On Wave-induced Ocean Mixing

---- sea wave-generated turbulence and sea wave mixings

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Basic Recognitions on Ocean Numerical Modeling:

Ocean motions are described by the governing equation set of the ocean dynamic system (with four subsystems).

Only the mixing terms are uncertain in the ocean motion equations and the flux ones in the boundary conditions.

Parameter tuning is the opposite of the dynamic (analytic) estimate of ocean mixing and boundary flux.

Over-parameterization leads to misuse and confidence lose in ocean numerical modeling.

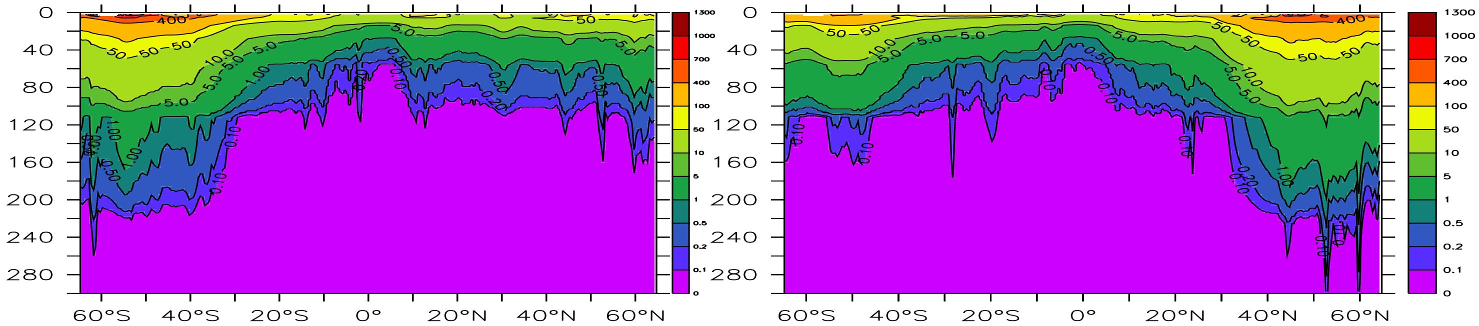
In 90s of last century we presented a sea wave-induced mixing coefficient based on the Prandtl mixing length theory

$$B_{WV} = 4\left[\iint_{k_\beta} E_W(k_1,k_2)\frac{sh\{2k(x_3+H)\}}{\left|sh\{kH\}\right|^2}dk_1dk_2\right]\left\{\frac{\partial}{\partial x_3}\left[\iint_{k_\beta} \omega^2 E_W(k_1,k_2)\frac{sh\{2k(x_3+H)\}}{\left|sh\{kH\}\right|^2}dk_1dk_2\right]^{1/2}\right\}$$

Yuan,Y.L., F.L.Qiao, et al., 1999: A numerical model for shallow water circulation, I. Wave mixing in ocean and wave-current interaction, J. Hydro., 14(4B), 1_8 (in Chinese with English abs.)

Qiao,F.L., Y.L.Yuan, et al., 2004: Wave-induced mixing in the upper ocean : distribution and application to a global ocean circulation model, Geophys. Res. Lett. 31, L11303

The coefficient is completely determined without any tunable parameter.



$$\frac{cm^2}{\sec}$$

winter

summer

Characteristics of the mixing coefficient:

1. The coefficient has exponential distribution vertically and two maximums in north and south high latitudes;
2. The effective value of the coefficient $0.1\frac{cm^2}{\sec}$ can reach depth more than 300 m in the two high latitudes and the surface value is about 1000 times larger than that.

Combining with

the third generation wave model (MASNUM…)

and substituting the mixing coefficient calculated into

the ocean circulation model (such as POM…)

and the climate model (such as NCAR-CCSM3…)

we get the model system used and called as

the coupled ocean circulation model

and the coupled climate model

without any additional tunable coefficient

These models have successful applications to the simulation

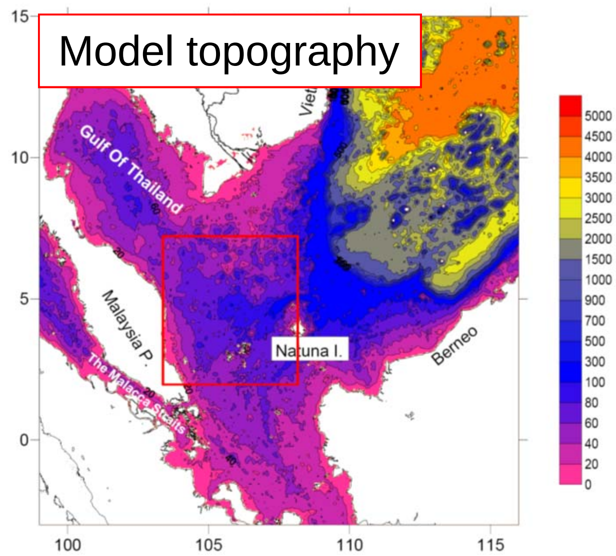
in regional seas, global ocean and global climate.

All the applications are positive.

I) Simulation in Regional Seas

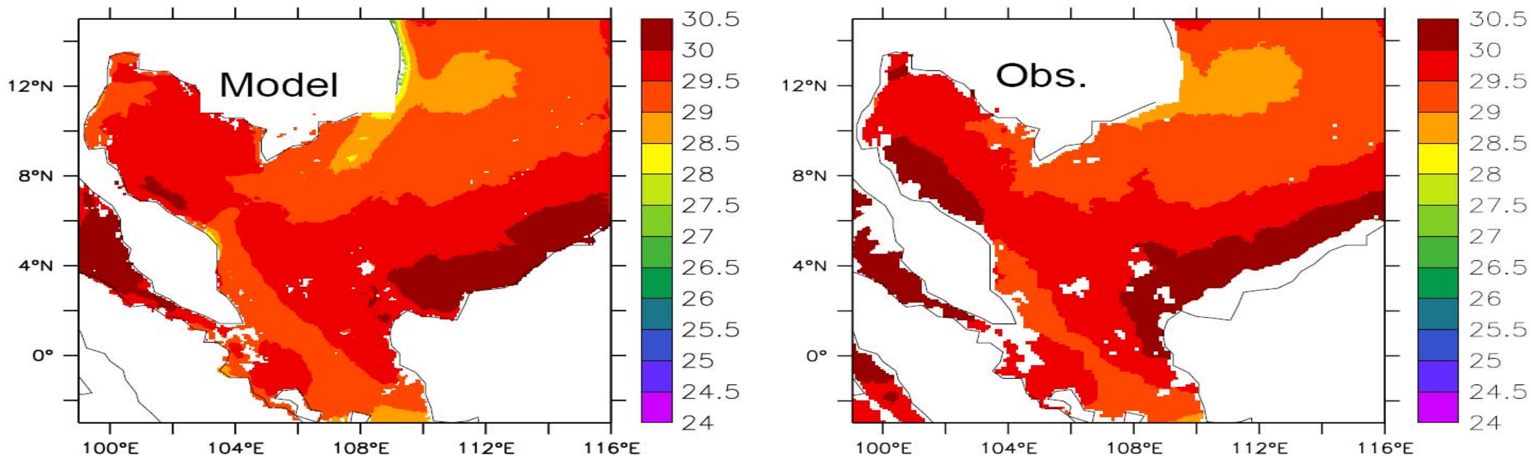
using the coupled ocean circulation model

1. The simulation in the offshore area of Malaysia

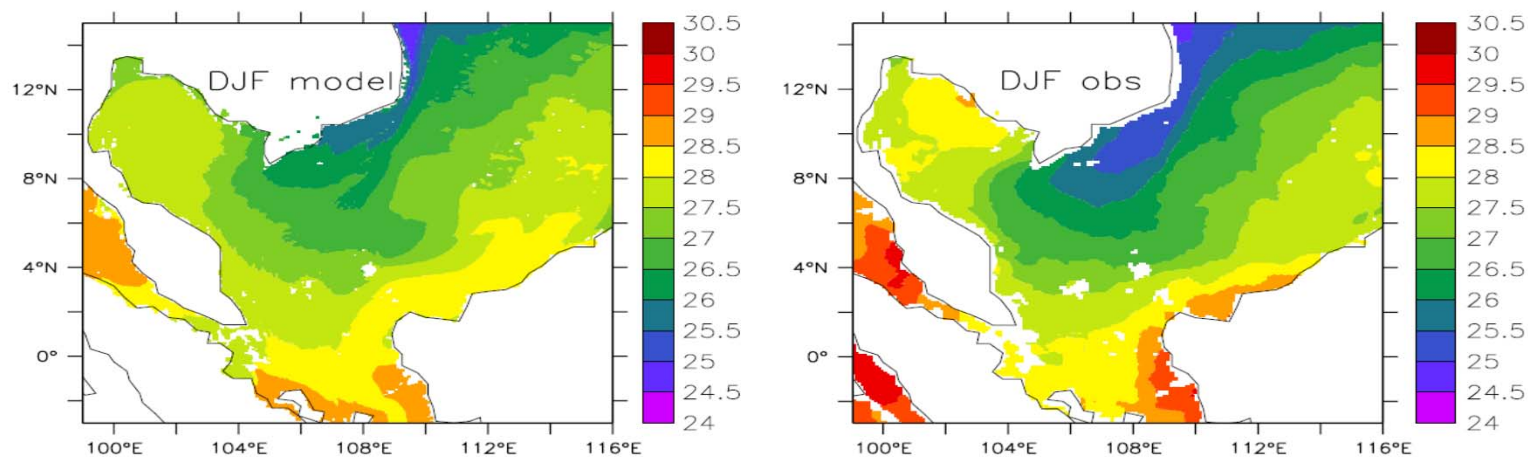


- The model domain is 3^0 S – 15^0N, 99^0 E – 116^0 E,
- Horizontal resolution $1/18^0 \times 1/18^0$ (6Km)
- 51 vertical sigma layers

1) The comparison of the modeled SST to the observed one

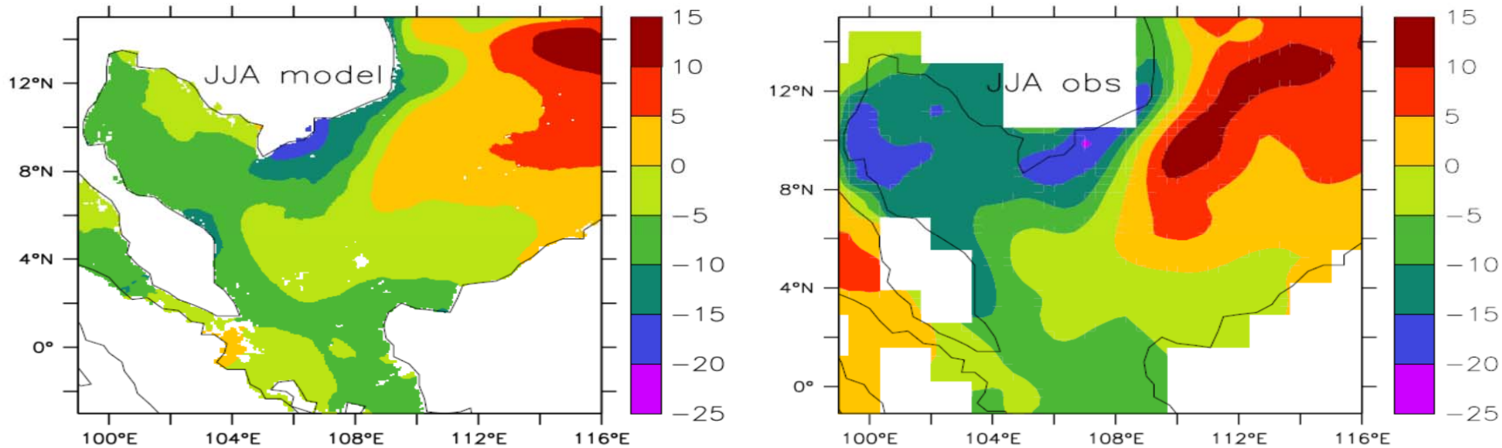


Modeled (left) and observed (right) SST in summer (JJA)

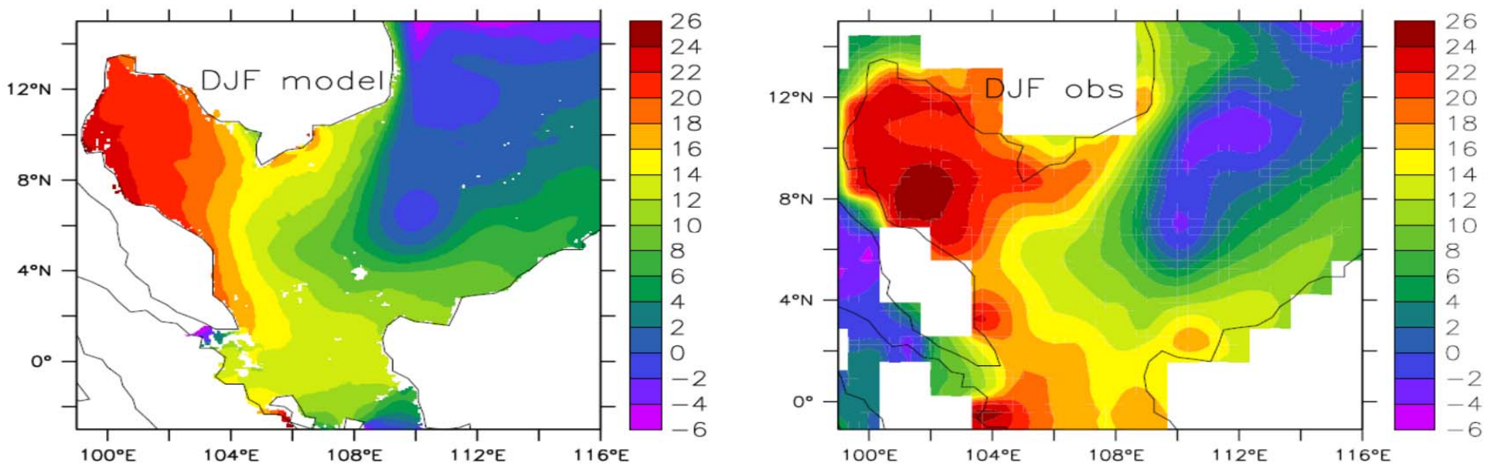


Modeled (left) and observed (right) SST in winter (DJF)

2) The comparison of the modeled SSH to the observed one

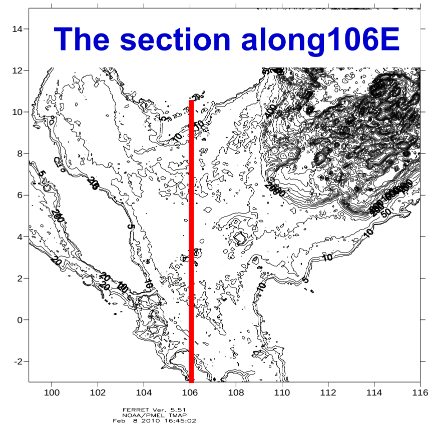


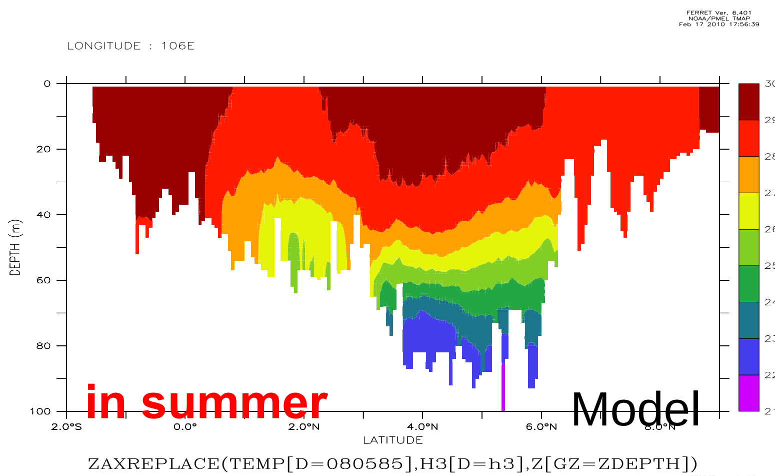
Modeled (left) and observed (right) SSH in summer (JJA) (units: cm).

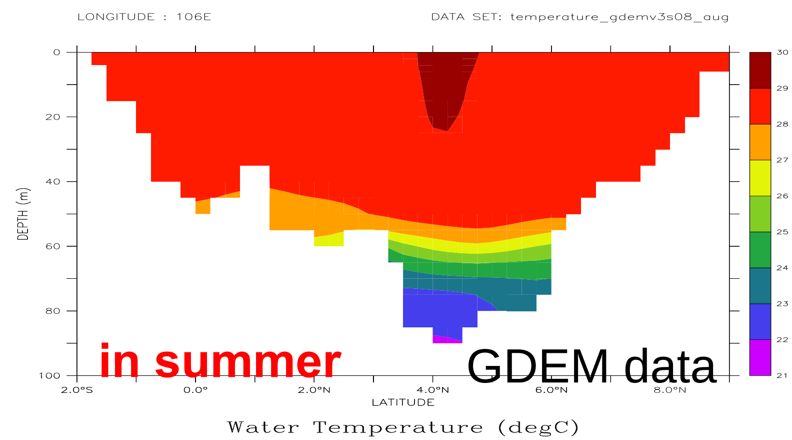


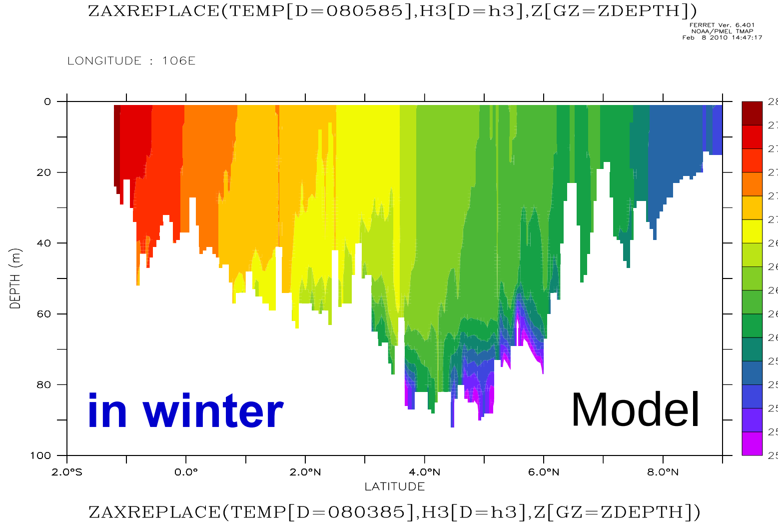
Modeled (left) and observed (right) SSH in winter (DJF) (units: cm).

