must be positive somewhere, and it is negative for the smallest relevant value of i, $A(i, k)$ must eventually be positive. □

Proof of Proposition 5.1. The proof is by induction on the length of w. The base case is $w = 1$ (the empty word), which is trivial, since $[a^k] = 1$ for all $k \ge 0$. For the induction step, assume the statement to hold for the word v and we show that it then also holds for $w = va$ and $w = vb$, which covers all possibilities. In the case $w = va$, the sequence $\{[wa^k]\}_{k\geq 0}$ coincides with $\{[va^k]\}_{k\geq 1}$, and so is log-concave. If $w = vb$ we have, by Proposition 2.5,

$$[wa^k] = [vba^k] = \sum_{i=0}^{n} \binom{n+1}{i} [va^i].$$

But by Lemma 5.2 this implies that the sequence $\{[wa^k]\}_{k\geq 0}$ is log-concave. $\square$

As a generalization of Proposition 5.1 we conjecture the following.

CONJECTURE 5.3. *For any three words u, v and w the following four inequalities hold*:

$$[uvw] \cdot [uavaw] \leq [uavw] \cdot [uvaw],$$
$$[uvw] \cdot [uavbw] \geq [uavw] \cdot [uvbw],$$
$$[uvw] \cdot [ubvaw] \geq [ubvw] \cdot [uvaw],$$
$$[uvw] \cdot [ubvbw] \leq [ubvw] \cdot [uvbw].$$

Observe that the first and fourth inequalities are equivalent by duality. Moreover, the first inequality implies Proposition 5.1.

One consequence of this conjecture is that the sequence $[ub^k va^{n-k} w]$, for $k = 0, 1, \ldots, n$, is unimodal. The argument is as follows. Let $\alpha_{i,j} = [ub^{k+i} va^{n-k+j} w]$ for $i + j \leq 2$. Conjecture 5.3 implies that

$$\alpha_{0,0} \cdot \alpha_{0,2} \leq \alpha_{0,1}^2, \qquad \alpha_{0,0} \cdot \alpha_{2,0} \leq \alpha_{1,0}^2, \qquad \text{and}$$
$$(\alpha_{0,1} \cdot \alpha_{1,0})^2 \leq (\alpha_{0,0} \cdot \alpha_{1,1})^2.$$

Multiplying these three inequalities together and canceling terms, we obtain $\alpha_{0,2} \cdot \alpha_{2,0} \leq \alpha_{1,1}^2$. This inequality implies that the sequence $[ub^k va^{n-k} w]$, for $k = 0, 1, \ldots, n$, is log-concave, and hence unimodal.

6. AN INCLUSION–EXCLUSION FORMULA FOR $[w]$

Recall that the excedance set of a permutation π is $E(\pi) = \{i : \pi(i) > i\}$. Abusing notation, define the excedance set of a word $u = u_1 u_2 \cdots u_{n-1}$ to be $E(u) = \{i : u_i = b\}$. Thus, if π is a permutation with $w(\pi) = w$, then $E(w) = E(\pi)$.

Let

$$w = a^{n_1} b a^{n_2} b a^{n_3} b \cdots a^{n_k} b a^{n_{k+1}}$$

and set $\mathbf{n}(w) = (n_1, n_2, \ldots, n_{k+1})$. Note that a pair of consecutive b's in w will correspond to a zero coordinate in the vector $\mathbf{n}(w)$. As an example, $\mathbf{n}(babbaaba) = (0, 1, 0, 2, 1)$.

We wish to compute $[w]$, but first we determine the number of permutations whose excedance set is *contained in* the excedance set of w.