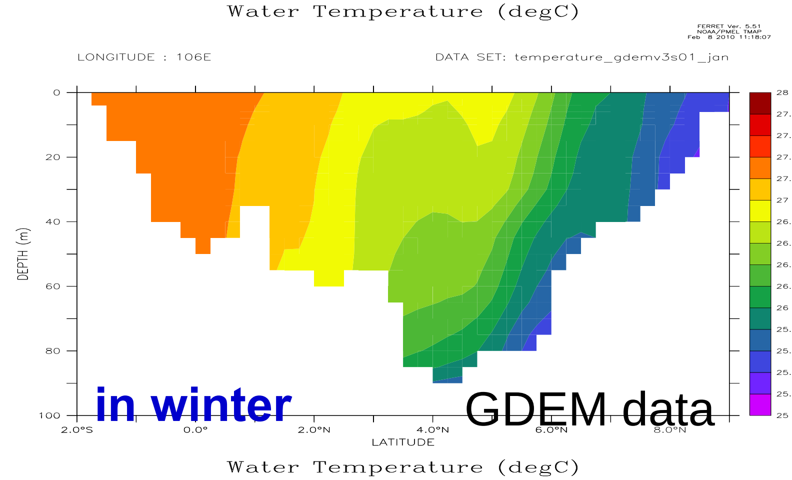
3) The comparison of the modeled temperature section to the observed one



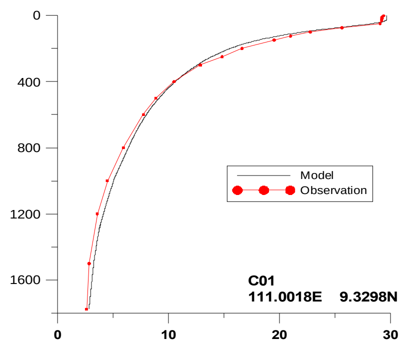


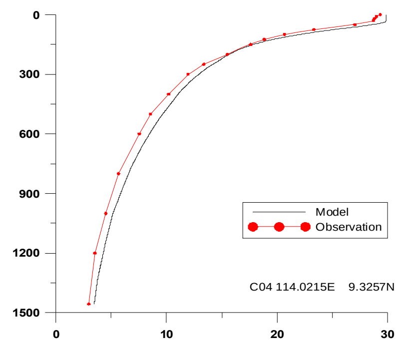


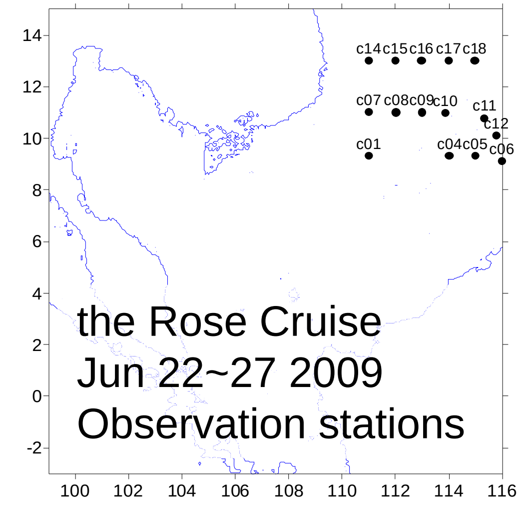


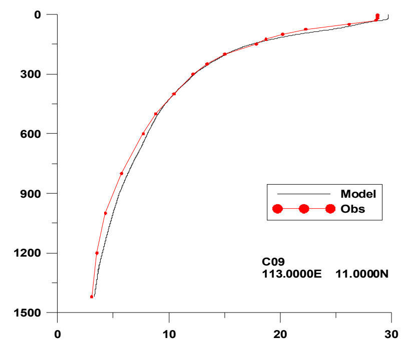


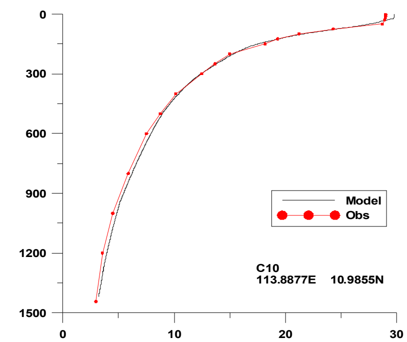
4) The comparison of the modeled temperature profiles to the observed ones



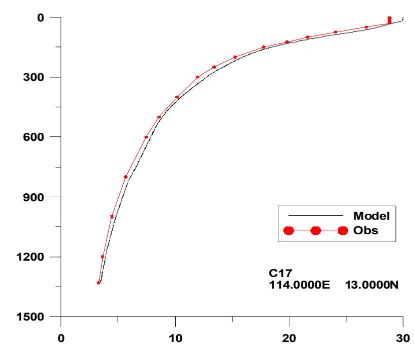


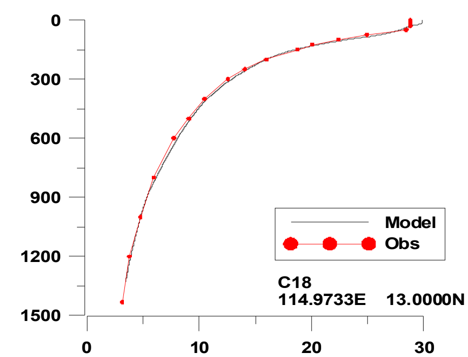




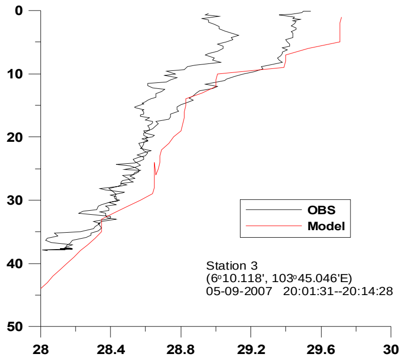


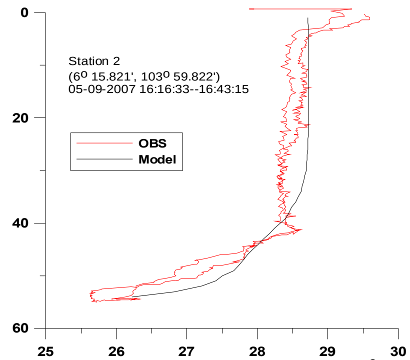
In Summer (June)

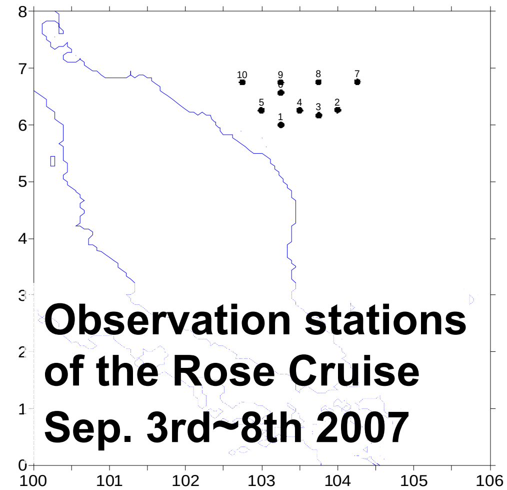


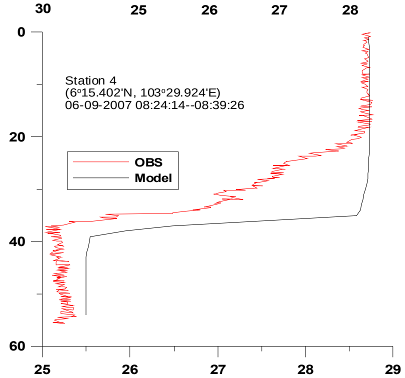


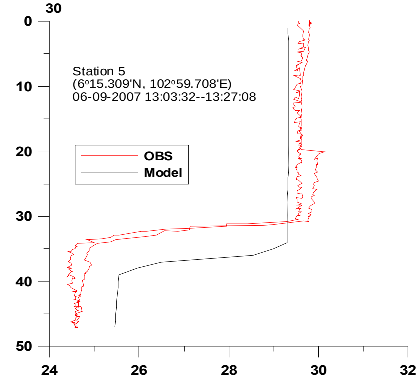
4) The comparison of the modeled temperature profile to the observed one

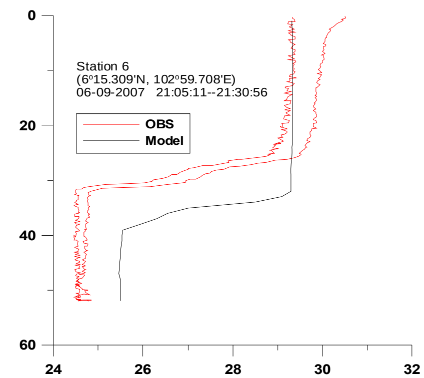








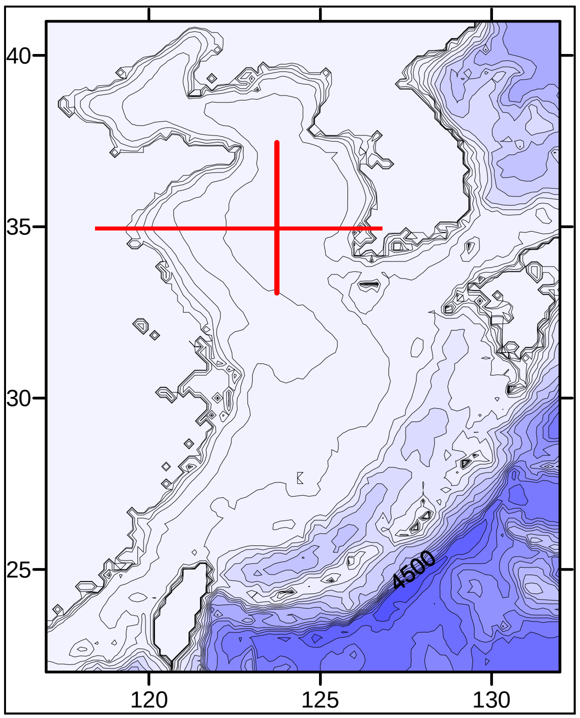




In Autumn (September)

Good simulation in upper uniform layer !

2. The simulation in the yellow sea

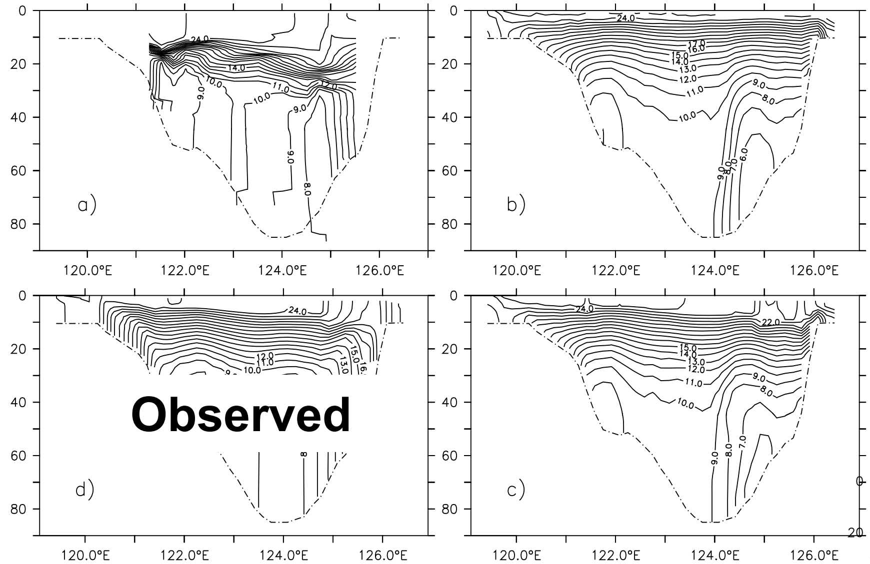


- The model domain is
  $22^0N – 41^0N, 117^0E – 132^0E$,
- Horizontal resolution
  $1/18^0 \times 1/18^0$ (6Km)
- 25 vertical sigma layers

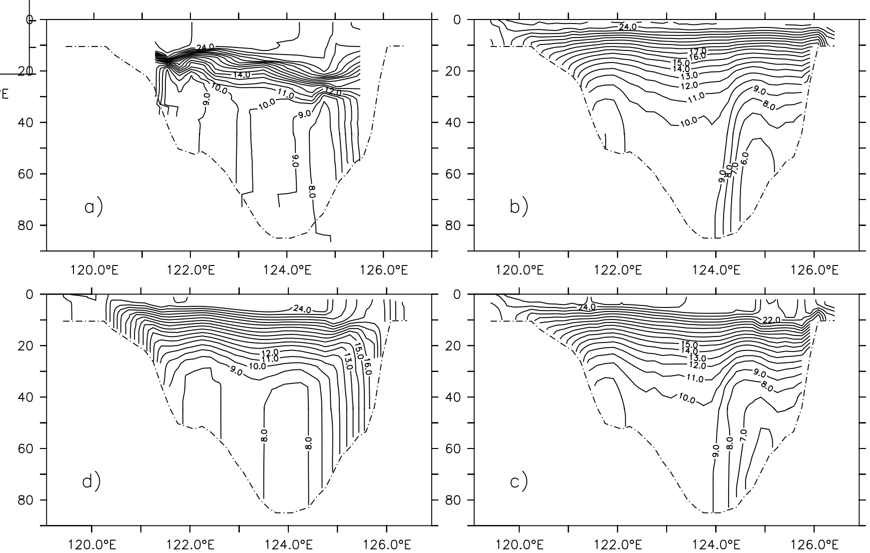
《 The cold water mass of the Yellow sea 》

The temperature

along the section 35^0N



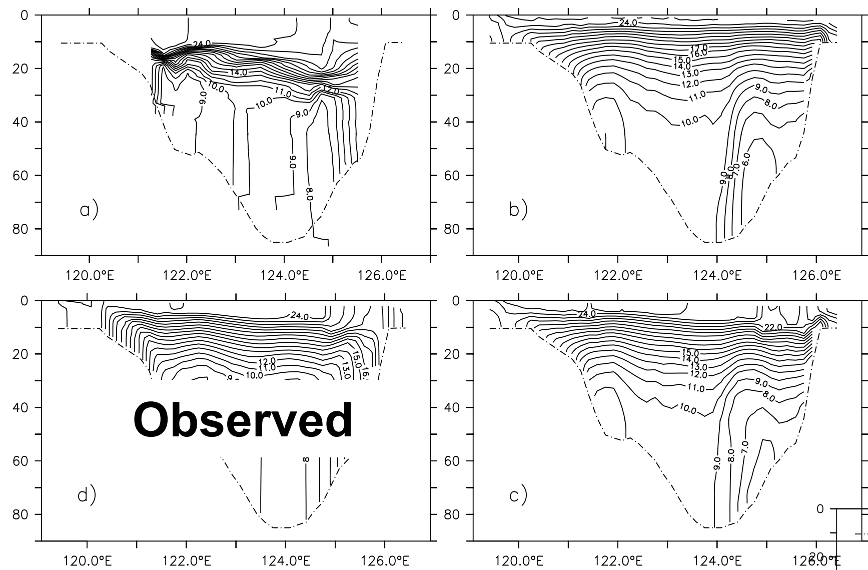
Observed



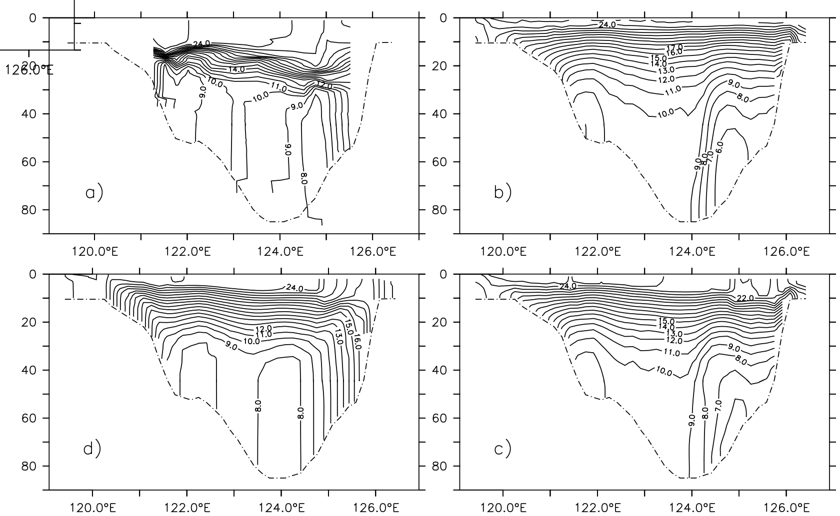
Modeled by POM model only

Not good in the upper and deep layers

as well as the transition one.



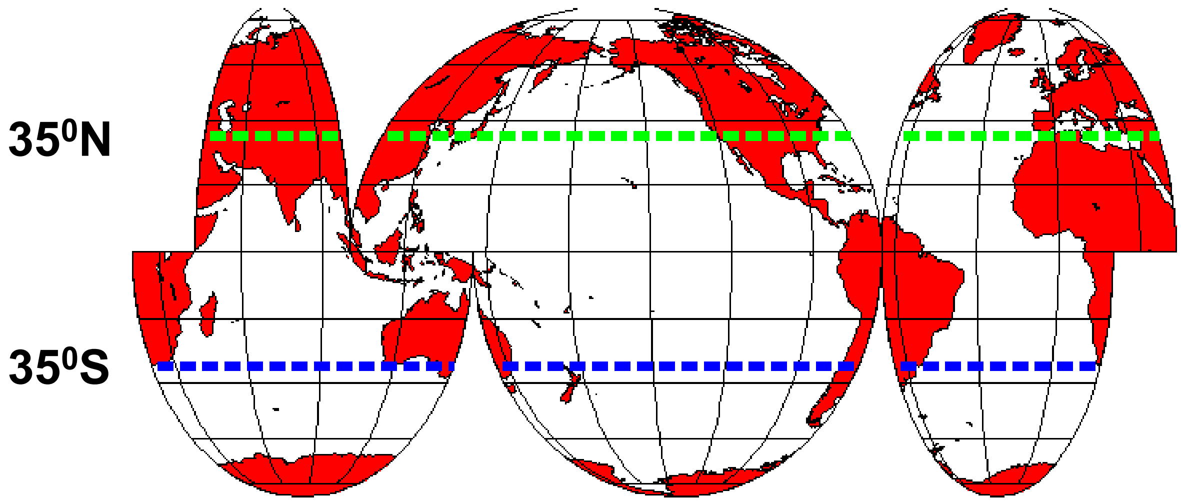
Good improvement in
the upper and deep layers
as well as the transition one.



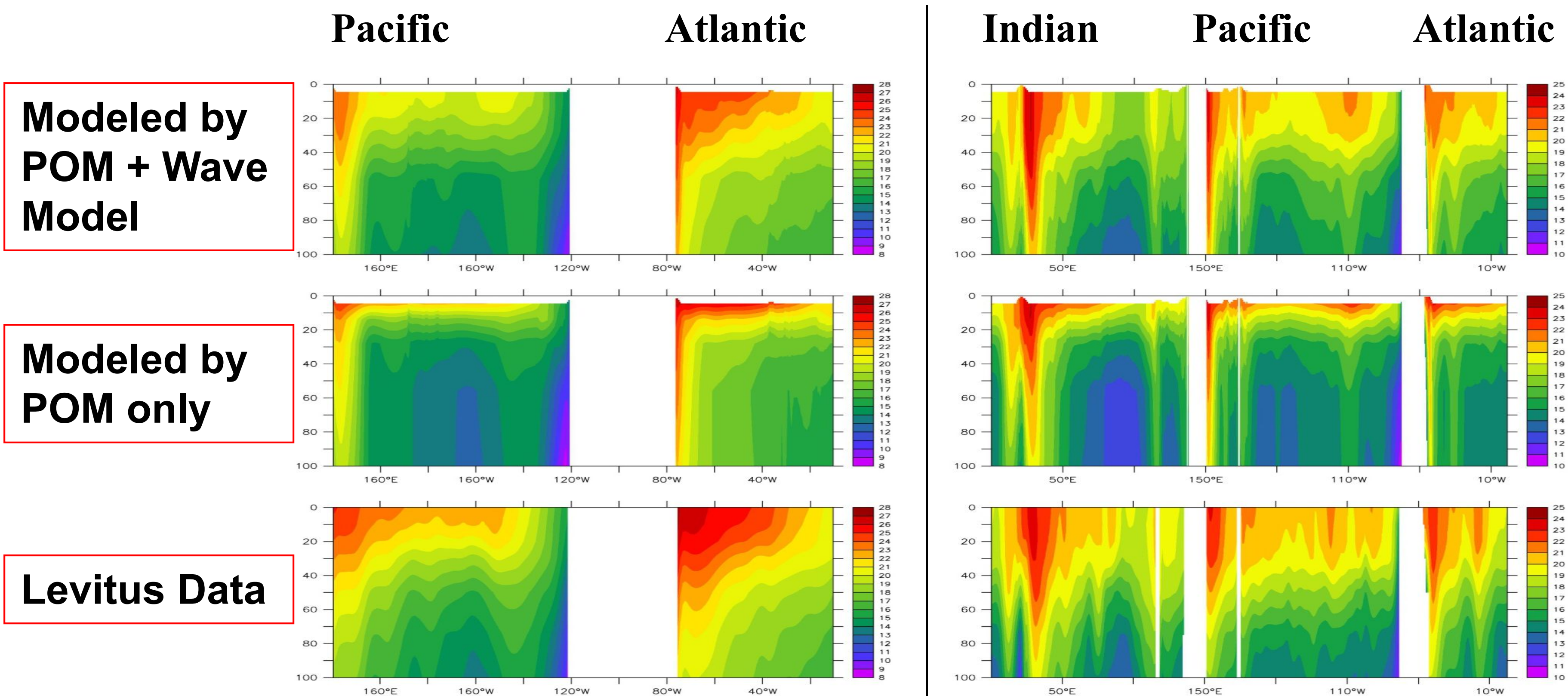
Modeled by POM + WAVE model with tide

II) Simulation in Global Ocean

using the coupled ocean circulation model



1. The comparison of the modeled temperature to the Levitus Data along the sections of 35^0N and 35^0S



in August (summer)

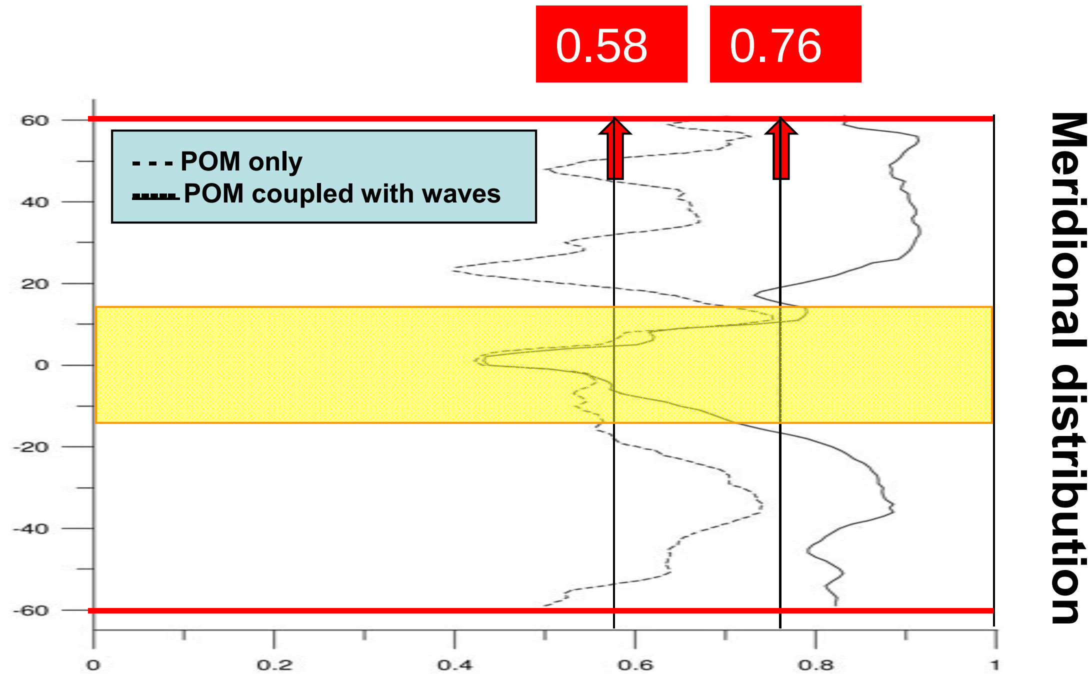
in February (winter)

Thick and uniform upper layer and strong transition are modeled.

2. The improvement shown with correlation coefficient

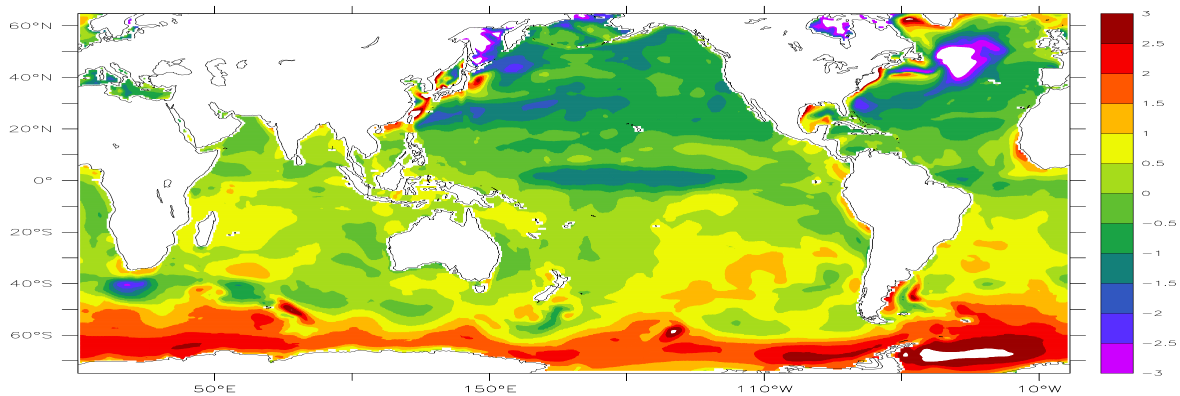
Good improvement in global modeling

But no improvement in low latitudes

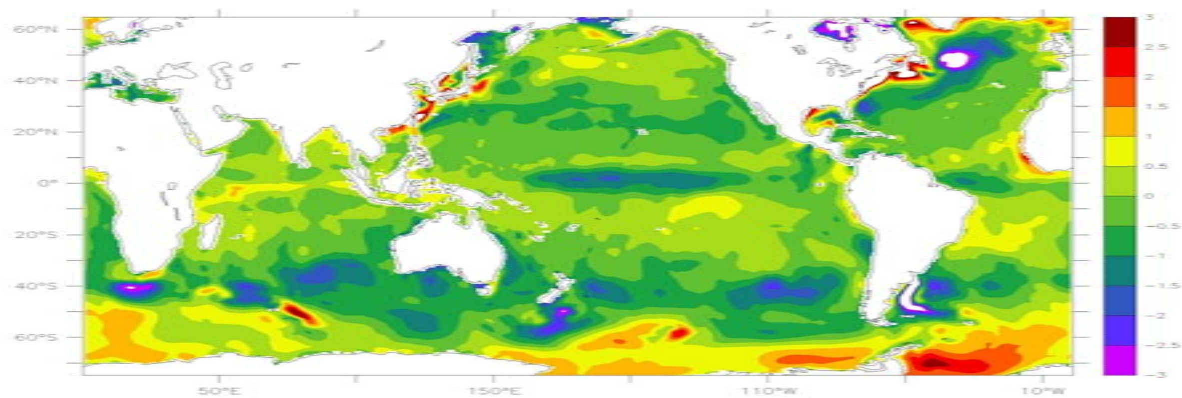


The correlation coefficient of the modeled temperature to the Levitus Data is raised from 0.58 to 0.76 for POM from 0.62 to 0.79 for ROMS in upper 100m layer.

3. The comparison of the SST modeled to the COADS climatology



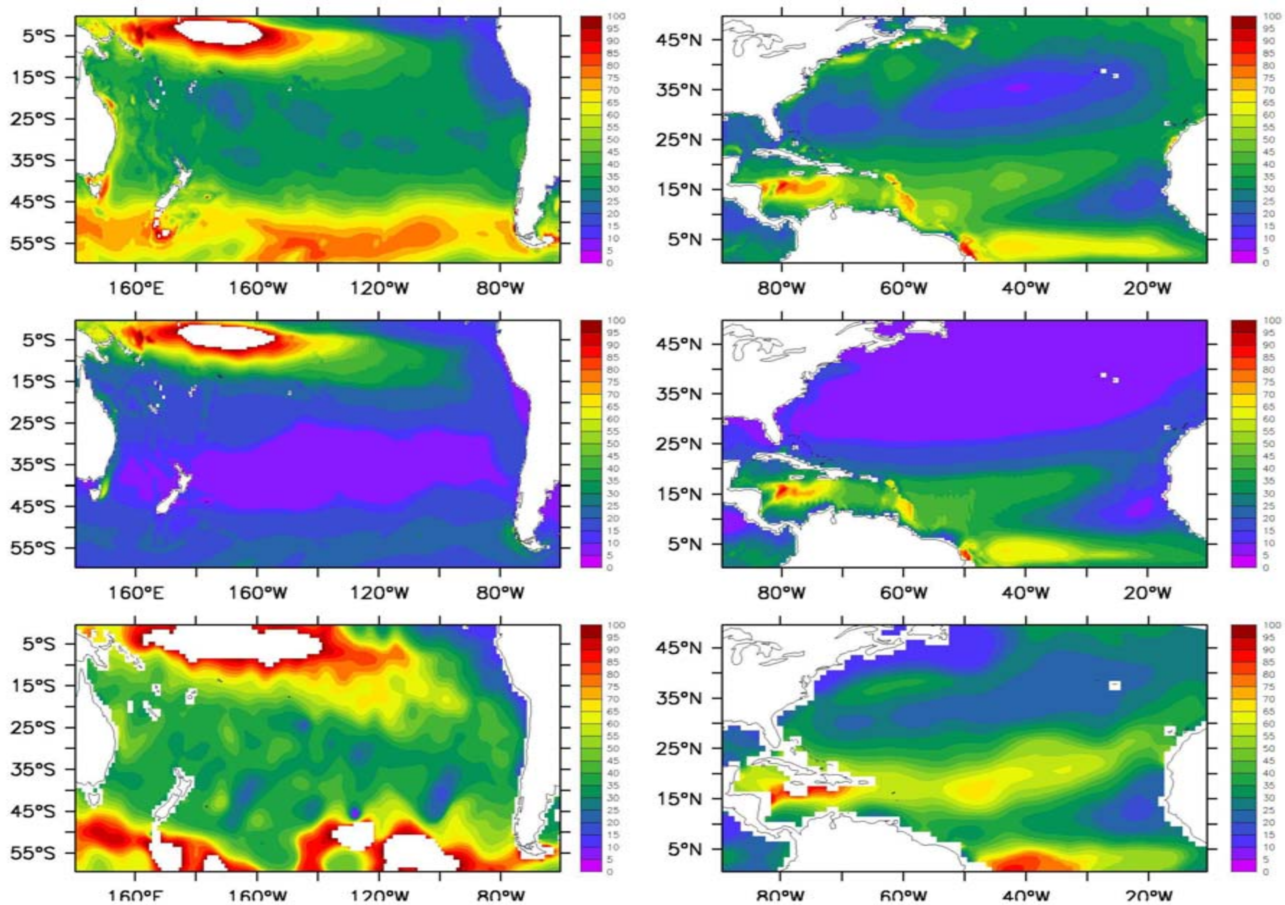
Deviation of the SST modeled by POM without sea wave-induced mixing from the COADS climatology



Deviation of the SST modeled by POM with sea wave-induced mixing from the COADS climatology

4. The comparison of the depth of mixing layer modeled to the Levitus Data

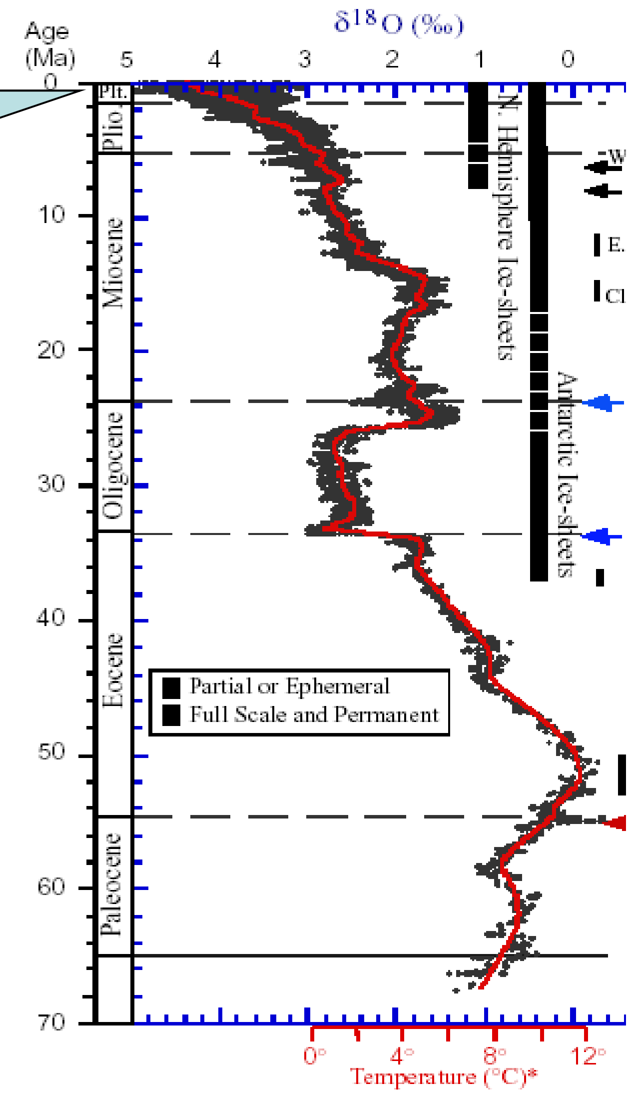
	Depth of mixing layer of the South Pacific in Feb.	Depth of mixing layer of the North Atlantic in Aug.
Levitus Data		
POM Model only		
POM + Wave Model		



III) Simulation in Global Climate

using the coupled climate model

Modeling in recent 1000 years



1. The improvement in the tropical bias indicated by the 27^0C isotherm

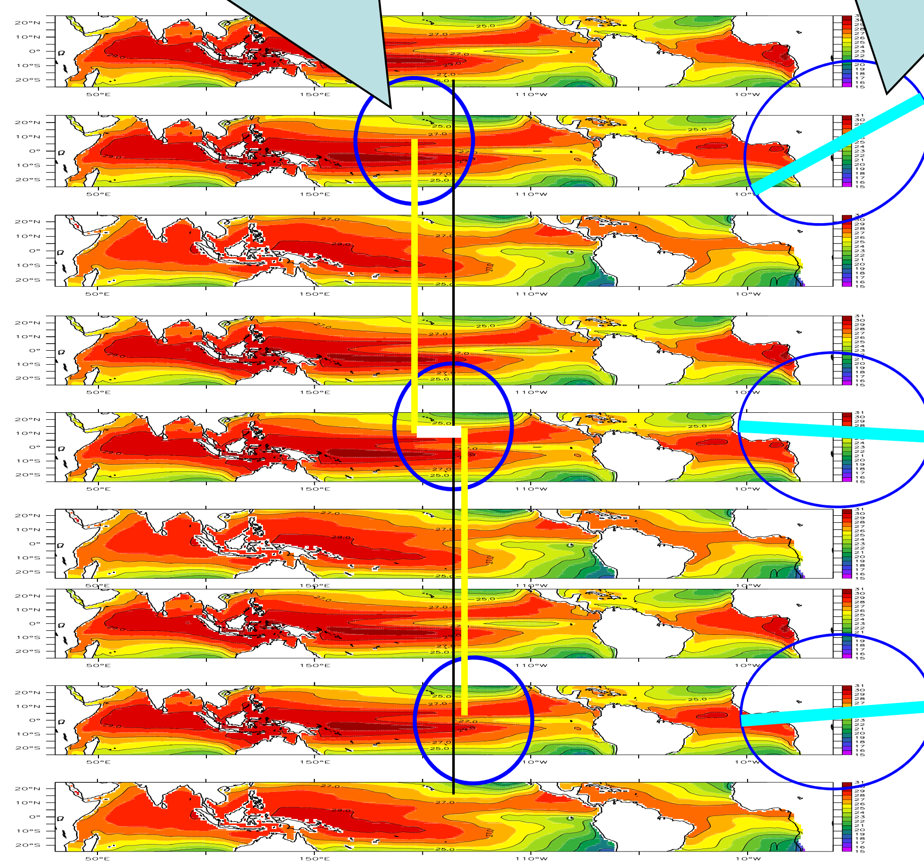
too forward cold tongue in the East Pacific

opposite SST slope in the Atlantic

NCAR-CCSM3 Model only

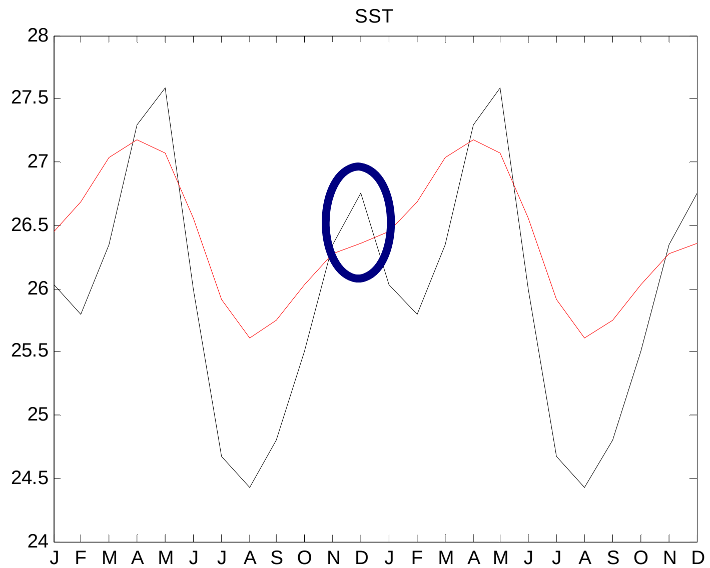
Observation

NCAR-CCSM3 + Sea Wave Model



2. The improvement in the SST time evolution in the equator area of the Eastern Pacific

(averaged in 110^0-90^0W, 5^0S-5^0N area)



Black : with two peaks a year

modeled by NCAR-CCSM3 Model

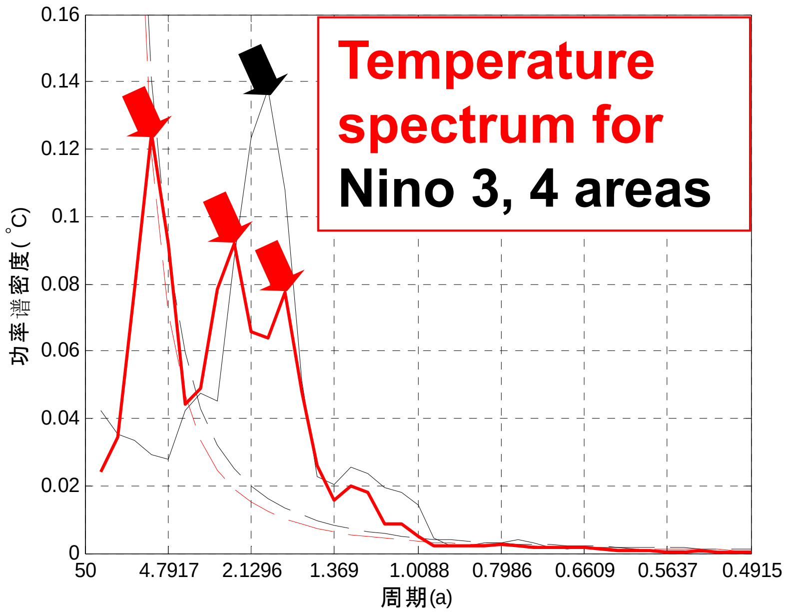
Red: with one peak a year

modeled by NCAR-CCSM3 Model

with sea wave mixing

2. The improvement in the time scale of inter-annual variation

(averaged in 1100-900W, 50S-50N area)

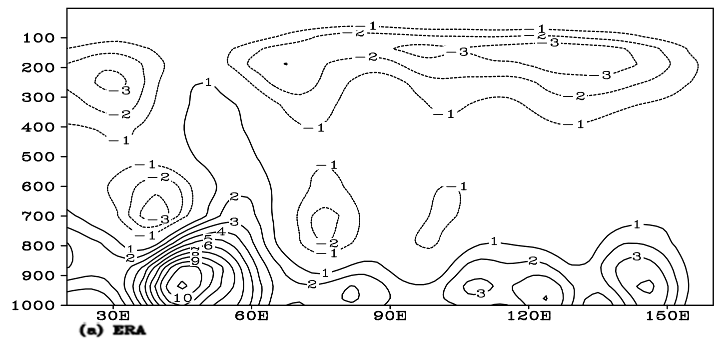


Black: with only one peak at 2.0a modeled by the CCSM3 Model

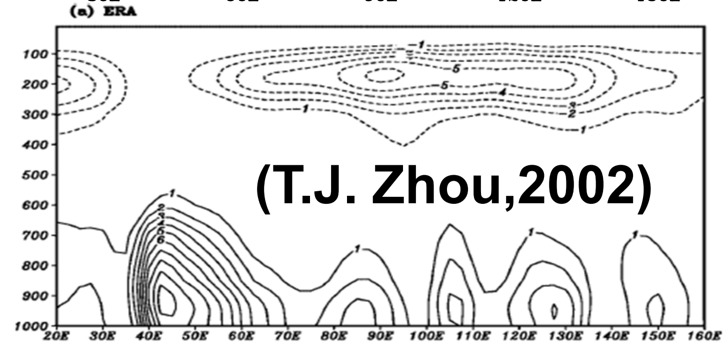
Red: with three peaks at 1.8a, 3.0a and 7.0a modeled by the CCSM3 Model with sea wave mixing