LEMMA 6.1. *Let w be a word and suppose $\mathbf{n}(w) = (n_1, \ldots, n_{k+1})$. Then we have*

$$\left|\{\pi \in \mathcal{S}_n : E(\pi) \subseteq E(w)\}\right| = 1^{n_1+1} \cdot 2^{n_2+1} \cdot 3^{n_3+1} \cdots (k+1)^{n_{k+1}+1}.$$

Proof. We wish to count the number of permutations in $\mathcal{S}_n$ whose excedance set is contained in $E(w)$. We do this in two steps. First we choose the entries $\pi(i)$ of the permutation π for i not in $E(w)$. That is, these entries are non-excedances of the permutation π, so we need $\pi(i) \leq i$. Then we choose the remaining entries of π. Since they may or may not be excedances, there are no restrictions on them and they can thus be chosen freely.

To choose $\pi(i)$ such that $\pi(i) \leq i$ for all $i \notin E(w)$, and such that all entries are distinct, is equivalent to choosing a rook placement on a Ferrers board of width $n - k - 1 = n_1 + \cdots + n_{k+1}$, where the set of heights is $E(w) = \{\lambda_1 < \cdots < \lambda_{n-k-1}\}$. By the same counting technique as in [14] (see also [23, Theorem 2.4.1]) this can be done in

$$\prod_{j=1}^{n-k-1} (\lambda_j - j) = 1^{n_1} \cdot 2^{n_2} \cdots (k+1)^{n_{k+1}}$$

different ways.

As for the places in π corresponding to b's in w, these are allowed to be either excedances or non-excedances. Hence, the remaining letters in π can be placed in any order. There are k such places, corresponding to the k b's, and there is also the last hidden position of π which does not correspond to a letter in the word w. The letters in these remaining $k+1$ positions can be ordered in $(k+1)!$ different ways. All in all, then, the permutation π can be constructed in $1^{n_1+1} \cdot 2^{n_2+1} \cdots (k+1)^{n_{k+1}}$ ways. $\square$

The bracket $[w]$ is defined on ab-monomials. By linearity we can extend the bracket to the ring $\mathbb{Z}\langle a, b\rangle$ of polynomials in the non-commuting variables a, b over $\mathbb{Z}$. We now reformulate Lemma 6.1 in this setting.

LEMMA 6.2. *For any vector $(n_1, \ldots, n_{k+1})$ we have*

$$[a^{n_1} \cdot (a+b) \cdot a^{n_2} \cdot (a+b) \cdots (a+b) \cdot a^{n_{k+1}}] =$$
$$1^{n_1+1} \cdot 2^{n_2+1} \cdots (k+1)^{n_{k+1}+1}.$$

Observe that having an $(a+b)$ in position i means that we can either have an excedance or a non-excedance at position i. Hence Lemma 6.2 follows directly from Lemma 6.1.

We now give an explicit formula for $[w]$, where w is an *ab*-word. This can be done either by Lemma 6.1, together with the principle of inclusion and exclusion, or we can use Lemma 6.2 and a change of basis. We do the latter here.

Consider the *ab*-word $w = a^{n_1} \cdot b \cdot a^{n_2} \cdot b \cdot a^{n_3}$. By writing $b = (a+b) - a$ and expanding, we can write w as a linear combination of monomials in the letters a and $(a+b)$. In our example,

$$\begin{aligned} & a^{n_1} \cdot b \cdot a^{n_2} \cdot b \cdot a^{n_3} \\ &\quad = a^{n_1} \cdot (a+b) \cdot a^{n_2} \cdot (a+b) \cdot a^{n_3} - a^{n_1} \cdot (a+b) \cdot a^{n_2} \cdot a \cdot a^{n_3} \\ &\qquad - a^{n_1} \cdot a \cdot a^{n_2} \cdot (a+b) \cdot a^{n_3} + a^{n_1} \cdot a \cdot a^{n_2} \cdot a \cdot a^{n_3} \\ &\quad = a^{n_1} \cdot (a+b) \cdot a^{n_2} \cdot (a+b) \cdot a^{n_3} - a^{n_1} \cdot (a+b) \cdot a^{n_2+1+n_3} \\ &\qquad - a^{n_1+1+n_2} \cdot (a+b) \cdot a^{n_3} + a^{n_1+1+n_2+1+n_3}. \end{aligned}$$