3. The improvement in the meridional wind speed along the equator

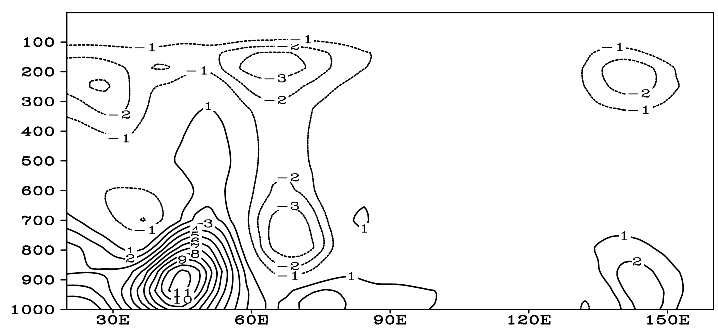
The modeled by
FGCM
+ Sea Wave Model



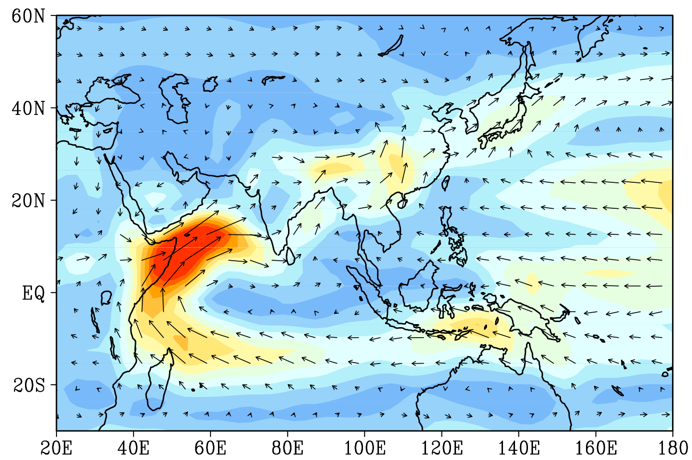
The data analysis
by T.J. Zhou, 2002



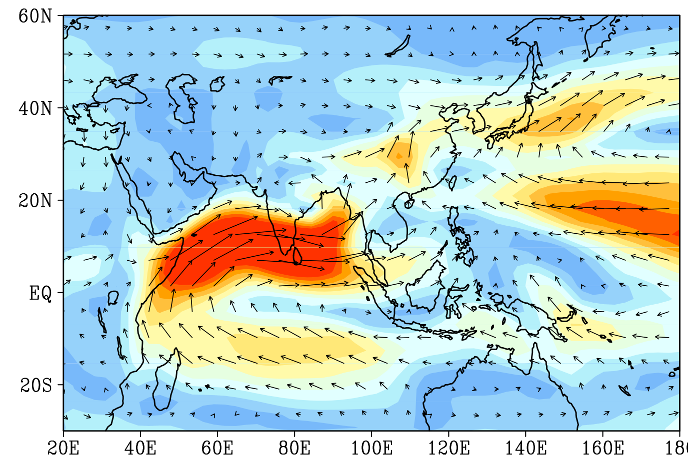
The modeled by
FGCM model



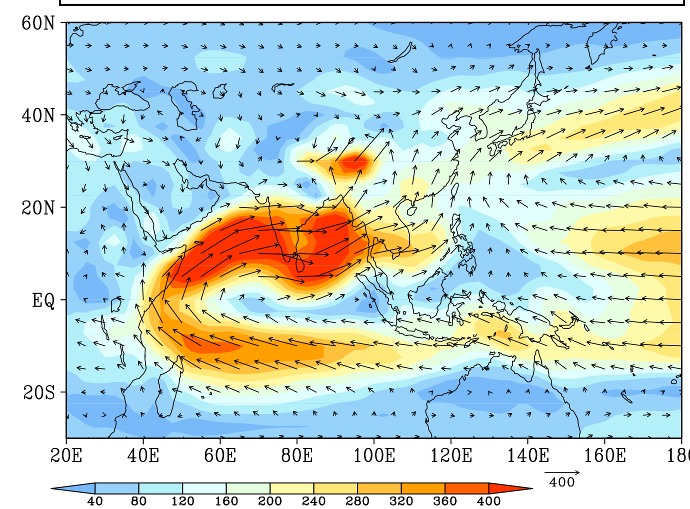
4. The improvement in the moisture transport



The modeled by FGCM
Model only

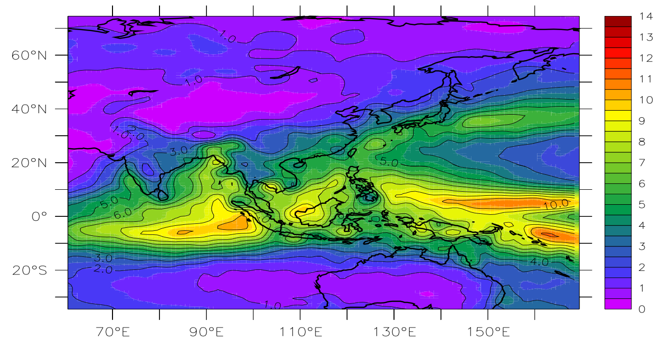


The modeled by FGCM
+ Sea Wave Model

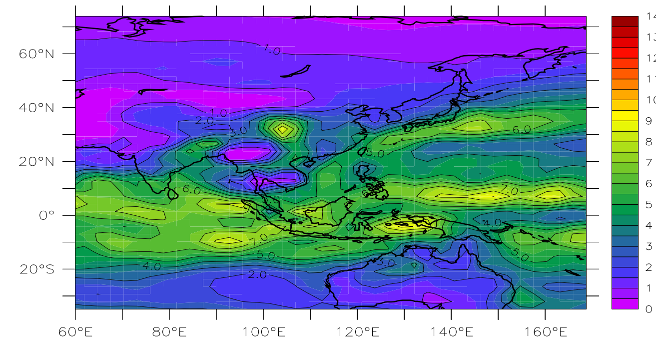


The observation

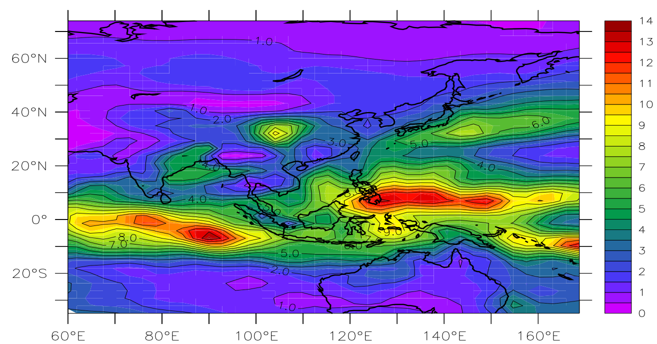
5. The improvement in the precipitation in Asia-Australia Monsoon area



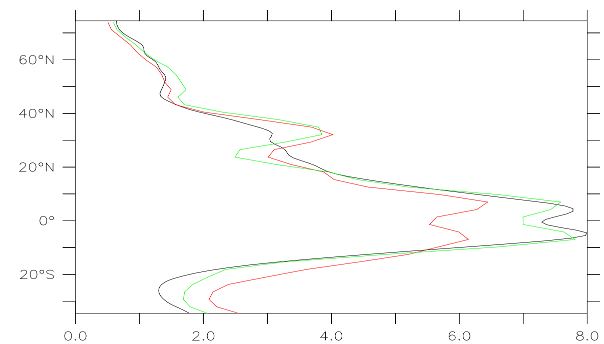
Xie-Arkin data (1979-2003)



Modeled by FGCM model only



Modeled by FGCM + Sea Wave model



Zonal averaged

Although we got good simulation results, we still need to answer the questions that

what the ocean mixing is,

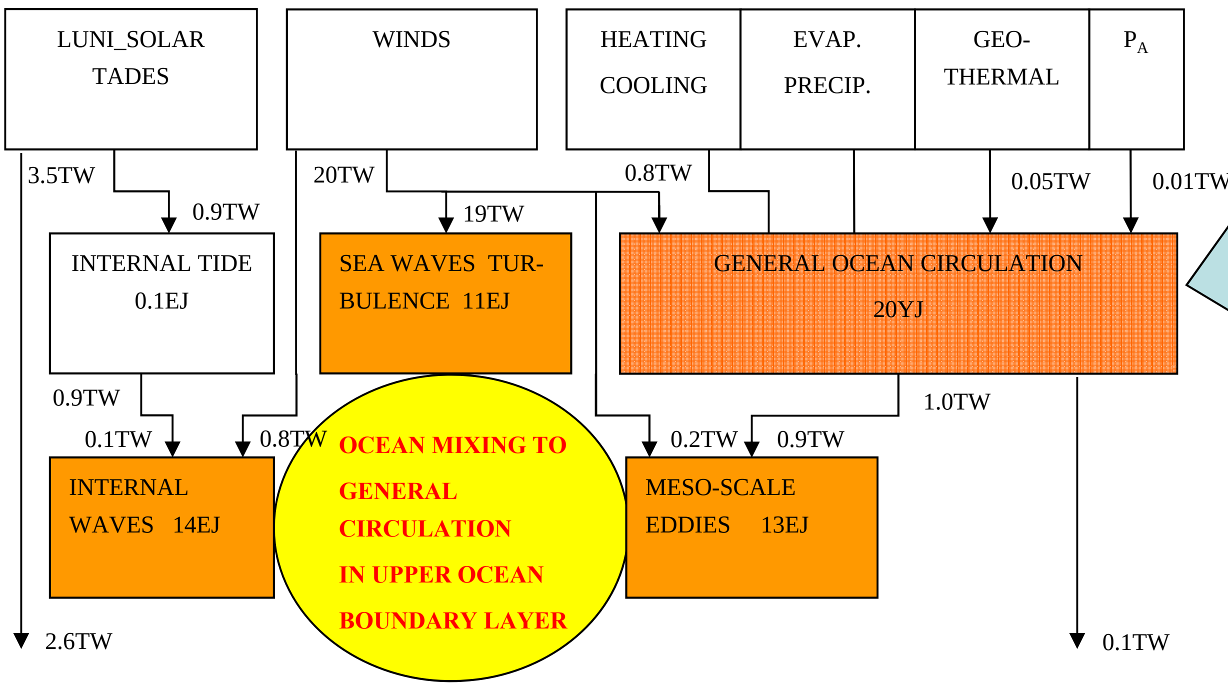
How to estimate analytically the mixing induced by sea waves.

Contents

I. Ocean dynamic system

and its governing equation set

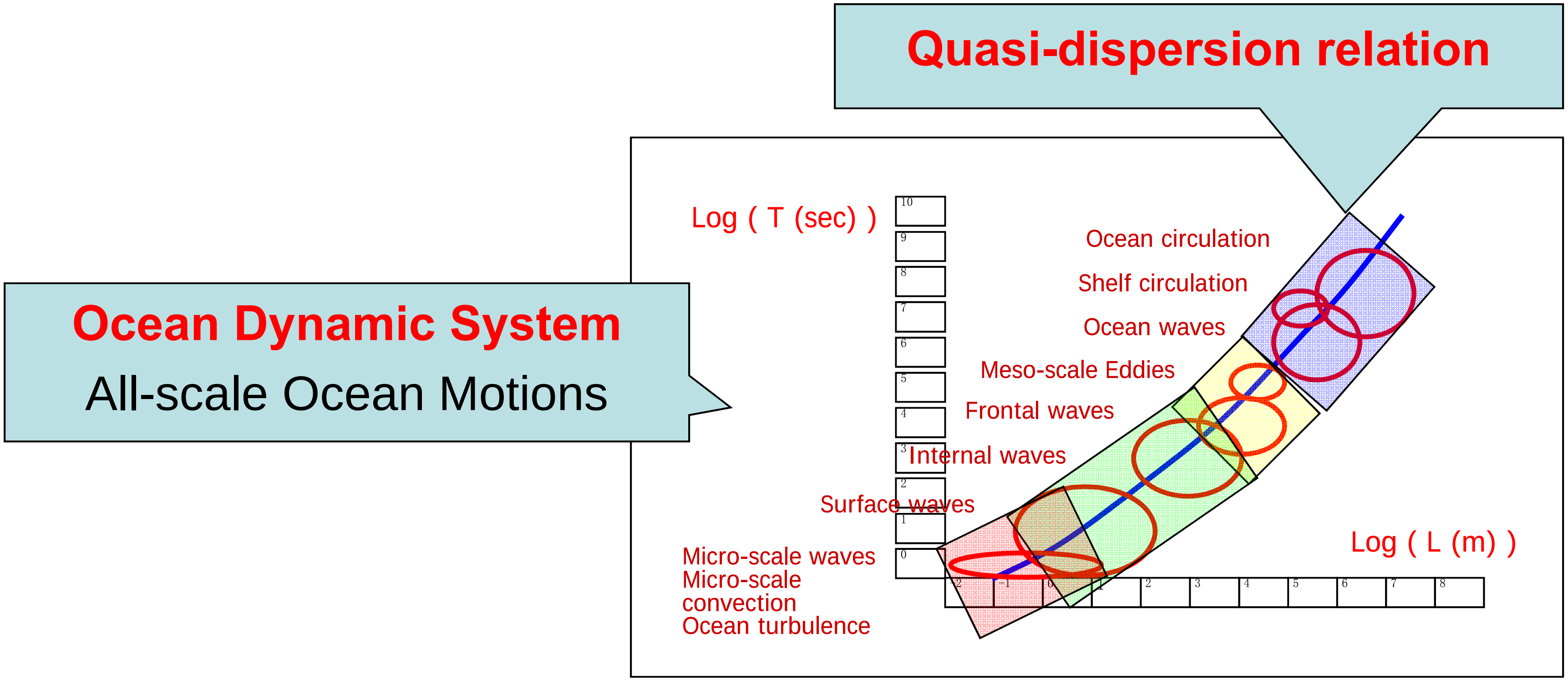
In fact, Wunsch presented the conception of ocean dynamic system for ocean mixing study in 2004, which includes four sub-systems of sea waves + turbulence, internal waves, meso-scale eddies and general ocean circulation.



The Wunsch' ocean dynamic system and the mixing effects of Sea waves + turbulence, Internal waves, Meso-scale eddies and Ocean circulation.

Wunsch C. and R. Ferrari, 2004: Vertical mixing, energy, and the general circulation of the oceans, Annu. Rev. Fluid Mech., 36:281-314.

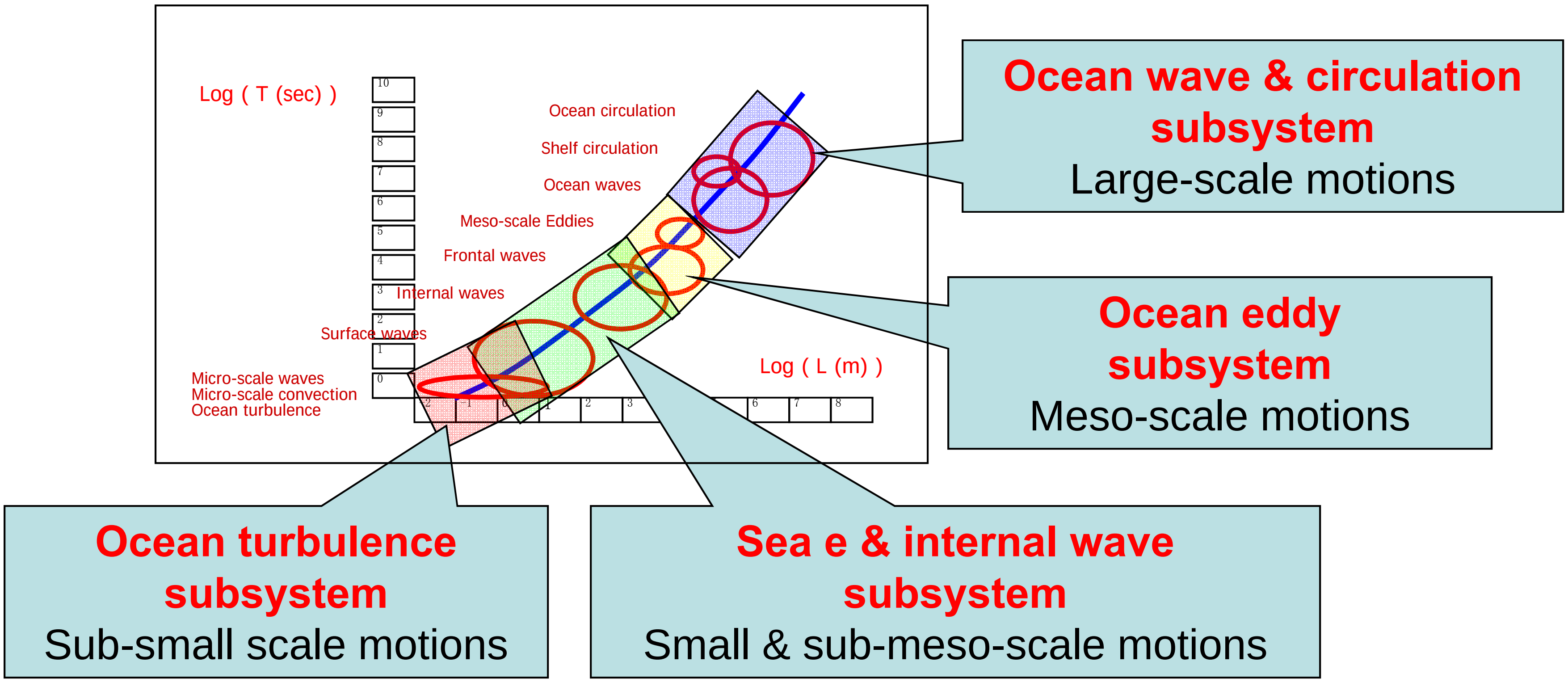
1. The quasi-dispersion relation
and the ocean dynamic system



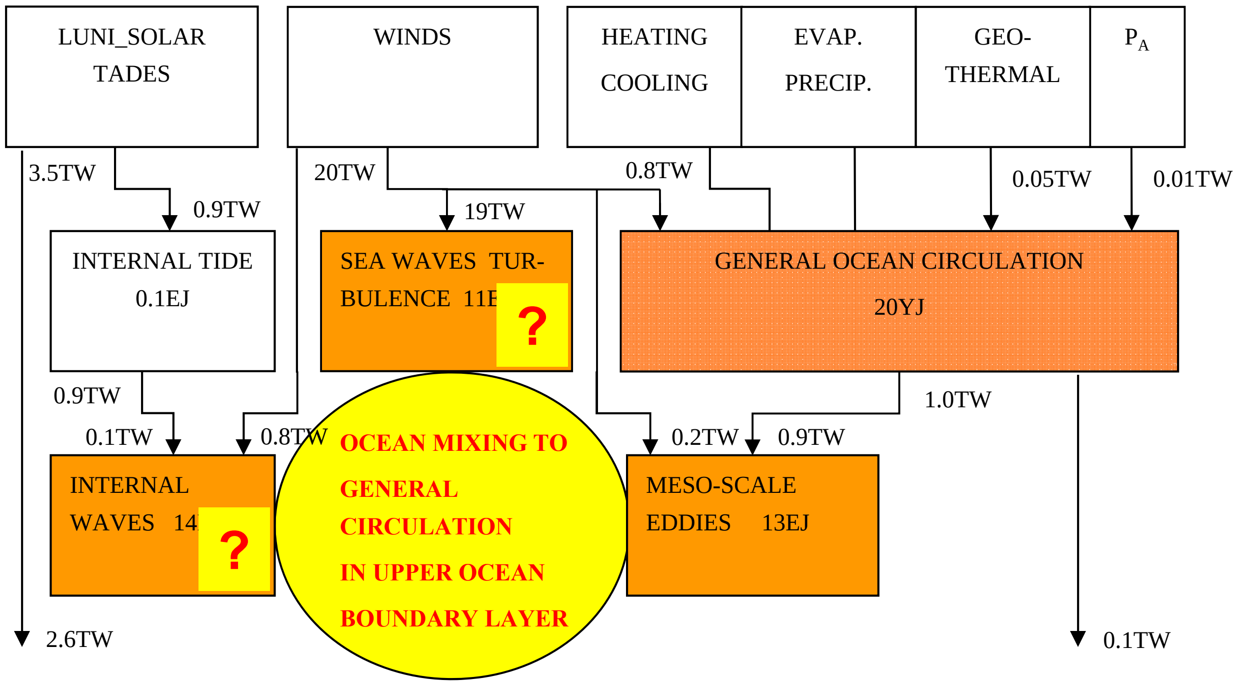
The quasi-dispersion relation shows the distribution of time- and space-scales of the ocean motions.

2. The four sub-system of the ocean dynamic system

According to the quasi-dispersion relation the ocean dynamic system can be divided in to four sub-systems.



3. The partition is consistent
with the Wunsch 's consideration



Wunsch C. and R. Ferrari, 2004: Vertical mixing, energy, and the general circulation of the oceans, Annu. Rev. Fluid Mech., 36:281-314.

In order to get the governing equation set for the ocean dynamic system **we define a three-fold Reynolds average treatment:**

$\langle X \rangle_{SS}$ **The first fold Reynolds average**

on the motion set of ocean turbulence

$\langle X \rangle_{SM}$ **The second fold Reynolds average**

on the motion set of sea and internal waves

$\langle X \rangle_{MM}$ **The third fold Reynolds average**

on the motion set of ocean eddies

I) The governing equations

of the ocean perturbation motions

1. The governing equations of ocean turbulence

1) The motion equations

$$\nabla \cdot \mathbf{u}_{SS}^{\prime} = 0$$

$$\frac{\partial \mathbf{u}_{SS}^{\prime}}{\partial t} + \left(\tilde{\mathbf{U}} \cdot \nabla\right)\mathbf{u}_{SS}^{\prime} + \left(\mathbf{u}_{SS}^{\prime} \cdot \nabla\right)\tilde{\mathbf{U}} + \Delta_{SS}\left[\left(\mathbf{u}_{SS}^{\prime} \cdot \nabla\right)\mathbf{u}_{SS}^{\prime}\right] + 2\bar{\mathbf{\Omega}} \times \mathbf{u}_{SS}^{\prime} = -\frac{1}{\rho_0}\nabla p_{SS}^{\prime} - g\frac{\rho_{SS}^{\prime}}{\rho_0}\mathbf{k} + \nu_0 \Delta \mathbf{u}_{SS}^{\prime}$$

$$\frac{\partial T_{SS}^{\prime}}{\partial t} + \left(\tilde{\mathbf{U}} \cdot \nabla\right)T_{SS}^{\prime} + \left(\mathbf{u}_{SS}^{\prime} \cdot \nabla\right)\tilde{T} + \Delta_{SS}\left[\left(\mathbf{u}_{SS}^{\prime} \cdot \nabla\right)T_{SS}^{\prime}\right] = \kappa_0 \Delta T_{SS}^{\prime} + Q_{SS}^{\prime}$$

$$\frac{\partial s_{SS}^{\prime}}{\partial t} + \left(\tilde{\mathbf{U}} \cdot \nabla\right)s_{SS}^{\prime} + \left(\mathbf{u}_{SS}^{\prime} \cdot \nabla\right)\tilde{s} + \Delta_{SS}\left[\left(\mathbf{u}_{SS}^{\prime} \cdot \nabla\right)s_{SS}^{\prime}\right] = D_0 \Delta s_{SS}^{\prime}$$

$$\rho_{SS}^{\prime} = \left[\rho\left(\tilde{s} + s_{SS}^{\prime}, \tilde{T} + T_{SS}^{\prime}, \tilde{p} + p_{SS}^{\prime}\right) - \rho\left(\tilde{s}, \tilde{T}, \tilde{p}\right)\right]$$

2) The boundary conditions 。。。。。。。

2. The governing equations of sea & internal waves

1) The motion equations

$$\nabla \cdot \mathbf{u}'_{SM} = 0$$

$$\frac{\partial \mathbf{u}'_{SM}}{\partial t} + \left(\hat{\mathbf{U}} \cdot \nabla\right)\mathbf{u}'_{SM} + \left(\mathbf{u}'_{SM} \cdot \nabla\right)\hat{\mathbf{U}} + \Delta_{SM}\left[\left(\mathbf{u}'_{SM} \cdot \nabla\right)\mathbf{u}'_{SM}\right] + 2\bar{\mathbf{\Omega}} \times \mathbf{u}'_{SM}$$

$$= -\frac{1}{\rho_0}\nabla p'_{SM} - g\frac{\rho'_{SM}}{\rho_0}\mathbf{k} + \left\{\nu_0 \Delta \mathbf{u}'_{SM} - \Delta_{SM}\left[\left\langle\left(\mathbf{u}'_{SS} \cdot \nabla\right)\mathbf{u}'_{SS}\right\rangle_{SS}\right]\right\}$$

$$\frac{\partial T'_{SM}}{\partial t} + \left(\hat{\mathbf{U}} \cdot \nabla\right)T'_{SM} + \left(\mathbf{u}'_{SM} \cdot \nabla\right)\hat{T} + \Delta_{SM}\left[\left(\mathbf{u}'_{SM} \cdot \nabla\right)T'_{SM}\right]$$

$$= \left\{\kappa_0 \Delta T'_{SM} - \Delta_{SM}\left[\left\langle\left(\mathbf{u}'_{SS} \cdot \nabla\right)T'_{SS}\right\rangle_{SS}\right]\right\} + Q'_{SM}$$

$$\frac{\partial s'_{SM}}{\partial t} + \left(\hat{\mathbf{U}} \cdot \nabla\right)s'_{SM} + \left(\mathbf{u}'_{SM} \cdot \nabla\right)\hat{s} + \Delta_{SM}\left[\left(\mathbf{u}'_{SM} \cdot \nabla\right)s'_{SM}\right]$$

$$= \left\{D_0 \Delta s'_{SM} - \Delta_{SM}\left[\left\langle\left(\mathbf{u}'_{SS} \cdot \nabla\right)s'_{SS}\right\rangle_{SS}\right]\right\}$$

$$\rho'_{SM} = \left[\rho\left(\left(\hat{s} + s'_{SM}\right), \left(\hat{T} + T'_{SM}\right), \left(\hat{p} + p'_{SM}\right)\right) - \rho\left(\hat{s}, \hat{T}, \hat{p}\right)\right]$$

2) The boundary conditions 。。。。。。。

3. The governing equations of ocean eddies

1) The motion equations

$$\nabla\cdot\mathbf{u}_{MM}^{/}=0$$

$$\frac{\partial\mathbf{u}_{MM}^{/}}{\partial t}+\left(\bar{\mathbf{U}}\cdot\nabla\right)\mathbf{u}_{MM}^{/}+\left(\mathbf{u}_{MM}^{/}\cdot\nabla\right)\bar{\mathbf{U}}+\Delta_{MM}\left[\left(\mathbf{u}_{MM}^{/}\cdot\nabla\right)\mathbf{u}_{MM}^{/}\right]+2\bar{\mathbf{\Omega}}\times\mathbf{u}_{MM}^{/}$$

$$=-\frac{1}{\rho_0}\nabla p_{MM}^{/}-g\frac{\rho_{MM}^{/}}{\rho_0}\mathbf{k}+\left\{\nu_0\Delta\mathbf{u}_{MM}^{/}-\Delta_{MM}\left[\left\langle\left(\mathbf{u}_{SM}^{/}\cdot\nabla\right)\mathbf{u}_{SM}^{/}\right\rangle_{SM}\right]-\Delta_{MM}\left[\left\langle\left\langle\left(\mathbf{u}_{SS}^{/}\cdot\nabla\right)\mathbf{u}_{SS}^{/}\right\rangle_{SS}\right\rangle_{SM}\right]\right\}$$

$$\frac{\partial T_{MM}^{/}}{\partial t}+\left(\bar{\mathbf{U}}\cdot\nabla\right)T_{MM}^{/}+\left(\mathbf{u}_{MM}^{/}\cdot\nabla\right)\bar{T}+\Delta_{MM}\left[\left(\mathbf{u}_{MM}^{/}\cdot\nabla\right)T_{MM}^{/}\right]$$

$$=\left\{\kappa_0\Delta T_{MM}^{/}-\Delta_{MM}\left[\left\langle\left(\mathbf{u}_{SM}^{/}\cdot\nabla\right)T_{SM}^{/}\right\rangle_{SM}\right]-\Delta_{MM}\left[\left\langle\left\langle\left(\mathbf{u}_{SS}^{/}\cdot\nabla\right)T_{SS}^{/}\right\rangle_{SS}\right\rangle_{SM}\right]\right\}+Q_{MM}^{/}$$

$$\frac{\partial s_{MM}^{/}}{\partial t}+\left(\bar{\mathbf{U}}\cdot\nabla\right)s_{MM}^{/}+\left(\mathbf{u}_{MM}^{/}\cdot\nabla\right)\bar{s}+\Delta_{MM}\left[\left(\mathbf{u}_{MM}^{/}\cdot\nabla\right)s_{MM}^{/}\right]$$

$$=\left\{D_0\Delta s_{MM}^{/}-\Delta_{MM}\left[\left\langle\left(\mathbf{u}_{SM}^{/}\cdot\nabla\right)s_{SM}^{/}\right\rangle_{SM}\right]-\Delta_{MM}\left[\left\langle\left\langle\left(\mathbf{u}_{SS}^{/}\cdot\nabla\right)s_{SS}^{/}\right\rangle_{SS}\right\rangle_{SM}\right]\right\}$$

$$\rho_{MM}^{/}=\left[\rho\left(\left(\bar{s}+s_{MM}^{/}\right),\left(\bar{T}+T_{MM}^{/}\right),\left(\bar{p}+p_{MM}^{/}\right)\right)-\rho\left(\bar{s},\bar{T},\bar{p}\right)\right]$$

2) The boundary conditions。。。。。。。

II) The governing equations of the ocean motion with scale larger than some one

1. The governing equations of the ocean motion with scale larger than sub-small one

1) The motion equations

$$\nabla \cdot \tilde{\mathbf{U}} = 0$$

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \left(\tilde{\mathbf{U}} \cdot \nabla\right)\tilde{\mathbf{U}} + 2\overline{\mathbf{\Omega}} \times \tilde{\mathbf{U}} = -\frac{1}{\rho_0}\nabla \tilde{p} - g\frac{\tilde{\rho}}{\rho_0}\mathbf{k} + \left[\nu_0 \Delta \tilde{\mathbf{U}} - \left\langle\left(\mathbf{u}_{SS}^{/} \cdot \nabla\right)\mathbf{u}_{SS}^{/}\right\rangle_{SS}\right]$$

$$\frac{\partial \tilde{T}}{\partial t} + \left(\tilde{\mathbf{U}} \cdot \nabla\right)\tilde{T} = \left[\kappa_0 \Delta \tilde{T} - \left\langle\left(\mathbf{u}_{SS}^{/} \cdot \nabla\right)T_{SS}^{/}\right\rangle_{SS}\right] + \tilde{Q}$$

$$\frac{\partial \tilde{s}}{\partial t} + \left(\tilde{\mathbf{U}} \cdot \nabla\right)\tilde{s} = \left[D_0 \Delta \tilde{s} - \left\langle\left(\mathbf{u}_{SS}^{/} \cdot \nabla\right)s_{SS}^{/}\right\rangle_{SS}\right]$$

$$\tilde{\rho} = \rho\left(\tilde{s}, \tilde{T}, \tilde{p}\right)$$

2) The boundary conditions 。。。。。。。

2. The governing equations of the ocean motion with scale larger than small & sub-meso one

1) The motion equations

$$\nabla\cdot\hat{\mathbf{U}}=0$$

$$\frac{\partial\hat{\mathbf{U}}}{\partial t}+\left(\hat{\mathbf{U}}\cdot\nabla\right)\hat{\mathbf{U}}+2\bar{\mathbf{\Omega}}\times\hat{\mathbf{U}}=-\frac{1}{\rho_0}\nabla\hat{p}-g\frac{\hat{\rho}}{\rho_0}\mathbf{k}+\left[\nu_0\Delta\hat{\mathbf{U}}-\left\langle\left\langle\left(\mathbf{u}_{SS}^{/}\cdot\nabla\right)\mathbf{u}_{SS}^{/}\right\rangle_{SS}\right\rangle_{SM}-\left\langle\left(\mathbf{u}_{SM}^{/}\cdot\nabla\right)\mathbf{u}_{SM}^{/}\right\rangle_{SM}\right]$$

$$\frac{\partial\hat{T}}{\partial t}+\left(\hat{\mathbf{U}}\cdot\nabla\right)\hat{T}=\left[\kappa_0\Delta\hat{T}-\left\langle\left\langle\left(\mathbf{u}_{SS}^{/}\cdot\nabla\right)T_{SS}^{/}\right\rangle_{SS}\right\rangle_{SM}-\left\langle\left(\mathbf{u}_{SM}^{/}\cdot\nabla\right)T_{SM}^{/}\right\rangle_{SM}\right]+\hat{Q}$$

$$\frac{\partial\hat{s}}{\partial t}+\left(\hat{\mathbf{U}}\cdot\nabla\right)\hat{s}=\left[D_0\Delta\hat{s}-\left\langle\left(\mathbf{u}_{SM}^{/}\cdot\nabla\right)s_{SM}^{/}\right\rangle_{SM}-\left\langle\left\langle\left(\mathbf{u}_{SS}^{/}\cdot\nabla\right)s_{SS}^{/}\right\rangle_{SS}\right\rangle_{SM}\right]$$

$$\hat{\rho}=\rho\left(\hat{s},\hat{T},\hat{p}\right)$$

2) The boundary conditions 。。。。。。。

3. The governing equations of the large scale motion

1) The motion equations

$$\nabla \cdot \overline{\mathbf{U}} = 0$$

$$\frac{\partial \overline{\mathbf{U}}}{\partial t} + \left(\overline{\mathbf{U}} \cdot \nabla\right)\overline{\mathbf{U}} + 2\overline{\mathbf{\Omega}} \times \overline{\mathbf{U}} = -\frac{1}{\rho_0}\nabla \overline{p} - g\frac{\overline{\rho}}{\rho_0}\mathbf{k} + \left[\nu_0 \Delta \overline{\mathbf{U}} - \left\langle\left\langle\left\langle\left(\mathbf{u}'_{SS} \cdot \nabla\right)\mathbf{u}'_{SS}\right\rangle_{SS}\right\rangle_{SM}\right\rangle_{MM} - \left\langle\left\langle\left(\mathbf{u}'_{SM} \cdot \nabla\right)\mathbf{u}'_{SM}\right\rangle_{SM}\right\rangle_{MM} - \left\langle\left(\mathbf{u}'_{MM} \cdot \nabla\right)\mathbf{u}'_{MM}\right\rangle_{MM}\right]$$

$$\frac{\partial \overline{T}}{\partial t} + \left(\overline{\mathbf{U}} \cdot \nabla\right)\overline{T} = \left[\kappa_0 \Delta \overline{T} - \left\langle\left\langle\left\langle\left(\mathbf{u}'_{SS} \cdot \nabla\right)T'_{SS}\right\rangle_{SS}\right\rangle_{SM}\right\rangle_{MM} - \left\langle\left\langle\left(\mathbf{u}'_{SM} \cdot \nabla\right)T'_{SM}\right\rangle_{SM}\right\rangle_{MM} - \left\langle\left(\mathbf{u}'_{MM} \cdot \nabla\right)T'_{MM}\right\rangle_{MM}\right] + \overline{Q}$$

$$\frac{\partial \overline{s}}{\partial t} + \left(\overline{\mathbf{U}} \cdot \nabla\right)\overline{s} = \left[D_0 \Delta \overline{s} - \left\langle\left\langle\left\langle\left(\mathbf{u}'_{SS} \cdot \nabla\right)s'_{SS}\right\rangle_{SS}\right\rangle_{SM}\right\rangle_{MM} - \left\langle\left\langle\left(\mathbf{u}'_{SM} \cdot \nabla\right)s'_{SM}\right\rangle_{SM}\right\rangle_{MM} - \left\langle\left(\mathbf{u}'_{MM} \cdot \nabla\right)s'_{MM}\right\rangle_{MM}\right]$$

$$\overline{\rho} = \rho\left(\overline{s}, \overline{T}, \overline{p}\right)$$

turbulence

sea & internal waves

eddies

2) The boundary conditions 。。。。。。。

the large scale ocean mixing is defined by the residues of fluxes produced by molecule, turbulence, sea & internal waves and eddies based on the motion equations derived.

II. The analytic estimate of the sea wave-generated turbulence mixing

$$\frac{\partial \bar{\mathbf{U}}}{\partial t} + \left(\bar{\mathbf{U}} \cdot \nabla\right)\bar{\mathbf{U}} + 2\bar{\mathbf{\Omega}} \times \bar{\mathbf{U}} = -\frac{1}{\rho_0}\nabla \bar{p} - g\frac{\bar{\rho}}{\rho_0}\mathbf{k} + \nu_0 \Delta \bar{\mathbf{U}}$$

$$\boxed{+\frac{\partial}{\partial x_j}\left[-\left\langle\left\langle\left\langle u'_{SS\,i} u'_{SS\,j}\right\rangle_{SS}\right\rangle_{SM}\right\rangle_{MM}\right]} + \frac{\partial}{\partial x_j}\left[-\left\langle\left\langle u'_{SM\,i} u'_{SM\,j}\right\rangle_{SM}\right\rangle_{MM}\right] + \frac{\partial}{\partial x_j}\left[-\left\langle u'_{MM\,i} u'_{MM\,j}\right\rangle_{MM}\right]$$

I) The $2\frac{1}{2}$-level second order moment closure model of ocean turbulence

Second order moment closure model is the basic treatment for turbulence mixing, which includes

1. The 2-level expressions of second order moments

$$\left\langle u_i u_j \right\rangle = -\frac{k}{\varepsilon}\left(\left\langle u_i u_k \right\rangle \frac{\partial \tilde{U}_j}{\partial x_k} + \left\langle u_j u_k \right\rangle \frac{\partial \tilde{U}_i}{\partial x_k} \right) - C_{1VP}\left(\left\langle u_i u_j \right\rangle - \frac{1}{3}\delta_{ij}\left(2k\right) \right) - \left(C_{2VP}\right)\frac{k}{\varepsilon}\left(P_{ij} - \frac{1}{3}\delta_{ij}P_{kk} \right)$$

$$\left\langle u_i \theta \right\rangle = \frac{\sigma k}{\pi \varepsilon}\left[-\left(\left\langle u_k \theta \right\rangle \frac{\partial \tilde{U}_i}{\partial x_k} + \left\langle u_i u_k \right\rangle \frac{\partial \tilde{T}}{\partial x_k} \right) + 2\left(\frac{\partial \tilde{U}_i}{\partial x_l} \right)_M \left\langle u_l \theta \right\rangle + \left\langle u_i Q' \right\rangle \right]$$

$$\left\langle u_i s \right\rangle = \frac{\sigma k}{\pi \varepsilon}\left[-\left(\left\langle u_i u_k \right\rangle \frac{\partial \tilde{S}}{\partial x_k} + \left\langle u_k s \right\rangle \frac{\partial \tilde{U}_i}{\partial x_k} \right) + 2\left(\frac{\partial \tilde{U}_i}{\partial x_l} \right)_M \left\langle u_l s \right\rangle \right]$$

$$\left\langle \theta s \right\rangle = \frac{k}{\varepsilon}\left[-\left(\left\langle u_k s \right\rangle \frac{\partial \tilde{T}}{\partial x_k} + \left\langle u_k \theta \right\rangle \frac{\partial \tilde{S}}{\partial x_k} \right) + \left\langle s Q' \right\rangle \right]$$

$$\left\langle \theta^2 \right\rangle = 2\frac{k}{\varepsilon}\left[-\left\langle u_k \theta \right\rangle \frac{\partial \tilde{T}}{\partial x_k} + \left\langle \theta Q' \right\rangle \right] \qquad \left\langle s^2 \right\rangle = 2\frac{k}{\varepsilon}\left[-\left\langle u_k s \right\rangle \frac{\partial \tilde{S}}{\partial x_k} \right]$$