Applying the bracket, which is a linear map, to the above equation we obtain

$$\begin{aligned} [a^{n_1} \cdot b \cdot a^{n_2} \cdot b \cdot a^{n_3}] &= 1^{n_1+1} \cdot 2^{n_2+1} \cdot 3^{n_3+1} - 1^{n_1+1} \cdot 2^{n_2+1+n_3+1} \\ &\quad - 1^{n_1+1+n_2+1} \cdot 2^{n_3+1} + 1^{n_1+1+n_2+1+n_3+1} \\ &= 1^{n_1+1} \cdot 2^{n_2+1} \cdot 3^{n_3+1} - 1^{n_1+1} \cdot 2^{n_2+1} \cdot 2^{n_3+1} \\ &\quad - 1^{n_1+1} \cdot 1^{n_2+1} \cdot 2^{n_3+1} + 1^{n_1+1} \cdot 1^{n_2+1} \cdot 1^{n_3+1}. \end{aligned}$$

Let $R_k = \{\mathbf{r} = (r_1, \ldots, r_{k+1}) : r_1 = 1, r_{i+1} - r_i \in \{0, 1\}\}$. Thus, each **r**-*vector* $\mathbf{r} = (r_1, \ldots, r_{k+1})$ in R_k has $r_1 = 1$ and increases by at most one at each coordinate. We say that i is a *horizontal step* in $\mathbf{r} = (r_1, \ldots, r_{k+1})$ if $r_i = r_{i+1}$. Let $h(\mathbf{r})$ be the number of horizontal steps in $\mathbf{r}$. We have that $h(\mathbf{r}) = k + 1 - r_{k+1}$. In particular, if $h(\mathbf{r}) = 0$, then $\mathbf{r} = (1, 2, \ldots, k+1)$.

Let now $\mathbf{1} = (1, 1, \ldots, 1)$, and set

$$\mathbf{r}^{\mathbf{n}(w)+\mathbf{1}} = r_1^{n_1+1} \cdot r_2^{n_2+1} \cdots r_k^{n_k+1}.$$

A straightforward argument, following the example after Lemma 6.2, proves the following theorem.

THEOREM 6.3. *Let w be an ab-word with exactly k b*'s. *Then*

$$[w] = \sum_{\mathbf{r} \in R_k} (-1)^{h(\mathbf{r})} \cdot \mathbf{r}^{\mathbf{n}(w)+\mathbf{1}}.$$

EXAMPLE 6.4. *Let* $w = babbaa$. *Then* $\mathbf{n}(w) = (0, 1, 0, 2)$ *and* $\mathbf{n}(w) + \mathbf{1} = (1, 2, 1, 3)$, *so*

$$\begin{aligned}[w] &= 1 \cdot 2^2 \cdot 3 \cdot 4^3 - 1 \cdot 2^2 \cdot 3 \cdot 3^3 - 1 \cdot 2^2 \cdot 2 \cdot 3^3 - 1 \cdot 1^2 \cdot 2 \cdot 3^3 \\ &\quad + 1 \cdot 2^2 \cdot 2 \cdot 2^3 + 1 \cdot 1^2 \cdot 2 \cdot 2^3 + 1 \cdot 1^2 \cdot 1 \cdot 2^3 - 1 \cdot 1^2 \cdot 1 \cdot 1^3 \\ &= 261.\end{aligned}$$