2. The 3-level governing equations of the characteristic quantities $k-\varepsilon$ of turbulence

1) The motion equations

$$\frac{\partial k}{\partial t}+U_\alpha\frac{\partial k}{\partial x_\alpha}=\frac{\partial}{\partial x_3}\left(\frac{k^2}{\pi^2\varepsilon}\frac{\partial k}{\partial x_3}\right)+\left[\left(-\langle u_\alpha u_3\rangle\frac{\partial\tilde{U}_\alpha}{\partial x_3}\right)-\varepsilon\right]$$

$$\frac{\partial \varepsilon}{\partial t}+\tilde{U}_\alpha\frac{\partial \varepsilon}{\partial x_\alpha}=\frac{\partial}{\partial x_3}\left(\frac{k^2}{\pi^2\varepsilon}\frac{\partial \varepsilon}{\partial x_3}\right)+2\frac{\varepsilon}{k}\left[\frac{3}{4}\left(-\langle u_\alpha u_3\rangle\frac{\partial\tilde{U}_\alpha}{\partial x_3}\right)-\varepsilon\right]$$

Two Generating terms by velocity shear of sea waves mainly

2) The boundary conditions on sea surface

$$\left.\frac{\partial k}{\partial x_3}\right|_{x_3=0}=\left(\frac{\pi^2\varepsilon}{k^2}\right)_{x_3=0}\left[\Phi E_t\left(H_S,\frac{H_S}{\bar{L}}\right)\right]$$

$$\varepsilon\Big|_{x_3=0}=\left(\frac{k}{\pi}\right)_{x_3=0}^{3/2}\left[\Psi h_{en}\left(\bar{L},\frac{H_S}{\bar{L}}\right)\right]^{-1}$$

Two Generating terms by sea wave breaking

Yuan, Y.L., L. Han, et al., 2009: The statistical theory of breaking entrain-ment depth and surface whitecap coverage of real sea waves, J. Phys. Ocean. vol.35, 143-161

Turbulence in upper ocean

≈ sea wave - generated turbulence

II) The balanced solution of ocean turbulence

Balanced solution is the basic description of ocean turbulence due to its scales are small

1. The definition of balanced solution

$$\because \quad \mathrm{Min}\left\{\left[\left(\frac{\overline{k}^2}{\pi^2\overline{\varepsilon}}\right)\left|\frac{\partial \tilde{U}_\alpha}{\partial x_3}\right|^2 - \overline{\varepsilon}\right]^2 + \left[\frac{3}{4}\left(\frac{\overline{k}^2}{\pi^2\overline{\varepsilon}}\right)\left|\frac{\partial \tilde{U}_\alpha}{\partial x_3}\right|^2 - \overline{\varepsilon}\right]^2\right\}$$

$$\therefore \quad \overline{\varepsilon} = \frac{7}{8}\frac{\overline{k}^2}{\pi^2\overline{\varepsilon}}\left|\frac{\partial \tilde{U}_\alpha}{\partial x_3}\right|^2$$

2. The determination of the balanced solution

In fact, the balanced solution allows a relation

$$\frac{\overline{k}^2}{\overline{\varepsilon}} = C_K \exp\{\gamma K x_3\}$$

so that we have a set of solutions

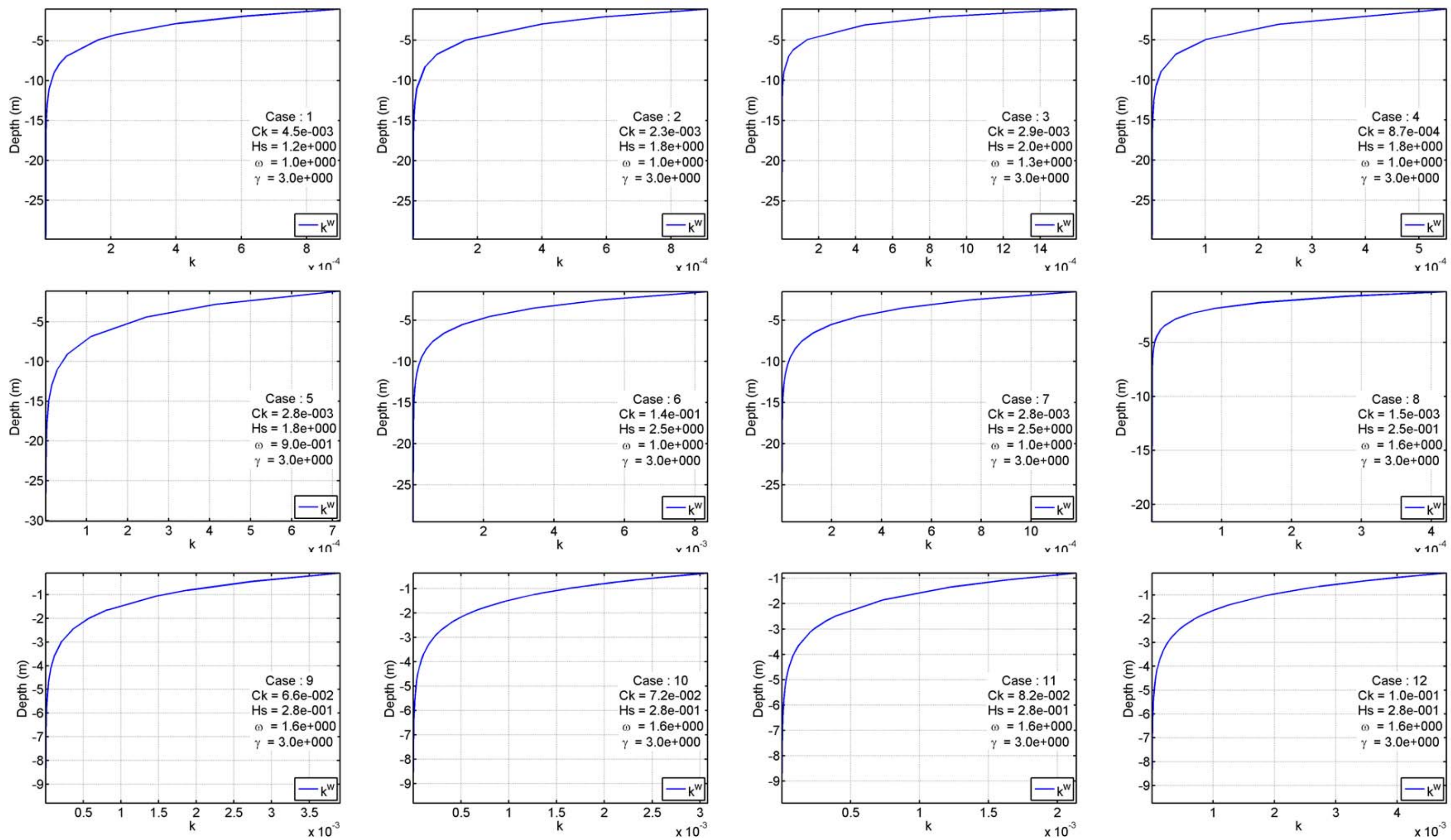
$$\overline{k} = \sqrt{\frac{7}{8}\frac{C_K}{\pi}\exp\{\gamma K x_3\}}\left|\frac{\partial \tilde{U}_\alpha}{\partial x_3}\right| \qquad \overline{\varepsilon} = \frac{7}{8}\frac{C_K}{\pi^2}\exp\{\gamma K x_3\}\left|\frac{\partial \tilde{U}_\alpha}{\partial x_3}\right|^2$$

The solution of $\gamma = 0$ was identified by Huang's data analysis.

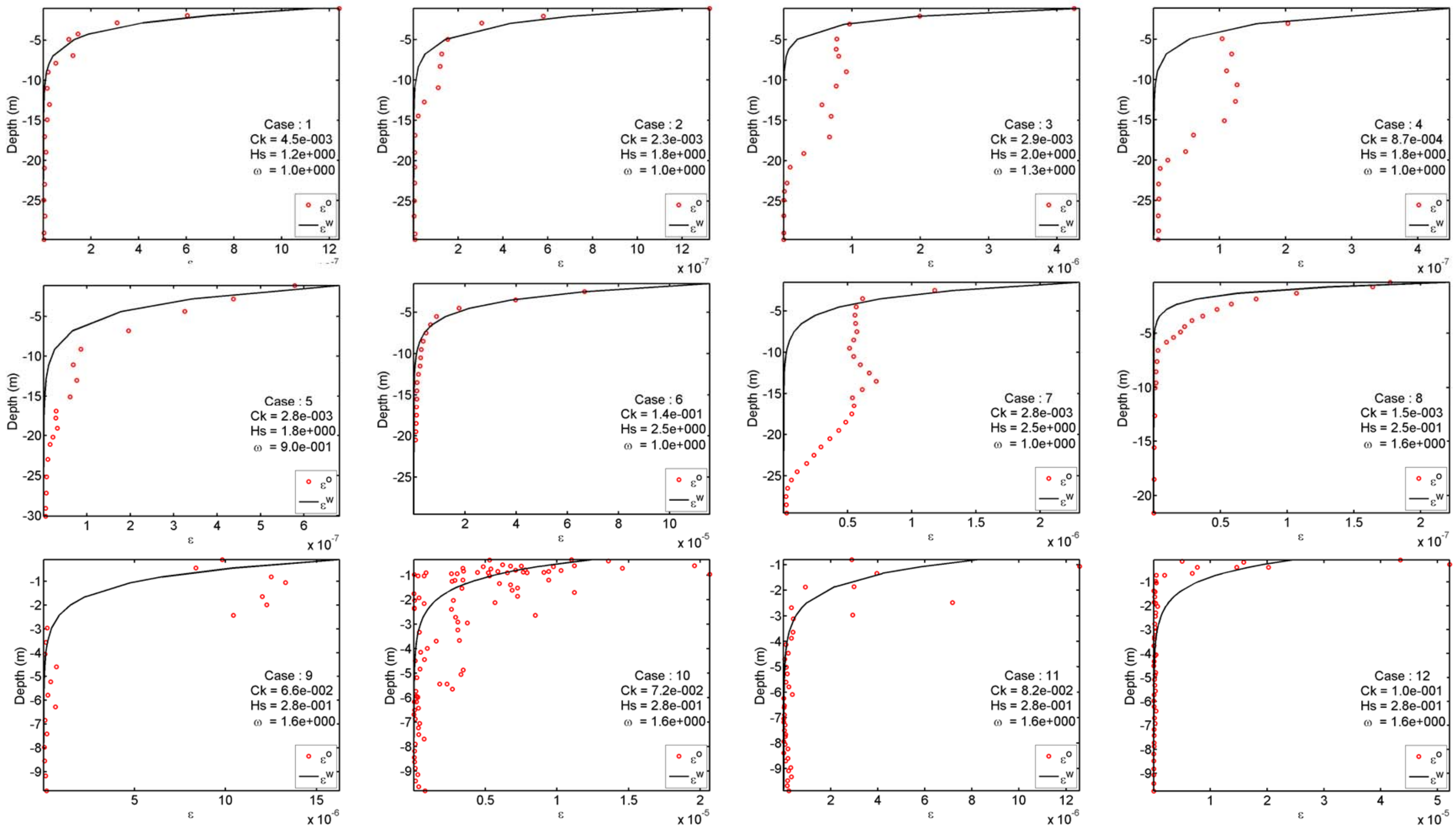
Huang, C. J. and F. L. Qiao, 2010: Wave-turbulence interaction and its induced mixing in the upper ocean, J. Geophys. Res., Vol.115, C04026

3. The comparison of the balanced theoretic solution to the field observation

1) The vertical distribution of theoretic kinetic energy k ($\gamma = 3$)



2) The vertical distribution of theoretic dissipation rate ε



The red dots are the measured data and the blue ones are the theoretic solution

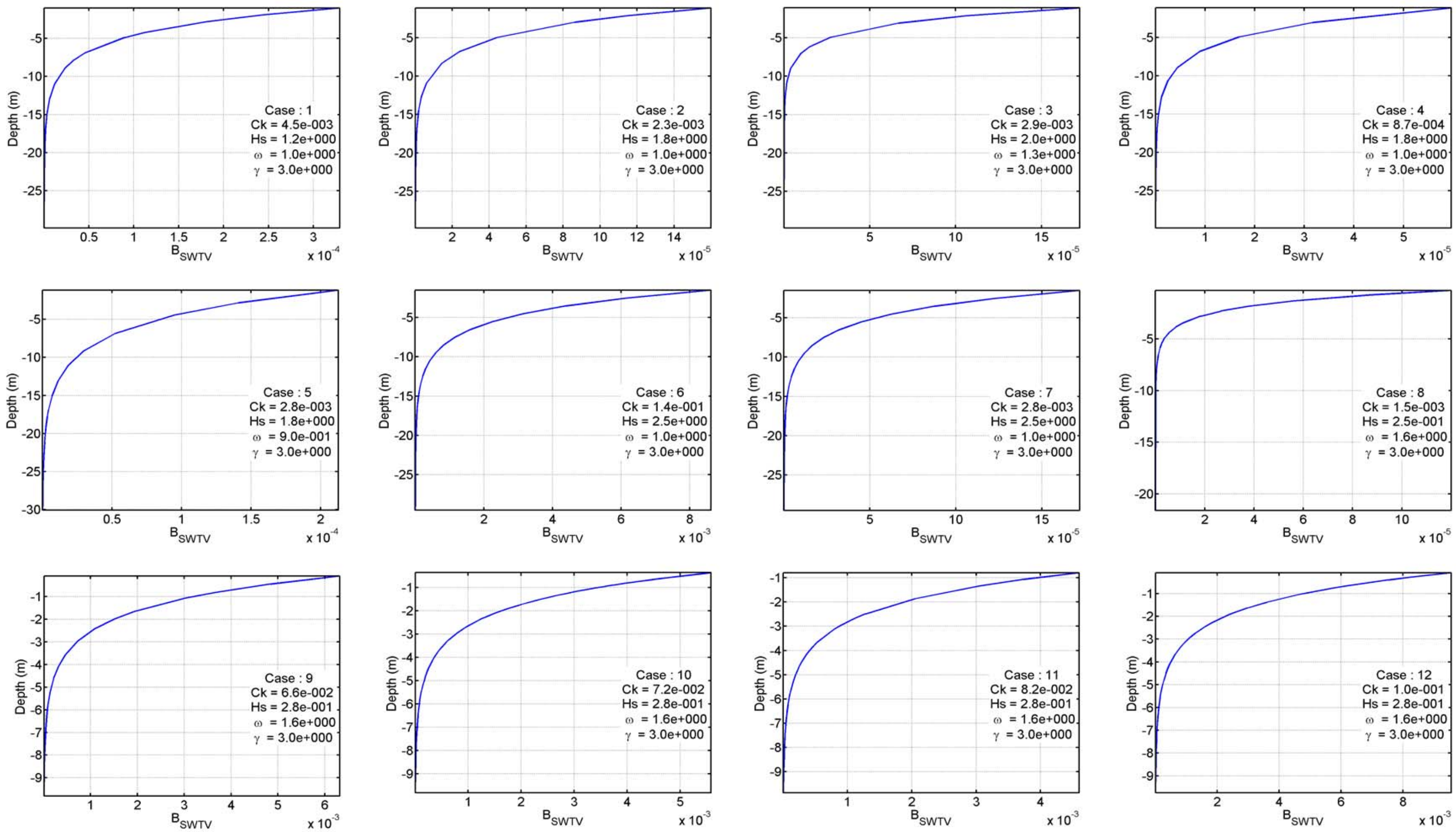
4. The theoretic mixing coefficient based on the balanced solution

$$\left(\bar{B}_{SWTV}\right)_{Theory} = \frac{\bar{k}^2}{\pi^2 \bar{\varepsilon}} = \frac{C_K}{\pi^2} \exp\left\{ \gamma K x_3 \right\}$$

$$\left(B_{WV}\right)_{\Pr andtl} = 4\left[\iint_{k_\beta} E_W\left(k_1, k_2\right) \frac{sh\left\{2k\left(x_3 + H\right)\right\}}{\left| sh\left\{ kH \right\}\right|^2} dk_1 dk_2 \right] \left\{ \frac{\partial}{\partial x_3} \left[\iint_{k_\beta} \omega^2 E_W\left(k_1, k_2\right) \frac{sh\left\{2k\left(x_3 + H\right)\right\}}{\left| sh\left\{ kH \right\}\right|^2} dk_1 dk_2 \right]^{1/2} \right\}$$

$$= 4\left(\overline{aK}\right) a^2 \omega \exp\left\{ 3K x_3 \right\}$$

1) The vertical distribution of the theoretic mixing coefficient $\left(\bar{B}_{SWTV}\right)_{Theory}$



2) The comparison of the theoretic mixing coefficient $\left(\bar{B}_{SWTV}\right)_{Theory}$ to the Prandtl's one $\left(B_{WV}\right)_{\Pr andtl}$

$$\left(\bar{B}_{SWTV}\right)_{Theory} = \frac{\bar{k}^2}{\pi^2 \bar{\varepsilon}} = \frac{C_K}{\pi^2} \exp\left\{\gamma K x_3\right\}$$

$$\left(B_{WV}\right)_{\Pr andtl} = 4\left(\overline{aK}\right) a^2 \omega \exp\left\{3 K x_3\right\}$$

or

$$\left(\tilde{B}_{SWTV}\right)_{Theory} = \frac{1}{4} \iint_{k_\alpha} \omega_{\mathrm{R}} E\left(k_1, k_2\right) \frac{\mathrm{sh}\left\{2k\left(x_3 + H\right)\right\}}{\left|\mathrm{sh}\left\{k\left(-H\right)\right\}\right|^2} dk_1 dk_2$$

$$\left(B_{WV}\right)_{\Pr andtl} = 4\left[\iint_{k_\beta} E_W\left(k_1, k_2\right) \frac{sh\left\{2k\left(x_3 + H\right)\right\}}{\left|sh\left\{kH\right\}\right|^2} dk_1 dk_2\right] \left\{\frac{\partial}{\partial x_3}\left[\iint_{k_\beta} \omega^2 E_W\left(k_1, k_2\right) \frac{sh\left\{2k\left(x_3 + H\right)\right\}}{\left|sh\left\{kH\right\}\right|^2} dk_1 dk_2\right]^{1/2}\right\}$$

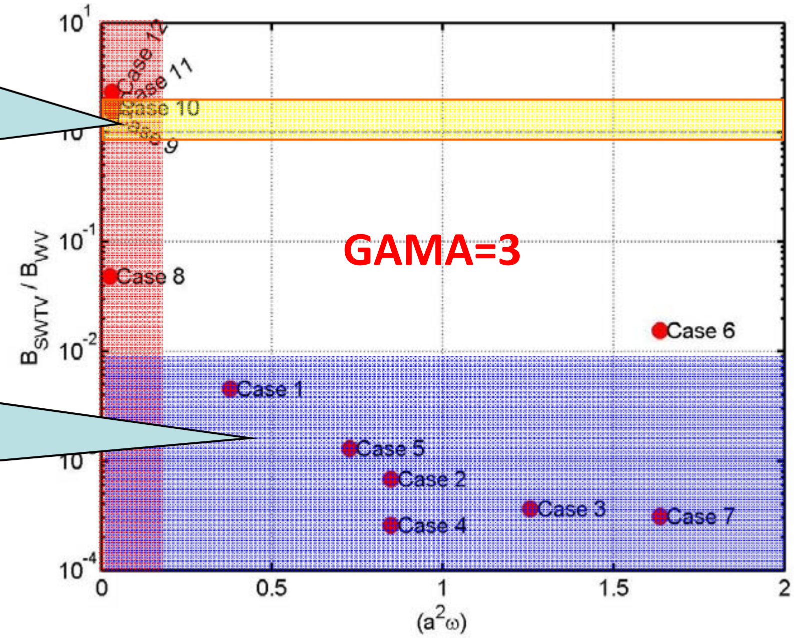
2) The comparison of the theoretic mixing coefficient $(\bar{B}_{SWTV})_{Theory}$ to the Prandtl's one $(B_{WV})_{Prandtl}$

In the range of small $(a^2\omega)$ value

$$\left[\frac{(\tilde{B}_{SWTV})_{Theory}}{(B_{WV})_{Prandtl}} \approx 1 \right]_{\text{CASES 9, 10, 11, 12}}$$

In the range of large $(a^2\omega)$ value

$$\left[\frac{(\tilde{B}_{SWTV})_{Theory}}{(B_{WV})_{Prandtl}} << 1 \right]_{\text{CASES 1, 2, 3, 4, 5, 6, 7}}$$



The mixing produced by sea wave-generated turbulence looks like large in a small range of $a^2\omega \approx 0$,

more observations and experiments are needed for understanding the turbulence mixing in whole range of $a^2\omega$

and the mixing produced by sea waves themselves should be studied deeply.

III. The analytic estimate of sea (and internal) wave mixing

$$\frac{\partial \bar{\mathbf{U}}}{\partial t} + \left(\bar{\mathbf{U}} \cdot \nabla\right)\bar{\mathbf{U}} + 2\bar{\mathbf{\Omega}} \times \bar{\mathbf{U}} = -\frac{1}{\rho_0}\nabla \bar{p} - g\frac{\bar{\rho}}{\rho_0}\mathbf{k} + \nu_0 \Delta \bar{\mathbf{U}}$$

$$+\frac{\partial}{\partial x_j}\left[-\left\langle\left\langle\left\langle u'_{SS\,i} u'_{SS\,j}\right\rangle_{SS}\right\rangle_{SM}\right\rangle_{MM}\right] + \boxed{\frac{\partial}{\partial x_j}\left[-\left\langle\left\langle u'_{SM\,i} u'_{SM\,j}\right\rangle_{SM}\right\rangle_{MM}\right]} + \frac{\partial}{\partial x_j}\left[-\left\langle u'_{MM\,i} u'_{MM\,j}\right\rangle_{MM}\right]$$

The original definition of the mixing produced by waves themselves

In order to calculate the mixing produced by waves in second order accuracy, we need to develop a wave theory without the ir-rotational assumption.

Yuan, Y.L., L. Han, et al., 2011: The unified linear theory of wavelike perturbations under general ocean conditions, Dyn. Atm. & Ocean., Vol.51, 55-74

I) A unified theory of wavelike perturbations in general ocean conditions

1. The governing equations of the motions

1) Motion equations

$$\gamma u_1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

$$\frac{\partial u_1}{\partial t} - F u_2 + \Delta_{M1} = -\frac{\partial}{\partial x_1}\left(\frac{p}{\rho_0}\right)$$

$$\frac{\partial u_2}{\partial t} + \left(F + \frac{\partial \bar{U}_2}{\partial x_1}\right)u_1 + \frac{\partial \hat{U}_2}{\partial x_3}u_3 + \Delta_{M2} = -\frac{\partial}{\partial x_2}\left(\frac{p}{\rho_0}\right)$$

$$\frac{\partial u_3}{\partial t} + \Delta_{M3} = -\frac{\partial}{\partial x_3}\left(\frac{p}{\rho_0}\right) - g\left(\frac{\rho}{\rho_0}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) + \frac{\partial}{\partial x_1}\left(\frac{\hat{\rho}}{\rho_0}\right)u_1 + \frac{\partial}{\partial x_3}\left(\frac{\hat{\rho}}{\rho_0}\right)u_3 + \Delta_D = 0$$

2) Boundary conditions

$$\left(\frac{p}{\rho_0}\right)_{x_3=0} = \left(\frac{p_A}{\rho_0}\right) + g\left(\frac{\hat{\rho}}{\rho_0}\right)_{x_3=0} h + \Delta_{S1}$$

$$\left(u_3\right)_{x_3=0} - \frac{\partial h}{\partial t} + \Delta_{S2} = 0$$

$$\left(u_3\right)_{x_3=-H} + \frac{\partial \hat{H}}{\partial x_1}\left(u_1\right)_{x_3=-H} + \Delta_{B1} = 0$$

Dynamic balance of gravity in general ocean conditions

2. The analytic solution of the linear theory

1) The expression of the wave motion

$$u_{SW1} = \iint_{\vec{k}} \eta_{SW}\mu_{SW1}(x_3)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1dk_2$$

$$u_{SW2} = \iint_{\vec{k}} \eta_{SW}\mu_{SW2}(x_3)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1dk_2$$

$$u_{SW3} = \iint_{\vec{k}} \eta_{SW}\,\mu_{SW3}(x_3)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1dk_2$$

$$\left(\frac{p_{SW}}{\rho_0}\right) = \iint_{\vec{k}} \eta_{SW}\phi_{SW}(x_3)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1dk_2$$

$$\left(\frac{\rho_{SW}}{\rho_0}\right) = \iint_{\vec{k}} \eta_{SW}\beta_{SW}(x_3)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1dk_2$$

$$h_{SW} = \iint_{\vec{k}} \eta_{SW}\exp\{i(k_\alpha x_\alpha - \omega_0 t)\}dk_1dk_2$$

in which

$$\mu_{SW1} = -\frac{\varpi_0}{(I_4)'}\exp\{i\chi_{31}(x_3)\}\left\{\begin{array}{l} i\left\{\begin{array}{l} -\dfrac{1}{2}\left[\dfrac{M^2}{\Omega^2}\left(\dfrac{k_1}{k}I_1\right)+i\dfrac{\omega}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}\left(\dfrac{k_2}{k}I_3\right)\right]\left(\dfrac{k_1}{k}I_1\right)-\dfrac{F}{\omega}\dfrac{\omega}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}I_4 \\ -\left\{\begin{array}{l} \dfrac{1}{2}\dfrac{F}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}\left[\dfrac{M^2}{\Omega^2}\left(\dfrac{k_1}{k}I_1\right)+i\dfrac{\omega}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}\left(\dfrac{k_2}{k}I_3\right)\right] \\ +\dfrac{1}{\Omega^2}\left[\omega^2-\left(N^2\right)'\right]\left(\dfrac{k_1}{k}I_1\right) \end{array}\right\}\bar{\delta}_{-H} \end{array}\right\}\dfrac{\mathrm{sh}\{\chi_{32}(x_3)-\chi_{32}(-H)\}}{\mathrm{sh}\{\chi_{32}(-H)\}} \\ +\left[\left(\dfrac{k_1}{k}I_1\right)-\dfrac{F}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}\bar{\delta}_{-H}\right]\left\{\dfrac{1}{\Omega^2}\left[\omega^2-\left(N^2\right)^*\right]I_4\right\}^{1/2}\dfrac{\mathrm{ch}\{\chi_{32}(x_3)-\chi_{32}(-H)\}}{\mathrm{sh}\{\chi_{32}(-H)\}} \end{array}\right\}$$

$$
\mu_{SW2} = -\frac{\varpi_0}{\left(I_4\right)^{/}}\exp\left\{iX_{31}\left(x_3\right)\right\}\left\{\begin{aligned}
& i\left\{\begin{aligned}
& -\frac{1}{2}\left[\frac{M^2}{\Omega^2}\left(\frac{k_1}{k}I_1\right)+i\frac{\omega}{\Omega^2}\frac{\partial\hat{U}_2}{\partial x_3}\left(\frac{k_2}{k}I_3\right)\right]\left(\frac{k_2}{k}I_2\right)+i\frac{\omega}{\Omega^2}\frac{\partial\hat{U}_2}{\partial x_3}I_4 \\
& +i\left\{\begin{aligned}
& \frac{1}{2}\frac{\omega}{\Omega^2}\frac{\partial\hat{U}_2}{\partial x_3}\left[\frac{M^2}{\Omega^2}\left(\frac{k_1}{k}I_1\right)+i\frac{\omega}{\Omega^2}\frac{\partial\hat{U}_2}{\partial x_3}\left(\frac{k_2}{k}I_3\right)\right] \\
& +i\frac{1}{\Omega^2}\left[\omega^2-\left(N^2\right)^{/}\right]\left(\frac{k_2}{k}I_2\right)
\end{aligned}\right\}\bar{\delta}_{-H}
\end{aligned}\right\}\frac{\mathrm{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\mathrm{sh}\left\{\chi_{32}\left(-H\right)\right\}} \\
& +\left[\left(\frac{k_2}{k}I_2\right)+i\frac{\omega}{\Omega^2}\frac{\partial\hat{U}_2}{\partial x_3}\bar{\delta}_{-H}\right]\left\{\frac{1}{\Omega^2}\left[\omega^2-\left(N^2\right)^{*}\right]I_4\right\}^{1/2}\frac{\mathrm{ch}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\mathrm{sh}\left\{\chi_{32}\left(-H\right)\right\}}
\end{aligned}\right.
$$

$$
\mu_{SW3} = i\frac{\varpi_0}{\left(I_4\right)^{/}}\exp\left\{i\chi_{31}\left(x_3\right)\right\}\left\{\begin{aligned}
& \left\{I_4+\frac{1}{2}\left[\frac{M^2}{\Omega^2}\left(\frac{k_1}{k}I_1\right)+i\frac{\omega}{\Omega^2}\frac{\partial\hat{U}_2}{\partial x_3}\left(\frac{k_2}{k}I_3\right)\right]\bar{\delta}_{-H}\right\}\frac{\mathrm{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\mathrm{sh}\left\{\chi_{32}\left(-H\right)\right\}} \\
& -i\left\{\frac{1}{\Omega^2}\left[\omega^2-\left(N^2\right)^{*}\right]I_4\right\}^{1/2}\bar{\delta}_{-H}\frac{\mathrm{ch}\left\{i\chi_{32}\left(x_3\right)-i\chi_{32}\left(-H\right)\right\}}{\mathrm{sh}\left\{i\chi_{32}\left(-H\right)\right\}}
\end{aligned}\right.
$$

$$\phi_{SW}=-\frac{\varpi_0}{\left(I_4\right)^{/}}\exp\left\{i\chi_{31}\left(x_3\right)\right\}\frac{\Omega^2}{\omega k}\left\{\begin{array}{l}-i\left\{\begin{array}{l}\dfrac{1}{2}\left[\dfrac{M^{2}}{\Omega^2}\left(\dfrac{k_1}{k}I_1\right)+i\dfrac{\omega}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}\left(\dfrac{k_2}{k}I_3\right)\right]\\+\dfrac{1}{\Omega^2}\left[\omega^2-\left(N^{2}\right)^{/}\right]\bar{\delta}_{-H}\end{array}\right\}\dfrac{\operatorname{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}}\\+\left\{\dfrac{1}{\Omega^2}\left[\omega^2-\left(N^{2}\right)^{*}\right]I_4\right\}^{1/2}\dfrac{\operatorname{ch}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}}\end{array}\right.$$

$$\beta_{SW}=-\frac{\varpi_0}{\left(I_4\right)^{/}}\exp\left\{i\chi_{31}\left(x_3\right)\right\}\frac{1}{g\omega}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(N^{2}\right)^{/}I_4+\dfrac{1}{2}M^{2}\left[\dfrac{M^{2}}{\Omega^2}\left(\dfrac{k_1}{k}I_1\right)+i\dfrac{\omega}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}\left(\dfrac{k_2}{k}I_3\right)\right]\left(\dfrac{k_1}{k}I_1\right)\\+\left\{\begin{array}{l}\dfrac{1}{2}\left[\dfrac{M^{2}}{\Omega^2}\left(\dfrac{k_1}{k}I_1\right)+i\dfrac{\omega}{\Omega^2}\dfrac{\partial\hat{U}_2}{\partial x_3}\left(\dfrac{k_2}{k}I_3\right)\right]\left(N^{2}\right)^{/}\\+\dfrac{M^{2}}{\Omega^2}\left[\omega^2-\left(N^{2}\right)^{/}\right]\left(\dfrac{k_1}{k}I_1\right)\end{array}\right\}\bar{\delta}_{-H}\end{array}\right\}\dfrac{\operatorname{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}}\\+i\left[M^{2}\left(\dfrac{k_1}{k}I_1\right)-\left(N^{2}\right)^{/}\bar{\delta}_{-H}\right]\left\{\dfrac{1}{\Omega^2}\left[\omega^2-\left(N^{2}\right)^{*}\right]I_4\right\}^{1/2}\dfrac{\operatorname{ch}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}}\end{array}\right.$$

2) The relations of complex frequency to wave-number

(1) The relations of the complex frequency to the wave-number on the sea surface

$$\left\{\left[\omega_0^2-\left(N^2\right)_0^*\right]\left[\omega_0^2-F\left(F+\frac{\partial\bar{U}_2}{\partial x_1}\right)_0\right]I_{40}\right\}^{1/2}$$

$$=\pm\left\{\left(\frac{\hat{\rho}_0}{\rho_0}\right)\left(\frac{\omega_0}{\varpi_0}\right)\left(I_4\right)_0' g\left(-k_0\right)+\frac{1}{2}\left[\omega_0\left(\frac{\partial\hat{U}_2}{\partial x_3}\right)_0\left(\frac{k_2}{k}I_3\right)_0-iM^2\left(\frac{k_1}{k}I_1\right)\right]-i\left(\omega_0^2-\left(N^2\right)_0'\right)\left(\bar{\delta}_{-H}\right)_0\right\}\left|\frac{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}}{\operatorname{ch}\left\{\chi_{32}\left(-H\right)\right\}}\right|$$

(2) The vertical invariance of the complex frequency

$$\frac{\partial\omega}{\partial x_3}=0$$

(3) The expression of the vertical wave-number

$$k_{31}=-\frac{1}{2\Omega^2}\left[M^2\left(\frac{k_1}{k}I_1\right)-i\omega\frac{\partial\hat{U}_2}{\partial x_3}\left(\frac{k_2}{k}I_3\right)\right]k$$

$$k_{32}=\left\{\frac{1}{\Omega^2}\left[\omega^2-\left(N^2\right)^*\right]I_4\right\}^{1/2}k$$

$$\chi_{31}\left(x_3\right)=-\int_0^{x_3}\frac{1}{2\Omega^2}\left[M^2\left(\frac{k_1}{k}I_1\right)-i\omega\frac{\partial\hat{U}_2}{\partial x_3}\left(\frac{k_2}{k}I_3\right)\right]k\,dx_3$$

$$\chi_{32}\left(x_3\right)=\int_0^{x_3}\left\{\frac{1}{\Omega^2}\left[\omega^2-\left(N^2\right)^*\right]I_4\right\}^{1/2}k\,dx_3$$

II) The derivation of the vertical wave mixing coefficients

1. The wave transport fluxes of density and velocity in second order accuracy

1) The expression of sea wave motions by the real part of the solution

$$u_{SW\,i}^{R} = \mathrm{Re}\left(u_{SWi}\right) = \frac{1}{2}\iint_{k_\alpha}\left[\eta_{SW}\mu_{SW\,i}\left(x_3\right)\exp\left\{i\left(k_\alpha x_\alpha - \omega t\right)\right\} + \eta_{SW}^{*}\mu_{SW\,i}^{*}\left(x_3\right)\exp\left\{-i\left(k_\alpha x_\alpha - \omega^{*} t\right)\right\}\right]\left(dk_1 dk_2\right)_\alpha$$

$$\left(\frac{\rho_{SW}^{R}}{\rho_0}\right) = \mathrm{Re}\left(\frac{\rho_{SW}}{\rho_0}\right) = \frac{1}{2}\iint_{k_\gamma}\left[\eta_{SW}\beta_{SW}\left(x_3\right)\exp\left\{i\left(k_\gamma x_\gamma - \omega t\right)\right\} + \eta_{SW}^{*}\beta_{SW}^{*}\left(x_3\right)\exp\left\{-i\left(k_\gamma x_\gamma - \omega^{*} t\right)\right\}\right]\left(dk_1 dk_2\right)_\gamma$$

2) The expression of the transport fluxes

of density and velocity by sea waves

$$-\left\langle \left(\frac{\rho_{SW}^{R}}{\rho_0} \right) u_{SWi}^{R} \right\rangle_{SW} \approx \frac{1}{2} \bar{N}_{GSW} \iint_{k_\alpha} E\left(k_1,k_2\right) \mathrm{Re} \begin{Bmatrix} -\beta_{SW}\left(x_3\right) \mu_{SW1}^{*}\left(x_3\right) \\ -\beta_{SW}\left(x_3\right) \mu_{SW2}^{*}\left(x_3\right) \\ -\beta_{SW}\left(x_3\right) \mu_{SW3}^{*}\left(x_3\right) \end{Bmatrix} \left(dk_1 dk_2\right)_\alpha$$

$$-\left\langle u_{SWi}^{R} u_{SWj}^{R} \right\rangle_{SW} \approx \frac{1}{2} \bar{N}_{GSW} \iint_{k_\alpha} E\left(k_1,k_2\right) \mathrm{Re} \begin{Bmatrix} -\mu_{SW1}\left(x_3\right)\mu_{SW1}^{*}\left(x_3\right) & -\mu_{SW1}\left(x_3\right)\mu_{SW2}^{*}\left(x_3\right) & -\mu_{SW1}\left(x_3\right)\mu_{SW3}^{*}\left(x_3\right) \\ -\mu_{SW2}\left(x_3\right)\mu_{SW1}^{*}\left(x_3\right) & -\mu_{SW2}\left(x_3\right)\mu_{SW2}^{*}\left(x_3\right) & -\mu_{SW2}\left(x_3\right)\mu_{SW3}^{*}\left(x_3\right) \\ -\mu_{SW3}\left(x_3\right)\mu_{SW1}^{*}\left(x_3\right) & -\mu_{SW3}\left(x_3\right)\mu_{SW2}^{*}\left(x_3\right) & -\mu_{SW3}\left(x_3\right)\mu_{SW3}^{*}\left(x_3\right) \end{Bmatrix} \left(dk_1 dk_2\right)_\alpha$$

in the derivation of the formulas, the statistical relations of homogeneous motion were used

$$\begin{aligned} \left\langle \eta_{SW}\left(k_1,k_2;t\right)\eta_{SW}^{*}\left(k_1',k_2';t\right)\right\rangle &= \delta\left(k_1-k_1'\right)\delta\left(k_2-k_2'\right)A_{SW}\left(k_1,k_2;t\right)A_{SW}^{*}\left(k_1,k_2;t\right) \\ &= \delta\left(k_1-k_1'\right)\delta\left(k_2-k_2'\right)E_{SW}\left(k_1,k_2;t\right) \end{aligned}$$

$$\left\langle \eta_{SW}\left(k_1,k_2;t\right)\eta_{SW}\left(k_1',k_2';t\right)\right\rangle = \delta\left(k_1-k_1'\right)\delta\left(k_2-k_2'\right)A_{SW}\left(k_1,k_2;t\right)A_{SW}\left(k_1,k_2;t\right)$$

3) The expression of vertical transport flux of density in second order accuracy

$$-\left\langle \left(\frac{\rho_{SW}^{R}}{\rho_0}\right) u_{SW3}^{R} \right\rangle_{SW} \approx \frac{1}{2} \bar{N}_{GSW} \iint_{k_\alpha} E(k_1,k_2)\,\mathrm{Re}\left\{-\beta_{SW}(x_3)\,\mu_{SW3}^{*}(x_3)\right\} dk_1 dk_2$$

in which

$$\mathrm{Re}\left\{-\beta_{SW}(x_3)\,\mu_{SW3}^{*}(x_3)\right\}$$

$$= \mathrm{Re}\left\{ \left[\frac{\frac{|\varpi_0|^2 \omega}{g}}{\exp\left\{-2\,\mathrm{Im}\left\{\chi_{31}(x_3)\right\}\right\}} \right] \left\{ \begin{array}{l} + \left[\begin{array}{l} \frac{M^2}{\omega^2}\left(\frac{k_1}{k}\right) \frac{\mathrm{ch}\left\{\chi_{32}(x_3)-\chi_{32}(-H)\right\}}{\mathrm{sh}\left\{\chi_{32}(-H)\right\}} \frac{\mathrm{sh}\left\{\chi_{32}^{*}(x_3)-\chi_{32}^{*}(-H)\right\}}{\mathrm{sh}\left\{\chi_{32}^{*}(-H)\right\}} \\ + \frac{N^2}{\omega^2}\bar{\delta}_{-H} \frac{\mathrm{sh}\left\{\chi_{32}(x_3)-\chi_{32}(-H)\right\}}{\mathrm{sh}\left\{\chi_{32}(-H)\right\}} \frac{\mathrm{ch}\left\{\chi_{32}^{*}(x_3)-\chi_{32}^{*}(-H)\right\}}{\mathrm{sh}\left\{\chi_{32}^{*}(-H)\right\}} \end{array} \right] \\ -i \left[\begin{array}{l} \frac{N^2}{\omega^2}\left|\frac{\mathrm{sh}\left\{\chi_{32}(x_3)-\chi_{32}(-H)\right\}}{\mathrm{sh}\left\{\chi_{32}(-H)\right\}}\right|^2 \\ -\frac{M^2}{\omega^2}\left(\frac{k_1}{k}\right)\bar{\delta}_{-H}\left|\frac{\mathrm{ch}\left\{\chi_{32}(x_3)-\chi_{32}(-H)\right\}}{\mathrm{sh}\left\{\chi_{32}(-H)\right\}}\right|^2 \end{array} \right] \end{array} \right\} \right\}$$