PROPOSITION 6.5. *The bracket evaluated on the word* $b^k a^m$ *is given by*

$$\left[b^k a^m\right] = \sum_{i=1}^{k+1} (-1)^{k+1-i} \cdot S(k+1, i) \cdot i! \cdot i^m,$$

where $S(k+1, i)$ *denotes the Stirling number of the second kind.*

Proof. Let $R_{k,i}$ be the set $\{\mathbf{r} \in R_k : r_k = i\}$. Then the cardinality of $R_{k,i}$ is given by $\binom{k}{i}$. Let $\mathbf{r}$ be an element of $R_{k,i}$. For $1 \leq q \leq i$ let a_q be the number of entries in $\mathbf{r}$ that are equal to q. That is, $a_q = |\{j : r_j = q\}|$. Now $a_q \geq 1$ and $a_1 + \cdots + a_i = k + 1$. That is, $(a_1, \ldots, a_i)$ is a composition of the integer $k + 1$. We now have

$$\begin{aligned}\sum_{\mathbf{r} \in R_{k,i}} r_1 \cdots r_k &= \sum 1^{a_1} \cdot 2^{a_2} \cdots i^{a_i} \\ &= S(k+1, i) \cdot i!,\end{aligned}$$

where the second sum ranges over all compositions $a_1 + \cdots + a_k = k + 1$, and the last equality is by Exercise 16 in [23, Chap. 1].

Observe that $\mathbf{n}(b^k a^m) = (0, \ldots, 0, m)$. Hence by Theorem 6.3 we have

$$\begin{aligned}\left[b^k a^m\right] &= \sum_{\mathbf{r} \in R_k} (-1)^{h(\mathbf{r})} \cdot \mathbf{r}^{\mathbf{n}(w)+\mathbf{1}} \\ &= \sum_{i=1}^{k+1} \sum_{\mathbf{r} \in R_{k,i}} (-1)^{h(\mathbf{r})} \cdot r_1 \cdots r_{k-1} \cdot r_k \cdot i^m \\ &= \sum_{i=1}^{k+1} (-1)^{k+1-i} \cdot S(k+1, i) \cdot i! \cdot i^m. \qquad \square\end{aligned}$$

The identity in Proposition 6.5 can be inverted to yield the following corollary.

COROLLARY 6.6. *Let* $c(n, k)$ *be the signless Stirling number of the first kind, that is, the number of permutations in* $\mathcal{S}_n$ *with exactly* k *cycles. Then for* $n \geq 1$

$$\sum_{k=0}^{n-1} c(n, k+1) \cdot [b^k a^m] = n! \cdot n^m.$$

This corollary is equivalent to

$$[(b+1)\cdots(b+n-1)\cdot a^m] = n!\cdot n^m,$$

which may be proven directly.

7. SOME REMARKS ON GENOCCHI NUMBERS

Finally we mention that the number of permutations in $\mathcal{S}_{2n+1}$ with *alternating excedances*, that is, permutations with excedance word $(ba)^n = baba\cdots ba$, is equal to the *Genocchi number* G_{2n+1}. This follows by comparing our Theorem 6.3 with Proposition 1 in [5].

It may also be interesting to note that studying the excedance set of a permutation π is equivalent to studying the *descent bottoms set* of π, defined by

$$\text{DESBOT}(\pi) = \{\pi_i : \pi_{i-1} > \pi_i\},$$

in the following sense: There are several bijections from $\mathcal{S}_n$ to $\mathcal{S}_n$ in the literature taking a permutation with excedance set S to a permutation with descent bottoms set S. This is essentially a property of the "fundamental transformation" of Foata and Schützenberger [11]. See also, for instance, [3]. Thus, all our results can be translated into results about permutations with given descent bottom sets. Only some of these translations, however, will yield any interesting information.

As an example, the Genocchi numbers, which count the permutations with alternating excedances, can also be seen to count the permutations for which $\pi_{i-1} > \pi_i$ if and only if π_i is odd (and $\pi_i < n$).

We treat this relationship in a forthcoming paper [10], where we study the distribution of permutations with alternating excedances according to the first letter of each permutation and deduce their relations to the *Seidel triangle* for the Genocchi numbers [6] and to recent work of Kreweras [16].

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