4) The expression of vertical transport flux of velocity in second order accuracy

$$-\left\langle u_{SW1}^{R} u_{SW3}^{R} \right\rangle_{SW} \approx \frac{1}{2}\bar{N}_{GSW} \iint_{k_\alpha} E_{SW}\left(k_1,k_2\right) \mathrm{Re}\left\{-\mu_{SW1}\left(x_3\right)\mu_{SW3}^{*}\left(x_3\right)\right\} dk_1 dk_2$$

$$-\left\langle u_{SW2}^{R} u_{SW3}^{R} \right\rangle_{SW} \approx \frac{1}{2}\bar{N}_{GSW} \iint_{k_\alpha} E_{SW}\left(k_1,k_2\right) \mathrm{Re}\left\{-\mu_{SW2}\left(x_3\right)\mu_{SW3}^{*}\left(x_3\right)\right\} dk_1 dk_2$$

in which

$$\mathrm{Re}\left\{-\mu_{SW1}\left(x_3\right)\mu_{SW3}^{*}\left(x_3\right)\right\}$$

$$= \mathrm{Re}\left\{ \left[\begin{array}{l} -\left|\varpi_0\right|^2 \\ \exp\left\{-2\,\mathrm{Im}\left\{\chi_{31}\left(x_3\right)\right\}\right\} \end{array}\right] \left(\left[\begin{array}{l} \left\{\frac{1}{2}\left[\frac{M^2}{\omega^2}\left(\frac{k_1}{k}\right) + i\frac{1}{\omega}\frac{\partial \hat{U}_2}{\partial x_3}\left(\frac{k_2}{k}\right)\right] + \bar{\delta}_{-H}\right\}\left(\frac{k_1}{k}\right)\left|\frac{\mathrm{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\mathrm{sh}\left\{\chi_{32}\left(-H\right)\right\}}\right|^2 \\ -\left\{\left(1-\frac{N^2}{\omega^2}\right)\frac{1}{I_4}\right\}^{1/2}\bar{\delta}_{-H}\left(\frac{k_1}{k}\right)\left|\frac{\mathrm{ch}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\mathrm{sh}\left\{\chi_{32}\left(-H\right)\right\}}\right|^2 \end{array}\right] + i\left[\begin{array}{l} \left\{\left(1-\frac{N^2}{\omega^2}\right)\frac{1}{I_4}\right\}^{1/2}\left(\frac{k_1}{k}\right)\frac{\mathrm{ch}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}\,\mathrm{sh}\left\{\chi_{32}^{*}\left(x_3\right)-\chi_{32}^{*}\left(-H\right)\right\}}{\mathrm{sh}\left\{\chi_{32}\left(-H\right)\right\}\mathrm{sh}\left\{\chi_{32}^{*}\left(-H\right)\right\}} \\ +\left\{\frac{1}{2}\left[\frac{M^2}{\omega^2}\left(\frac{k_1}{k}\right)^2 + i\frac{1}{\omega}\frac{\partial \hat{U}_2}{\partial x_3}\left(\frac{k_1}{k}\right)\left(\frac{k_2}{k}\right)\right] + \bar{\delta}_{-H}\left(\frac{k_1}{k}\right)\right\}\bar{\delta}_{-H}\frac{\mathrm{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}\,\mathrm{ch}\left\{\chi_{32}^{*}\left(x_3\right)-\chi_{32}^{*}\left(-H\right)\right\}}{\mathrm{sh}\left\{\chi_{32}\left(-H\right)\right\}\mathrm{sh}\left\{\chi_{32}^{*}\left(-H\right)\right\}} \end{array}\right] \right) \right\}$$

$$
\begin{aligned}
&\operatorname{Re}\left\{-\mu_{SW\,2}\left(x_3\right)\mu^*_{SW\,3}\left(x_3\right)\right\} \\
&=\operatorname{Re}\left\{
\begin{bmatrix} -\left|\varpi_0\right|^2 \\ \exp\left\{-2\operatorname{Im}\left\{\chi_{31}\left(x_3\right)\right\}\right\} \end{bmatrix}
\left\{
\begin{array}{l}
\left[
\begin{array}{l}
\left\{\left[\frac{1}{2}\frac{M^2}{\omega^2}\left(\frac{k_1}{k}\right)+\bar{\delta}_{-H}\right]\left(\frac{k_2}{k}\right)-i\frac{1}{\omega}\frac{\partial\hat{U}_2}{\partial x_3}\left[1-\frac{1}{2}\left(\frac{k_2}{k}\right)^2\right]\right\}\left|\frac{\operatorname{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}}\right|^2 \\
-\left\{\left(1-\frac{N^2}{\omega^2}\right)\frac{1}{I_4}\right\}^{1/2}\bar{\delta}_{-H}\left[\left(\frac{k_2}{k}\right)-i\frac{1}{\omega}\frac{\partial\bar{U}_2}{\partial x_1}\left(\frac{k_1}{k}\right)\right]\left|\frac{\operatorname{ch}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}}\right|^2
\end{array}
\right] \\
+i\left[
\begin{array}{l}
\left\{\left(1-\frac{N^2}{\omega^2}\right)\frac{1}{I_4}\right\}^{1/2}\left[\left(\frac{k_2}{k}\right)-i\frac{1}{\omega}\frac{\partial\bar{U}_2}{\partial x_1}\left(\frac{k_1}{k}\right)\right]\frac{\operatorname{ch}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}\operatorname{sh}\left\{\chi^*_{32}\left(x_3\right)-\chi^*_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}\operatorname{sh}\left\{\chi^*_{32}\left(-H\right)\right\}} \\
+\left\{
\begin{array}{l}
\frac{1}{2}\frac{M^2}{\omega^2}\left(\frac{k_1}{k}\right)\left(\frac{k_2}{k}\right)-i\frac{1}{\omega}\frac{\partial\hat{U}_2}{\partial x_3}\left[1-\frac{1}{2}\left(\frac{k_2}{k}\right)^2\right] \\
+\bar{\delta}_{-H}\left(\frac{k_2}{k}\right)
\end{array}
\right\}\bar{\delta}_{-H}\frac{\operatorname{sh}\left\{\chi_{32}\left(x_3\right)-\chi_{32}\left(-H\right)\right\}\operatorname{ch}\left\{\chi^*_{32}\left(x_3\right)-\chi^*_{32}\left(-H\right)\right\}}{\operatorname{sh}\left\{\chi_{32}\left(-H\right)\right\}\operatorname{sh}\left\{\chi^*_{32}\left(-H\right)\right\}}
\end{array}
\right]
\end{array}
\right\}
\right\}
\end{aligned}
$$

2. The wave transport fluxes of density and velocity in first order accuracy

1) The expression of the vertical transport flux of density in first order accuracy

$$\boxed{-\left\langle \left(\frac{\rho_{SW}^{R}}{\rho_0}\right) u_{SW3}^{R} \right\rangle_{SW} \approx \left\{ \frac{1}{4} \bar{N}_{GSW} \hat{R}_D \iint_{k_\alpha} \left(\frac{k_1}{k}\right) \mathrm{Re}\{\omega\} E(k_1,k_2) \frac{\mathrm{sh}\{2k(x_3+H)\}}{\left|\mathrm{sh}\{-kH\}\right|^2} dk_1 dk_2 \right\} \frac{\partial}{\partial x_3}\left(\frac{\hat{\rho}}{\rho_0}\right)} = \hat{B}_{SWWVD} \frac{\partial}{\partial x_3}\left(\frac{\hat{\rho}}{\rho_0}\right)$$

in which

$$\bar{N}_{GSW} \equiv \exp\left\{ 2\omega_I t \big|_{t=TGSW} \right\}$$

the times of wave growth

$$\hat{R}_D \equiv \left[-\frac{\partial}{\partial x_1}\left(\frac{\hat{\rho}}{\rho_0}\right) \Big/ \frac{\partial}{\partial x_3}\left(\frac{\hat{\rho}}{\rho_0}\right) \right]$$

the Richardson number of density

$$\hat{B}_{SWWVD} = \frac{1}{4} \bar{N}_{GSW} \hat{R}_D \iint_{k_\alpha} \left(\frac{k_1}{k}\right) \mathrm{Re}\{\omega\} E(k_1,k_2) \frac{\mathrm{sh}\{2k(x_3+H)\}}{\left|\mathrm{sh}\{k(-H)\}\right|^2} dk_1 dk_2$$

the vertical mixing coefficient for density

2) The expression of the vertical transport flux of velocity in first order accuracy

$$-\left\langle u^R_{SW1} u^R_{SW3} \right\rangle_{SW} \approx 0$$

$$\boxed{-\left\langle u^R_{SW2} u^R_{SW3} \right\rangle_{SW} \approx \frac{1}{4} \bar{N}_{GSW} \hat{R}_U \iint_{k_\alpha} \left(\frac{k_1}{k}\right) \mathrm{Re}\{\omega\} E_{SW}(k_1,k_2) \frac{\mathrm{sh}\{2k(x_3+H)\}}{\left|\mathrm{sh}\{k(-H)\}\right|^2} dk_1 dk_2 \left(\frac{\partial \hat{U}_2}{\partial x_3}\right)} = \tilde{B}_{SWWVU} \left(\frac{\partial \hat{U}_2}{\partial x_3}\right)$$

in which

$$\hat{R}_U \equiv \left(-\frac{\partial \bar{U}_2}{\partial x_1} \Big/ \frac{\partial \hat{U}_2}{\partial x_3}\right)$$

the Richardson number of velocity

$$\tilde{B}_{SWWVU} = \frac{1}{4} \bar{N}_{GSW} \hat{R}_U \iint_{k_\alpha} \left(\frac{k_1}{k}\right) \mathrm{Re}\{\omega\} E_{SW}(k_1,k_2) \frac{\mathrm{sh}\{2k(x_3+H)\}}{\left|\mathrm{sh}\{k(-H)\}\right|^2} dk_1 dk_2$$

the vertical mixing coefficient for velocity (momentum)

3. The comparison of the theoretic mixing coefficients to the Prandtl's one

1) Comparison in the expression form

$$\left(\hat{B}_{SWWVD}\right)_{Theory}=\frac{1}{4}\bar{N}_{GSW}\left[-\frac{\partial}{\partial x_1}\left(\frac{\hat{\rho}}{\rho_0}\right)\Big/\frac{\partial}{\partial x_3}\left(\frac{\hat{\rho}}{\rho_0}\right)\right]\iint_{k_\alpha}\left(\frac{k_1}{k}\right)\mathrm{Re}\{\omega\}E(k_1,k_2)\frac{\mathrm{sh}\{2k(x_3+H)\}}{\left|\mathrm{sh}\{k(-H)\}\right|^2}dk_1dk_2$$

$$\left(\tilde{B}_{SWWVU}\right)_{Theory}=\frac{1}{4}\bar{N}_{GSW}\left[-\frac{\partial\bar{U}_2}{\partial x_1}\Big/\frac{\partial\hat{U}_2}{\partial x_3}\right]\iint_{k_\alpha}\left(\frac{k_1}{k}\right)\mathrm{Re}\{\omega\}E_{SW}(k_1,k_2)\frac{\mathrm{sh}\{2k(x_3+H)\}}{\left|\mathrm{sh}\{k(-H)\}\right|^2}dk_1dk_2$$

$$\left(B_{WV}\right)_{\mathrm{Prandtl}}=4\left[\iint_{k_\beta}E_W(k_1,k_2)\frac{sh\{2k(x_3+H)\}}{\left|sh\{kH\}\right|^2}dk_1dk_2\right]\left\{\frac{\partial}{\partial x_3}\left[\iint_{k_\beta}\omega^2E_W(k_1,k_2)\frac{sh\{2k(x_3+H)\}}{\left|sh\{kH\}\right|^2}dk_1dk_2\right]^{1/2}\right\}$$

They have same dimension and similar expression with sea wave spectrum.

The former two show the expressions of density and velocity respectively.

2) Comparison of the theoretic surface value to the Prandtl's one

$$\left(\tilde{B}_{SWWVD}\right)_{Theory} \approx \frac{1}{4}\bar{N}_{GSW}\hat{R}_D a^2\omega \exp\left\{2Kx_3\right\}\Big|_{x_3=0}$$

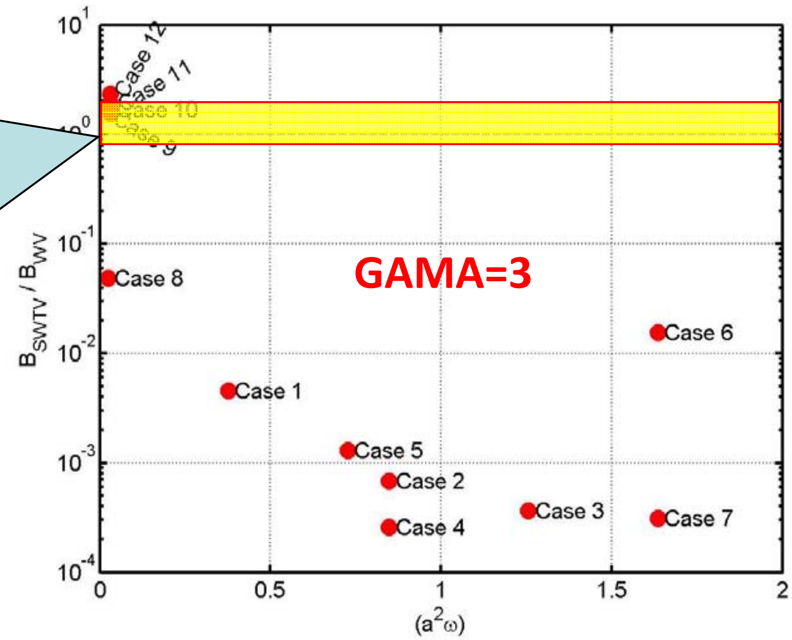
$$\left(B_{WV}\right)_{Prandtl} = 4\left(\overline{aK}\right)a^2\omega \exp\left\{3Kx_3\right\}\Big|_{x_3=0}$$

Due to $\bar{N}_{GSW} = O\left(10^2 \sim 10^3\right)$ **,** $R_{D,U} = o\left(10^{-2} \sim 10^{-3}\right)$

and $16\left(\overline{aK}\right) = O\left(10^0\right)$

so in all the range of $\left(a^2\omega\right)$ **we have**

$$\frac{\left(\tilde{B}_{SWWV}\right)_{Theory}}{\left(B_{WV}\right)_{Prandtl}} = \frac{\bar{N}_{GSW}\hat{R}_D}{16\left(\overline{aK}\right)} \approx 1$$



IV. Conclusion

1. **The ocean mixing is the residues of fluxes which are divided into three parts corresponding to turbulence-, wave- and eddy-perturbations.**

Ocean mixing $\left[-\left\langle \left\langle \left\langle u'_{SS\,i} u'_{SS\,j} \right\rangle_{SS} \right\rangle_{SM} \right\rangle_{MM} \quad -\left\langle \left\langle u'_{SM\,i} u'_{SM\,j} \right\rangle_{SM} \right\rangle_{MM} \quad -\left\langle u'_{MM\,i} u'_{MM\,j} \right\rangle_{MM} \right]$

It is not only produced by turbulence in narrow sense or in ambiguous sense

It is produced by three kinds of perturbation motions respectively.

2. The analytic estimates of sea wave-generated turbulence mixing and sea wave mixing have been made and are consistent with the Prandtl's one.

Ocean mixing

$$\left[-\left\langle\left\langle\left\langle u'_{SS\,i}u'_{SS\,j}\right\rangle_{SS}\right\rangle_{SM}\right\rangle_{MM} \quad -\left\langle\left\langle u'_{SM\,i}u'_{SM\,j}\right\rangle_{SM}\right\rangle_{MM} \quad -\left\langle u'_{MM\,i}u'_{MM\,j}\right\rangle_{MM}\right]$$

Sea-wave-generated turbulence mixing

$$\left(\bar{B}_{SWTV}\right)_{Theory}=\frac{\bar{k}^2}{\pi^2\bar{\varepsilon}}=\frac{C_K}{\pi^2}\exp\left\{\gamma K x_3\right\}$$

Sea-wave mixing

$$\left(\tilde{B}_{SWWV}\right)_{Theory}=\frac{1}{4}\bar{N}_{GSW}\left(\hat{R}_D,\hat{R}_U\right)\iint_{k_\alpha}\left(\frac{k_1}{k}\right)\mathrm{Re}\left\{\omega\right\}E_{SW}\left(k_1,k_2\right)\frac{\mathrm{sh}\left\{2k\left(x_3+H\right)\right\}}{\left|\mathrm{sh}\left\{k\left(-H\right)\right\}\right|^2}dk_1dk_2$$

So the theoretic wave-induced mixing includes two parts:

the wave - generated turbulence mixing

and the wave mixing,

and the mixing coefficient of the theoretic sea wave-induced mixing is consistent with the Prandtl's one quantitatively and qualitatively.

$$\left(B_{SW\ induced}\right)_{Theort} = \left(\bar{B}_{SWTV}\right)_{Theory} + \left(\tilde{B}_{SWWV}\right)_{Theory}$$

$$\left(B_{SW\ induced}\right)_{Theort} = \left(\bar{B}_{SWTV}\right)_{Theory} + \left(\tilde{B}_{SWWV}\right)_{Theory} \approx \left(B_{SWV}\right)_{\mathrm{Pr}\,andtl}$$

Comparison of the theoretical results of sea wave-induced mixing to that derived from the Prandtl mixing length theory

$$\left(\hat{B}_{SWD}\right)_{Theory}=\frac{1}{4}\left(1+\bar{N}_{GSW}\hat{R}_D\right)\iint_{k_\alpha}\omega_{\mathrm{R}}E\left(k_1,k_2\right)\frac{\mathrm{sh}\left\{2k\left(x_3+H\right)\right\}}{\left|\mathrm{sh}\left\{k\left(-H\right)\right\}\right|^2}dk_1dk_2$$

$$\left(\tilde{B}_{SWU}\right)_{Theory}=\frac{1}{4}\left(1+\bar{N}_{GSW}\hat{R}_U\right)\iint_{k_\alpha}\omega_{\mathrm{R}}E_{SW}\left(k_1,k_2\right)\frac{\mathrm{sh}\left\{2k\left(x_3+H\right)\right\}}{\left|\mathrm{sh}\left\{k\left(-H\right)\right\}\right|^2}dk_1dk_2$$

$$\left(B_{WV}\right)_{\mathrm{Pr}\,andtl}=4\left[\iint_{k_\beta}E_W\left(k_1,k_2\right)\frac{sh\left\{2k\left(x_3+H\right)\right\}}{\left|sh\left\{kH\right\}\right|^2}dk_1dk_2\right]\left\{\frac{\partial}{\partial x_3}\left[\iint_{k_\beta}\omega^2E_W\left(k_1,k_2\right)\frac{sh\left\{2k\left(x_3+H\right)\right\}}{\left|sh\left\{kH\right\}\right|^2}dk_1dk_2\right]^{1/2}\right\}$$

3. We still have works to do about the analytic estimate, specially, of the internal wave mixing and the ocean eddy one.

going to work on the mixing produced by ocean eddies themselves

Ocean mixing $\left[-\left\langle \left\langle \left\langle u'_{SS\,i} u'_{SS\,j} \right\rangle_{SS} \right\rangle_{SM} \right\rangle_{MM} \quad \boxed{ -\left\langle \left\langle u'_{SM\,i} u'_{SM\,j} \right\rangle_{SM} \right\rangle_{MM} \quad -\left\langle u'_{MM\,i} u'_{MM\,j} \right\rangle_{MM} } \right]$

going to work on the mixings produced by internal wave-generated turbulence and internal waves themselves

Thanks !!